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Solving Nonlinear Klein-Gordon Equation with a Quadratic Nonlinear term using Homotopy Analysis Method

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Abstract

In this paper, nonlinear Klein-Gordon equation with quadratic term is solved by means of an analytic technique, namely the Homotopy analysis method (HAM). Comparisons are made between the Adomian decomposition method (ADM), the exact solution and homotopy analysis method. The results reveal that the proposed method is very effective and simple.

Keywords: Klein-Gordon; Homotopy analysis method; Adomian decomposition method; partial differential equation; Homotopy perturbation method.

1 Introduction

Nonlinear partial differential equations are useful in describing the various phenomena in many disciplines. Apart of a limited number of these problems, most of them do not have a precise analytical solution, so these nonlinear equations should be solved using approximate methods. In 1992, Liao [13] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely Homotopy Analysis Method (HAM), [13, 14, 15, 16, 17, 18, 19]. This method has been successfully applied to solve many types of nonlinear problems [1, 4, 5, 6, 7, 8, 20, 21]. The HAM offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy, and require large computer power and time. HAM is better since it does not involve discretization of the variables hence is free from rounding off errors and does not require large computer memory or time. Since the beginning of the 1980s, the Adomian decomposition method (ADM) has been applied to a wide class of functional equations [2, 3]. In this method the solution is given as an infinite series usually converging to an accurate solution. K. Chandra Basak et al. [12] applied the multistage Adomians decomposition method for solving Klein-Gordon differential equation and compared the exact solution. In this paper, we propose homotopy analysis method to solve quadratic Klein-Gordon differential equation. Comparisons are made between Adomian decomposition method, the exact solution and the proposed method. The paper has been organized as follows. Section 2 basic idea of HAM. Section 3 applying HAM for Klein-Gordon equation. Section 4 the comparison of the results of HAM with the exact solution for different values of h. Section 5 is conclusion.

2 Basic idea of HAM

We apply the HAM to the Klein-Gordon equation with initial conditions .We consider the following differential equation

$$N(\tau) = 0, \tag{1}$$

where N is a nonlinear operator, τ denotes independent variable, u(τ) is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [2] constructs the so-called zero-order deformation equation

$$(1-p)L[\phi(\tau;p) - u_0(\tau)] = phH(\tau)N[\phi(\tau;p)], \tag{2}$$

where p$\in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, H(τ) $\neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, $u(\tau;p)$, is an unknown function. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and 1, it holds

$$\phi(\tau;0) = u_0(\tau), \phi(\tau;1) = u(\tau). \tag{3}$$

Thus, as p increases from 0 to 1, the solution $u(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$.

Expanding $u(\tau;p)$ in Taylor series with respect to p, we have

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=0}^{\infty} u_m(\tau)\, p^m \qquad (4)$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau;p)}{\partial p^m} \qquad (5)$$

According to the definition (6), the governing equation can be deduced from the zero-order deformation equation (4). Define the vector

$$\overrightarrow{u_n} = \{u_0(\tau), u_1(\tau), \dots, u_n(\tau)\} \qquad (6)$$

Differentiating equation (4) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = hH(\tau)R_m(\overrightarrow{u_{m-1}}), \qquad (7)$$

Where

$$R_m(\overrightarrow{u_{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(\tau;p)]}{\partial p^{m-1}}\Big|_{p=0} \quad , \qquad (8)$$

And

$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases}$$

It should be emphasized that $u_{m-1}(\tau)$ for $m > 1$ is governed by the linear equation (9) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as *Mathmatica*. If Eq. (3) admits unique solution, then this method will produce the unique solution. If equation (3) does not possess unique solution, the HAM will give a solution among many other (possible) solutions. For the convergence of the above method we refer the reader to Liao's paper [2]. In chapter (4) in [2] Liao shows that ADM is obtained from HAM.

3 Applying HAM for Klein-Gordon equation

Let us consider the non-linear Klein-Gordon equation

$$u_{tt} - u_{xx} + \frac{\pi^2}{4}u - u^2 = x^2 sin^2(\frac{\pi t}{2}) \quad (9)$$

with initial conditions:

$$u(x, 0) = 0, \qquad u_t(x, 0) = x. \quad (10)$$

To solve the Eq. (11), by means of homotopy analysis method, according to the initial conditions denoted in (12), it is natural to choose $u_0 = \frac{\pi t x}{2}$.

We choose the linear oprator

$$L[\emptyset(x,t;p)] = \frac{\partial^2 \emptyset(x,t;p)}{\partial t^2} \quad (11)$$

with the property

$$L[(c_1 t + c_2)] = 0,$$

where c_1 and $c_2$are constants. We now define a nonlinear operator as

$$N[\emptyset(x,t;p)] = \frac{\partial^2 \emptyset(x,t;p)}{\partial t^2} - \frac{\partial^2 \emptyset(x,t;p)}{\partial x^2} + \frac{\pi^2}{4}\emptyset(x,t;p) - \emptyset^2(x,t;p) - x^2 sin^2(\frac{\pi t}{2})$$

(12)

Using above definition, we construct the zeroth-order deformation equation

$$(1-p)L[(\emptyset(x,t;p) - u_0(x,t)] = phN[\emptyset(x,t;p)].$$

$$\emptyset(x,0;p) = 0, \qquad \emptyset(x,0;p)_t = \frac{\pi t x}{2}.$$

For p = 0 and p = 1, we can write

$$\emptyset(x,t;0) = u_0(x,t) = u(x,0), \qquad \emptyset(x,t;1) = u(x,t). \quad (13)$$

Thus, we obtain the mth-order deformation equations

$$(1-p)L[(u_m(x,t)-\chi_m u_{m-1}(x,t)] = hR_m[\overrightarrow{u_{m-1}}(x,t)],(m \geq 1),$$

$$u_m(x,0) = 0, \qquad (u_m(x,0))_t = 0. \tag{14}$$

Where

$$R_m[\overrightarrow{u_{m-1}}(x,t)] = \frac{\partial^2 u_{m-1}(x,t)}{\partial t^2} - \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} + \frac{\pi^4}{4}u_{m-1}(x,t)$$

$$-\sum_{n=0}^{m-1} u_n(x,t)u_{m-1-n}(x,t) - (1-\chi_m)sin^2(\frac{\pi t}{2}) \tag{15}$$

Now the solution of the mth-order deformation equations (16)

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + hL^{-1}[R_m[\overrightarrow{u_{m-1}}(x,t)],(m \geq 1). \tag{16}$$

In view of Eq.(6) we get

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) \tag{17}$$

The exact solution of the differential equation (3.1) is given by [12]

$$u(x,t) = x\sin(\frac{\pi t}{2}). \tag{18}$$

In table (1) and (2) we have presented approximate solution by 3th-order HAM for different value of x

4 The comparison of the results of HAM with the exact solution for different values of h

The validity of the method is based on such an assumption that the series (4) converges at $q = 1$. It is the auxiliary parameter h which ensures that is assumption can be satisfied. In general, by means of the so-called h-curve, it is straightforward to choose a proper value of h which ensure that the solution series is convergent. The h-curves for the first example considered in this paper are presented in Fig.1, which were obtained based on the 4th-order and the 6th-order HAM approximations solutions. By HAM, it is easy to discover the valid region of h , which corresponds to the line segments nearly parallel to the horizontal axis.

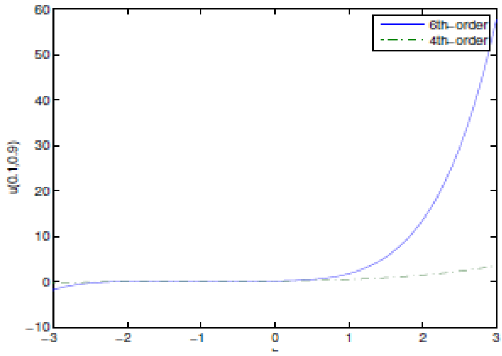

Fig1. The curve of u(0.1,0.9)given by (17) based on the 4th-order and 6th-order HAM approximations.

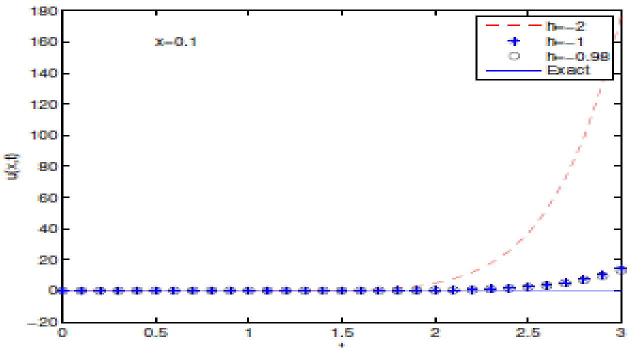

Fig 2. The 3th-order HAM solution (9) for different values of h

Table 1 The comparison of the results of the HAM ($h = -0.98$), the ADM [12] and the exact solution considering $x = 0.1$ for the Klein-Gordon equation with initial condition (10).

t	HAM(3th-order)	ADM[12]	Exact	Error(HAM)	Error(ADM)
0.1	0.015643	0.015643	0.015643	0.000000	0.000000
0.2	0.030902	0.030902	0.030902	0.000000	0.000000
0.3	0.045399	0.045399	0.045399	0.000000	0.000000
0.4	0.058778	0.058779	0.058778	0.000000	0.000000
0.5	0.070710	0.070711	0.070711	0.000001	0.000000
0.6	0.080902	0.080904	0.080902	0.000000	0.000002
0.7	0.089104	0.089113	0.089101	0.000003	0.000012
0.8	0.095124	0.095153	0.095106	0.000018	0.000048
0.9	0.098842	0.098925	0.098769	0.000073	0.000156
1.0	0.100236	0.100448	0.100000	0.000236	0.000448

Table 2 The comparison of the results of the HAM ($h = -0.98$), the ADM [12] and the exact solution considering $x = 0.4$ for the Klein-Gordon equation with initial condition(10).

t	HAM(3th-order)	ADM[12]	Exact	Error(HAM)	Error(ADM)
0.1	0.062574	0.062574	0.062574	0.000000	0.000000
0.2	0.123607	0.123607	0.123607	0.000000	0.000000
0.3	0.181596	0.181596	0.181596	0.000000	0.000000
0.4	0.235114	0.235114	0.235114	0.000000	0.000000
0.5	0.282843	0.282843	0.282843	0.000000	0.000000
0.6	0.323607	0.323608	0.232607	0.000000	0.000001
0.7	0.356403	0.356408	0.356403	0.000000	0.000005
0.8	0.380427	0.380446	0.380423	0.000004	0.000024
0.9	0.395094	0.395160	0.395075	0.000019	0.000085
1.0	0.400060	0.400264	0.400000	0.000060	0.000264

5 Conclusion

In this Letter, we have successfully used HAM for solving Klein-Gordon. It is apparently seen that HAM is a very powerful and efficient technique in finding analytical solutions for wide classes of nonlinear problems. It is worth pointing out that this method presents a rapid convergence for the solutions. In conclusion, HAM provides accurate numerical solution for nonlinear problems in comparison with other methods. They also do not require large computer memory and discretization of the variables t and x. The result shows that HAM is powerful mathematical tool for solving nonlinear partial differential equations and systems of nonlinear partial differential equations having wide applications in engineering. MATLAB has been used for computations in this paper.

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