AVAILABLE ONLINE AT IJO.IAURASHT.AC.IR

IRANIAN JOURNAL OF OPTIMIZATION

Volume 2, issue 2, spring 2010,150-158
Research Paper

Islamic Azad University
Rasht Branch

# *Decomposition Method for Solving Fully Fuzzy Linear Systems*

M. MOSLEH[1], M. OTADI [2*], A. KHANMIRZAIE [3]

[1,2]Department of mathematics, Islamic Azad University,Firuozkooh Branch,Firuozkooh,Iran[3] Department of mathematics, Islamic Azad University,Kermanshah Branch,kermanshah,Iran

*Correspondence E-mail: M. OTADI, mahmood_otadi@yahoo.com



***Abstract***

In this paper, we investigate the existence of a positive solution of fully fuzzy linear equation systems. This paper mainly to discuss a new decomposition of a nonsingular fuzzy matrix, the symmetric times triangular (ST) decomposition. By this decomposition, every nonsingular fuzzy matrix can be represented as a product of a fuzzy symmetric matrix S and a fuzzy triangular matrix T

***Keywords**:* Symmetric and triangular decomposition, Fuzzy system, Symmetric positive definite and triangular decomposition.

## *1 Introduction*

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [27], Dubois and Pard [10].We refer the reader to [21] for more information on fuzzy numbers and fuzzy arithmetic. Fuzzy systems are used to study a variety of problems ranging from fuzzy topological spaces [7] to control chaotic systems [17,20], fuzzy metric spaces [24], fuzzy differential equations [3], fuzzy linear and nonlinear systems [1,2,5,6], and particle physics [12,13,14,15,16,23,25].

One of the major applications of fuzzy number arithmetic is treating fuzzy linear systems and fully fuzzy linear systems, several problems in various areas such as economics, engineering and physics boil down to the solution of a linear system of equations. In many applications, at least some of the parameters of the system should be represented by fuzzy rather than crisp numbers. Thus, it is immensely important to develop numerical procedures that would appropriately treat fuzzy linear systems and solve them.

Friedman, et al. introduced a general model for solving a fuzzy $n \times n$ linear system whose coefficient matrix is crisp and the right –hand side column is an arbitrary fuzzy number vector [18]. They used the parametric from of fuzzy numbers and replaced the original fuzzy $n \times n$ linear system by a crisp $2n \times 2n$ linear system and studied duality in fuzzy linear systems $Ax = Bx + y$ where A, B are real $n \times n$ matrix, the unknown vector x is vector consisting of n fuzzy numbers and the constant y is vector consisting of n fuzzy numbers, in [19]. In [1,2,6] the authors presented conjugate gradient, LU decomposition method for solving general fuzzy linear systems or symmetric fuzzy linear systems. Also, Wang, et al. [26] presented an iterative algorithm for solving dual linear system of the form $x = Ax + u$ ,where A is real $n \times n$ matrix , the unknown vector x and the constant u are all vectors consisting of fuzzy numbers and Abbasbandy et al.[4] investigated the existence of a minimal solution of general dual fuzzy linear equation system of the form $Ax + f = Bx + c$ ,where A,B are real $m \times n$ matrix, the unknown vector x is vector consisting of n fuzzy numbers and the constant, f and c are vectors consisting of m fuzzy numbers. Recently, Muzziloi et al.[22] considered fully fuzzy linear systems of the form $A_1 x + b_1 = A_2 x + b_2$ with $A_1, A_2$ square matrices of fuzzy coefficients and $b_1, b_2$ fuzzy number vectors and Dehghan et al.[8] considered fully fuzzy linear systems of the form $Ax = b$, where A is a fuzzy matrix, the unknown vector x is vector consisting of n fuzzy numbers and the constant b are vectors consisting of n fuzzy numbers.

In this paper we intend to solve a fuzzy linear system $\tilde{A} \otimes \tilde{x} = \tilde{b}$ , where $\tilde{A}$ is a fuzzy $n \times n$ matrix, the unknown vector $\tilde{x}$ is vector consisting of n fuzzy numbers and the constant $\tilde{b}$ are vectors consisting of n fuzzy numbers. This paper mainly discusses a new decomposition of a nonsingular fuzzy matrix, the symmetric times triangular (ST) decomposition .By this decomposition every nonsingular fuzzy matrix can be represented as a product of a fuzzy symmetric matrix S and a fuzzy triangular matrix T.

## *2 Fully fuzzy linear system*

***Definition 2.1:*** A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix, if each element of $\tilde{A}$ is a fuzzy number [11]. $\tilde{A}$ will be positive (negative) and denoted by $\tilde{A} > 0 (\tilde{A} < 0)$. If each element of $\tilde{A}$ be positive (negative). Similarly, non-negative and non-positive fuzzy matrices may be defined.

Let each element of $\tilde{A}$ be a LR fuzzy number. We may represent $\tilde{A} = (\tilde{a}_{ij})$ that $\tilde{a}_{IJ} = (a_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, with new notation $\tilde{A} = (A, M, N)$, where A, M and N are three crisp matrices, with the same size of $\tilde{A}$, such that $A=(a_{ij})$, and $N=(\beta_{ij})$ are called the center matrix and the right and left spread matrices, respectively.

***Definition 2.2:*** A square fuzzy matrix $\tilde{A}=(\tilde{a}_{ij})$ will be an upper (lower) triangular fuzzy matrix, if

$$\tilde{a}_{ij}=\tilde{0}=(0,0,0) \quad ,\forall i>j(\forall i<j).$$

***Definition 2.3:*** Let $\tilde{A}=(\tilde{a}_{ij})$ and $\tilde{B}=(\tilde{b}_{ij})$ be two $m\times n$ and $n\times p$ fuzzy matrices. We define $\tilde{A}\otimes\tilde{B}=\tilde{C}=(\tilde{c}_{ij})$ which is the $m\times p$ matrix where

$$\tilde{c}_{ij}=\sum_{k=1}^{n}\tilde{a}_{ik}\otimes\tilde{b}_{kj}.$$

Up to rest of this paper we use Dubois and Prade's approximate multiplication $\otimes$.

***Definition 2.4***: Consider the $n\times n$ linear system of equation:

$$\begin{cases}(\tilde{a}_{11}\otimes\tilde{x}_1)\oplus(\tilde{a}_{12}\otimes\tilde{x}_2)\oplus\ldots\oplus(\tilde{a}_{1n}\otimes\tilde{x}_n)=\tilde{b}_1,\\(\tilde{a}_{21}\otimes\tilde{x}_1)\oplus(\tilde{a}_{22}\otimes\tilde{x}_2)\oplus\ldots\oplus(\tilde{a}_{2n}\otimes\tilde{x}_n)=\tilde{b}_2,\\\vdots\\(\tilde{a}_{n1}\otimes\tilde{x}_1)\oplus(\tilde{a}_{n2}\otimes\tilde{x}_2)\oplus\ldots\oplus(\tilde{a}_{nn}\otimes\tilde{x}_n)=\tilde{b}_n.\end{cases}$$

The matrix form of the above equation is

$$\tilde{A}\otimes\tilde{x}=\tilde{b}.$$

Or simply $\tilde{A}\tilde{x}=\tilde{b}$, where the coefficient matrix $\tilde{A}=(\tilde{a}_{ij}), 1\le i,j\le n$ is a $n\times n$ fuzzy matrix, and $\tilde{x}_i=(x_i,y_i,z_i),\ \tilde{b}_i=(b_i,h_i,g_i),\ 1\le i\le n$ are fuzzy numbers. This system is called a fully fuzzy linear system (FFLS). Also if $\tilde{A}$ and $\tilde{b}$ be a positive LR fuzzy number, we call the system (2) a positive FFLS. In many applied problems, engineers have some information about the range of fuzzy solution. In these cases with fixed y and z as the left and right spreads. The original problem is transformed to finding a vector x which satisfies in the following systems:

$$\begin{cases}Ax=b,\\Mx=h-Ay,\\Nx=g-Az.\end{cases}$$

***Definition 2.***5: Consider the positive FFLS (1). $\tilde{x}$ is a solution , if and only if

$$\begin{cases}Ax=b,\\Mx+Ay=h,\\Nx+Az=g.\end{cases}$$

In addition, if $y \geq 0, z \geq 0$ and $x - y \geq 0$. We say $\tilde{x} = (x, y, z)$ is a consistent solution of positive FFLS or for abbreviation consistent solution. Otherwise, it will be called dummy solution.

## *3 ST method for solving FFLS*

We shall present our main results on symmetric and triangular decomposition in this section.

***Theorem 1:*** (See [20]). For every nonsingular and non-symmetric $n \times n$ matrix A, whose leading principal sub-matrices are nonsingular, there exists a decomposition $A = ST$, where S is symmetric and T is unit triangular.

***Proof***: We shall prove the triangular matrix T is unit upper triangular. For $n = 2$ and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We can obtain

$$S = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} - a_{21}t_{12} \end{pmatrix}$$

And

$$T = \begin{pmatrix} 1 & t_{12} \\ 0 & 1 \end{pmatrix}$$

Such that $A = ST$, where $t_{12} = (a_{12} - a_{21}) / a_{11}$. Here, S is symmetric and nonsingular form the non-singularity of A.

Suppose that $A_n = S_n T_n$ holds for $n = k$. Now we like to show that it is still true for $n = k + 1$. For $n = k + 1$, we write

$$A = \begin{pmatrix} A_k & a_{k+1} \\ \hat{a}_{k+1}^T & \alpha_{k+1} \end{pmatrix}$$

and take

$$S = \begin{pmatrix} s_k & s_{k+1} \\ s_{k+1}^T & \beta_{k+1} \end{pmatrix}$$

$$T = \begin{pmatrix} T_K & t_{k+1} \\ 0 & 1 \end{pmatrix}$$

It follows form $A = ST$, that

$$\begin{cases} S_k T_k = A_k, \\ s_{k+1}^T T_k = \hat{a}_{k+1}^T, \\ S_k t_{k+1} + s_{k+1} = a_{k+1}, \\ \beta_{k+1} + s_{k+1}^T t_{k+1} = \alpha_{k+1}. \end{cases}$$

Since $A_k$ is nonsingular, $S_k$ and $T_k$ are nonsingular, by the induction assumption, we get the unique solution from above equations as follows:

$$\begin{cases} s_{k+1} = T_k^{-T} \hat{a}_{k+1}, \\ t_{k+1} = S_k^{-1}(a_{k+1} - s_{k+1)}, \\ \beta_{k+1} = \alpha_{k+1} - s_{k+1}^T t_{k+1}. \end{cases}$$

Therefore, $A = ST$ is well defined for $n = k+1$. From the non-singularity of A, it is easy to check that S is nonsingular. Therefore we will solve $\tilde{A}\tilde{x} = \tilde{b}$ , by using definition 2.5 we have

$$\begin{cases} Ax = b, \\ Mx + Ay = h, \\ Nx + Az = g. \end{cases}$$

By replace $A = ST$

$$\begin{cases} x = T^{-1}S^{-1}b, \\ y = T^{-1}S^{-1}(h - MT^{-1}S^{-1}b), \\ z = T^{-1}S^{-1}(g - NT^{-1}S^{-1}b). \end{cases}$$

***Theorem2***: Let $\tilde{A} = (A, M, N)$ and $\tilde{b} = (b, h, g)$ be a non-negative fuzzy matrix and a non-negative fuzzy vector. Let A be the product of a permutation matrix by a diagonal matrix with positive diagonal entries, and there exist a decomposition $A = ST$, where S is symmetric and T is unit triangular. Also, let $h \geq MT^{-1}S^{-1}b,\ g \geq NT^{-1}S^{-1}b$ and $(MT^{-1}S^{-1} + I)b \geq h.$ Then the system $\tilde{A}\tilde{x} = \tilde{b}$ has a positive fuzzy solution.

***Proof***: Our hypotheses on $A = ST$, imply that $A^{-1} = T^{-1}S^{-1}$ exists and is a nonnegative matrix [9]. So, $x = T^{-1}S^{-1}b \geq 0$. On the other hand, $h \geq MT^{-1}S^{-1}b$ and $g \geq NT^{-1}S^{-1}b.$ Thus with $y = T^{-1}S^{-1}h - T^{-1}S^{-1}MT^{-1}S^{-1}b$ and $z = T^{-1}S^{-1}g - T^{-1}S^{-1}NT^{-1}S^{-1}b$, we have $y \geq 0$ and $z \geq 0$. So $\tilde{x} = (x, y, z)$ is a fuzzy vector which satisfies $\tilde{A}\tilde{x} = \tilde{b}$ .Since $x - y = T^{-1}S^{-1}(b - h + MT^{-1}S^{-1}b)$, the positivity property of $\tilde{x}$ can be obtained from $\left(MT^{-1}S^{-1} + I\right) \geq h.$

### *4 Numerical examples*

***Example 1- C***onsider the fully fuzzy linear system in the follow form:

$$\begin{cases}(2,1,1)x_1+(3,1,2)x_2=(14,6,8),\\(4,1,2)x_1+(5,2,2)x_2=(26,8,12).\end{cases}$$

Therefore, we have

$$A=\begin{pmatrix}2&3\\4&5\end{pmatrix},\ M=\begin{pmatrix}1&1\\1&2\end{pmatrix},\ \ N=\begin{pmatrix}1&2\\2&2\end{pmatrix},$$
$$b=\begin{pmatrix}14\\26\end{pmatrix},\ g=\begin{pmatrix}6\\8\end{pmatrix},\ h=\begin{pmatrix}8\\12\end{pmatrix}.$$

Therefore, by deriving ST decomposition of A, we obtain:

$$S=\begin{pmatrix}2&4\\4&7\end{pmatrix},T=\begin{pmatrix}1&-1/2\\1&1\end{pmatrix},$$

Furthermore we have:

$$S^{-1}=\begin{pmatrix}-7/2&2\\2&1\end{pmatrix},$$
$$T^{-1}=\begin{pmatrix}1&1/2\\0&1\end{pmatrix}.$$

By using ST decomposition, we have

$$x=\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix},$$
$$y=\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$
$$z=\begin{pmatrix}z_1\\z_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Therefore, the solution of fully fuzzy linear system is crisp vector.

***Example 2-*** Consider the following $2\times 2$ fuzzy linear system:

$$\begin{cases}(3,1,4)x_1+(6,4,7)x_2=(18,19,55),\\(3,2,5)x_1+(4,2,6)x_2=(14,15,47).\end{cases}$$

Therefore, we have

$$A=\begin{pmatrix}3 & 6\\ 3 & 4\end{pmatrix},\ M=\begin{pmatrix}1 & 4\\ 2 & 2\end{pmatrix},\ N=\begin{pmatrix}4 & 7\\ 5 & 6\end{pmatrix},$$

$$b=\begin{pmatrix}18\\ 14\end{pmatrix},\ g=\begin{pmatrix}19\\ 15\end{pmatrix},\ h=\begin{pmatrix}55\\ 47\end{pmatrix}.$$

ST decomposition of A leads to

$$A=\begin{pmatrix}3 & 6\\ 3 & 4\end{pmatrix}=\begin{pmatrix}3 & 3\\ 3 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},$$

Furthermore, we have:

$$S^{-1}=\begin{pmatrix}-1/6 & 1/2\\ 1/2 & -1/2\end{pmatrix},$$

$$T^{-1}=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}.$$

By using ST decomposition, we have

$$x=\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}2\\ 2\end{pmatrix},$$

$$y=\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix},$$

$$z=\begin{pmatrix}z_1\\ z_2\end{pmatrix}=\begin{pmatrix}3\\ 4\end{pmatrix}$$

Therefore, the fuzzy solution of fully fuzzy linear system is fuzzy vector.

## *5 Summary and conclusions*

In this paper, we propose decomposition for the non-symmetric or the symmetric indefinite of coefficient matrix of fully fuzzy linear systems. We obtain a fuzzy solution for fully fuzzy linear system by decomposition coefficient matrix to the symmetric times triangular (ST), that S is the symmetric matrix and T is the triangular matrix.

## *References*

[1] Abbasbandy S., Jafarian A., and Ezzati R., Conjugate gradient method for fuzzy symmetric positive definite system of linear equations, Appl. Math.

Comput. 171, 1184-1191, 2005.

[2] Abbasbandy S., Ezzati R., and Jafarian A., LU decomposition method for solving fuzzy system of linear equations, Appl. Math. Comput. 172, 633-643, 2006.

[3] Abbasbandy S., Nieto J. J., and Alavi M., Tuning of reachable set in one dimensional fuzzy differential inclusions, Chaos, Solitons &Fractals 26, 1337-1341, 2005.

[4] Abbasbandy S., Otadi M., and Mosleh M., Minimal solution of general dual fuzzy linear systems, Chaos Solutions &Fractals, in press.

[5] Abbasbandy S., Otadi M., and Mosleh M., Numerical solution of a system of fuzzy polynomials by fuzzy neural network, Inform. Sci., In press.

[6] Asady B., Abbasbandy S., and Alavi M., Fuzzy general linear systems,Appl.Math.Comput.169, 34-40, 2005.

[7] Caldas M., and Jafari S., $\theta$-Compact fuzzy topological spaces, Chaos Solutions &Fractals 25, 229-232, 2005.

[8] Dehghan M., Hashemi B., and Ghatee M., Solution of the fully fuzzy linear systems using iterative techniques, Chaos Solution & Fractals, in press.

[9] Demarr R., Nonnegative matrices with nonnegative inverses. Proc Amer Math Soc, 35, 307-308, 1972.

[10] Dubois D., and Prade H., Operations on fuzzy numbers, J. Systems Sci., 9, 613-626, 1978.

[11] Dbois D., and Prade H., Systems of linear fuzzy constraints. Fuzzy Sets and Systems, 3, 37-48, 1980.

[12] Elnaschie M. S., A review of E-infinity theiry and mass spectrum of high energy particle physics, Chaos, Solutions &Fractals 19, 209-236, 2004.

[13] Elnaschie M. S., The concepts of E infinity: An elementary introduction to the Cantorian-fractal theory of quantum physics, Chaos, Solution & Fractals 22, 495-511, 2004.

[14] Elnaschie M. S., On a fuzzy Kahler manifold which is consistent with the two slit experiment, Int. J. Nonlinear Science and Numerical Simulation, 6, 95-98, 2005.

[15] Elnaschie M. S., Elementary number theory in superstrings, loop quantum mechanics, twisters and E-infinity high energy physics, Chaos, Solutions & Fractals 27, 297-330, 2006.

[16] Elnaschie M. S., Superstrings, entropy and the elementary particles content

of the standard model, Chaos, Solutions &Fractals 29, 48-54, 2006.

[17] Feng G. and Chen G., Adaptive control of discrete – time chaotic systems: a fuzzy control approach, Chaos Solutions & Fractals 23, 459-467, 2005.

[18] Friedman M., Ming M., and Kandel A., Fuzzy linear systems, Fuzzy Sets and Systems 96(1998)201-209.

[19] Friedman, M. Ming M., and Kandel A., Duality in fuzzy linear systems, Fuzzy Sets and Systems 109(2000) 55-58.

[20] Golub G. H., and Yun J., Symmetric-triangular decomposition and its applications, Swets & Zeilinger, 42, 814-822, 2002.

[21] Jiang W., and Guo-Dong Q., and Bin D., $H_\infty$ Variable universe adaptive fuzzy control for chaotic system, Chaos Solutions & Fractals 24, 1075-1086, 2004.

[22] Kaufmann A., and Gupta M. M., Introduction Fuzzy Arithmetic, Van Nostrand Reinhold, New York, 1985.

[23] Muzzioli S., and Reynaerts H., Fuzzy linear systems of the form $A_1x+b_1=A_2x+b_2$, Fuzzy Sets and Systems, In press.

[24] Nozavi K., and Fazlpour B., Some consequences of space time fuzziness, Caos, Solutions & Fractals, in press.

[25] J.H.Park,Intuitionistic fuzzy metric spaces,Chaos Solutions & Fractals 22, 1039-1046, 2004.

[26] Tanaka Y., Mizuno Y., and T.Kado,Chaotic dynamics in the Fridman equation, Chaos Solutions & Fractals 24, 407-422, 2005.

[27] Wang X., Zhong Z., and Ha M., Iteration algorithms for solving a system of fuzzy linear equations, Fuzzy Sets and Systems 119, 121-128, 2001.

[28] Zadeh L. A., The concept of a linguistic variable and its application to approximate reasoning, Inform. Sci. 8, 199-249, 1975.