Optimal Inventory Control of Obsolete Products with Permissible Delay in Payments and Time-dependent Demand using PSO Algorithm

Abstract

This study provides an inventory control model for determining the optimal replenishment cycle of obsolete items, in which the customer demand is regarded as a decreasing function of time in sudden obsolescence. Further, to encourage the buyer for more purchasing, the seller can allow the buyer to pay the cost with a delay. Accordingly, the present paper focuses on investigating an inventory control model for obsolete items while considering a trade credit policy with time-dependent demand and sudden obsolescence. Given the nonlinearity of the proposed model, the Taylor series approximation was used to solve it. In addition, to avoiding the effect of Taylor series approximation on the optimal solution, an efficient particle swarm optimization meta-heuristic algorithm was applied to find the near-optimal solution, indicating better answers. Numerical examples in the case study of mobile phone wholesale industry, were considered and solved to demonstrate the validation of the proposed model. Finally, a sensitivity analysis was performed on the effects of the main parameters on the total profit and replenishment cycle time. The numerical results indicated that the inclusion of obsolescence risk in the inventory model with obsolete items has a significant effect on increasing profits while reducing the costs of these items.

Keywords: Particle Swarm Optimization (PSO); Inventory Control; Obsolete Items; Time-dependent Demand; Delay in Payment;

Introduction

The classical economic order quantity (EOQ) model is based on implicit assumptions. For example, the demand rate is considered constant over an infinite planning horizon. This assumption is valid only at the maturity stage of the product lifecycle, while not applying to the introduction and growth stages of this lifecycle, when companies face high demands whereas less competition. Contrarily, the demand decreases while competition increases at the decline stage of the product life cycle. Therefore, different modelling patterns of time-varying demands should be considered to reflect product sales at different product lifecycle stages.

In this respect, the current paper evaluates an inventory control model for obsolete items where the optimal replenishment cycle time is achieved, aiming to reduce inventory costs under trade credit conditions.

To this end, a general optimization model was proposed with a decreasing demand function over time, considering that after occurring obsolescence, the demand will become zero immediately. In addition, the demand is considered time-dependent due to the importance of reviewing the lifecycle of obsolete items. Further, considering that the demand reduces over time, especially at the decline stage of the lifecycle, the trade credit policy has been used to encourage customers to buy more.

Obsolescence, especially in the mobile phone industry as the case study of this research, usually decreases the sales value of the product, and ultimately, the volume of the product that can be sold. Thus, the speed of product obsolescence affects the seller's decision and the amount of inventory in the sales channel.

Due to complexity and nonlinearity, it is impossible to solve the original model as the closed-form in such models. Thus, Taylor series are generally applied to simplify the model. Additionally, the first two or three terms of series are frequently employed, considering that writing and calculating the infinite terms of the Taylor series is impossible.

Nonetheless, it should be noted that this approximation regarding using the first two or three terms of the Taylor series at a point is only valid around this point. At the same time, it results in a more significant difference value between the real function and its approximation at farther points. So, meta-heuristics were used as the replacement solution of Taylor approximations. In other words, the model was solved with meta-heuristics instead of adding approximations to the model and then solving it with exact methods. In this study, the particle swarm optimization (PSO) algorithm was employed explicitly as the solution method, and numerical results demonstrated the performance of this replacement. The remaining sections of the paper are organized as follows.

The literature on time-dependent demands and studies on obsolete items are reviewed in section 2. Section 3 describes the applied assumptions and notations in the modelling in this study. In addition, the mathematical model is provided in section 4, followed by proving that the optimal value of the replenishment cycle exists along the planning horizon and is also unique. Finally, section 5 concludes the study.

Literature review

This section consists of three research parts related to the inventory models of obsolete items, inventory models with time-dependent demands, and inventory models with payment delays. The inventory models of obsolete items were first introduced by Brown (1964). In this study, the multi-period inventory control model was considered for sudden obsolescence items. In the multi-period modelling of obsolescence items, the probability of obsolescence in all periods was assumed to be equal to one, and the modelling was performed using dynamic programming. Cobbaert et al. (1996) proposed an inventory control model for the obsolescence items considering an exponential lifetime of products up to sudden obsolescence. They then generalized the proposed model to a situation by considering shortage and lead-time. Similarly, Arcelus et al. (2002) presented a model to maximize profits by considering Cobbaert’s model, in which the demand is a function of the sales price and time. In another study, Song et al. (2004) introduced a periodic inventory model for sudden obsolete items based on the concept of dynamic programming. In this model, obsolete costs in each period are calculated according to the conditional probability of non-obsolescence in the previous period. Wang et al. (2011) also proposed and solved an inventory control model in the case of obsolete items by considering time-dependent demands during the product lifecycle as a function of population growth and offering discounts during the demand decline period. In another study, Delft et al. (1996) added the descending discount function to the obsolete inventory model by considering Cobbaert's model. Likewise, Persona et al. (2005) developed the consignment stock policy concerning obsolete items. Further, Joglekar et al. (1993, 1996) modelled the case of sudden obsolescence by considering the sales price in a revenue function instead of a cost function.

Barron (2018) introduced an EOQ inventory model including revenues, losses, shortages, and ordering costs, under unexpected obsolescence conditions in which the returns are governed by a Markov additive process (MAP).

Inventory models with time-dependent demands were also reviewed in this study. As previously mentioned, a limited number of studies have addressed this concept in obsolescence models, mainly focusing on simple structures. Thus, the topic was reviewed in other types of models. Various studies have been conducted with different decreasing and increasing time functions in changing the demand relative to time regarding deteriorating items. Some studies have been performed by considering increasing geometric demand function over time. For instance, Pal et al. (2015) proposed an economic production quantity model with a time-dependent geometric demand function including shortage and inflation in a fuzzy environment for deteriorating items with variable deterioration rates. Prasad and Mukhergee (2014) also introduced a model for inventory control of perishable products by considering the geometric demand relying on the inventory level and time along with a shortage. In another study, Sunni and Chukwu (2013) provided a model of perishable items by considering the decreasing geometric demand function over time and time-dependent deterioration rate with the Weibull distribution function and the permissible shortage.

On the other hand, several studies examined the incremental time-dependent quadratic demand function. For example, Begum et al. (2012) offered an inventory model including a time-dependent quadratic demand function with partial shortage. Furthermore, Sett et al. (2012) suggested a two-warehouse inventory model for perishable items based on the incremental time-dependent quadratic demand. Similarly, Sarkar et al. (2012) submitted an inventory model for deteriorating items with variable deterioration rates, demand rates with a time-dependent quadratic function, and delays in payment between the retailer and the supplier.

Some other studies also evaluated the increasing linear dependence of demands on time. For instance, Chauhan and Singh (2015) presented an inventory model for perishable items with a linear time-dependent demand function, linear time-dependent deterioration rate, and inflation. Moreover, Dutta and Kumar (2015) introduced an inventory model for perishable items with linear demands over time, linear holding costs over time, and partial shortage. Mishra et al. also proposed a model similar to Butar and Kumar (2015) with linear demand functions over time and different holding costs over time. Additionally, Pervin et al. (2015) provided an inventory model for perishable items with delays in payment and a linear time-dependent demand function.

Some studies also investigated the demand function based on the product lifetime. For example, Avinadav et al. (2013) suggested an inventory model to determine the optimal price and order quantity of perishable items based on time- and price-dependent demands. The demand function is linear and polynomial relative to cost and time, respectively. Likewise, Herbon (2013) presented a dynamic inventory model for perishable items where the demand function depends on the product lifetime, and the demand is higher for a newer product.

As shown, most studies on the state of demand over time are increasing functions. However, some studies reported decreased demands over time that can be included in the obsolescence research area. For instance, Ghoreishi et al. (2013) provided an inventory model with a decreasing demand function on both time and price. In this paper, the cost and optimal order quantity was examined in an inflationary environment. Ghoreishi et al. (2013) also developed the previous model by adding the shortage to the model. Further, Maihami and Nakhaei (2012) introduced a model for simultaneously optimizing the order quantity and selling price of perishable items by considering demand reductions over time and, with cost in which delay in payment is allowable. Maihami and Nakhaei (2012) further presented the demand

model suggested in the previous study by considering partial shortage for non-instantaneous deteriorating items.
Akhtar et al. (2023) proposed an inventory model with partial backlogging for deteriorating items in a limited time horizon. The study considered the deterioration rate as a random variable that follows the three-parameter Weibull distribution. In addition, demand rate was considered to be dependent on the time and the selling price of the item. The purpose was to determine the optimal selling price of the product, the optimal number of ordering cycles, and the optimal level of shortage that maximizes the retailer's total profit in a limited time horizon.
Another classic inventory control model assumption indicates that the buyer immediately pays for the purchase after receiving the items. At the same time, the seller may consider more time for the buyer to pay the purchase cost. This is known as delay-in-payment policy and is generally used as an incentive policy to attract more customers by the seller. Regarding this policy's category of inventory models, Ouyang et al. (2006) evaluated an optimal inventory policy for non-instantaneous deteriorating items considering this policy. In another study, Moussawi et al. (2014) extended a three-level supply chain including the customer, vendor, and the bank under the conditions that the vendor's trade credit is given to the customer, and the vendor-related bank controls the cash. They considered that the bank would provide the vendor with a specific discount on the loan rate, and this coordination will reduce the costs. Jamal et al. (1997) also developed the ordering policy under credit periods when the shortage was allowable. Chang and Dye [29] offered an inventory model in which the items were perishable, and delays in payment and shortage were permissible. Furthermore, Huang (2003) provided a two-level trade credit model where the supplier and retailer could give trade credits to the retailer and the customer, respectively. Similarly, Soni et al. (2010) reviewed inventory models with trade credits consisting of topics such as perishability, stochastic demands, discounting, and cash present values.
Some studies have also investigated the decrease in inventory values over the time. For example, Khouja and Goyal (2006) obtained the optimal order quantity, considering continuously decreasing inventory costs over the time. Additionally, Yu et al. (2011) addressed the continuous product price reduction in an inventory model with a two-level supply chain. Vandana (2016) also suggested an ordering model under trade credit with a shortage in a two-level supply chain, where the retailer prepaid a percentage of costs.
Zahran et al. (2016) presented a trade credit inventory model with the consignment stock policy in their study. Pourmohammad Zia and Taleizadeh (2016) also introduced a three-level inventory model with a combination of shortage, prepayment, and delay in payment. Moreover, Amin Khan et al. (2021) expanded an EOQ model including advanced payments, partial delays, all-unit discounts, partial shortage, and warehouse limits. The retailer tends to optimize the number of rented warehouses in addition to own warehouses. Likewise, Ahmed et al. (2019) proposed a multi-period inventory model based on reworking defective items and considering delay in payment in the supply chain. In a similar study, Taleizadeh et al. (2020) provided an EOQ model with delay in payment, partial shortage, and probabilistic replenishment intervals.
Duary et al. (2022) presented a model of inventory control for perishable items with two warehouses, delay in payment, partial shortage, and advertisement activities.
Kaushik (2023) provided an inventory control model based on profit maximization policy considering two different interest rates, where ramp-type demand has been employed with a delay in payment approach.
Amin Khan et al. (2022) proposed a perishable inventory control model for the era of the Corona pandemic, with time-dependent demand and combined payment conditions, where the deterioration is considered to be non-instantaneous and the shortage is also allowed.

Gap Analysis:

Unlike the former inventory control models in which by simplifying assumptions the model can be easily solved, in the models with obsolescence, parameters, restrictions, decision variables, objective functions, and newer assumptions should be considered in inventory ordering models according to the prevailing conditions in today's world. Pricing, delay in payment, incentive policies such as discounts on the purchase price, and considering more other realistic functions on product demand are among the most important of these factors in the field of obsolete items. According to the conducted studies, the three characteristics of pricing, quantity discount, and delay in payment are significant in the subject of inventory control of obsolete items. Based on the previous literature, so far, no model is provided considering these three altogether. It is noteworthy that this topic is an essential request in the obsolete product market (specifically, mobile phone market in this study) considering the decreasing demand over time in these products and requiring an incentive policy such as trade credit to sell the products as much as possible. The other specifications of the developed model are as follows:

Assumptions:

The single product model is considered.

Lead time and shortage are not allowed.

The inventory level at the end of the ordering cycle is set to zero, if obsolescence does not occur and if obsolescence occurs; the inventory level at the end of the ordering cycle is set to the remained items.

To show the obsolescence in demand level, the decreasing demand function over time is used. Therefore, it is assumed that the demand rate function follows the exponential function $D_0 e^{-\lambda t}$; however, the mean lifetime of product is also considered equal to L, $\lambda = \frac{1}{L}$

$$D = D_0 e^{-\lambda t} \tag{1}$$

The replenishment cycle is considered to be continuous.

Obsolescence happens suddenly.

The probability distribution function of the obsolescence lifetime is assumed exponential

The planning horizon is unlimited

Parameters:

t: The time that the obsolescence occurs (year)

L: Expected lifetime of product (years)

D: The time-dependent annual amount of demand

D_0: The fixed demand rate per time unit.

h: The rate of holding costs per unit time of an item ($/year)

M: The payment time after receipt the order (years)

I_e: The interest rate earned in a cycle

I_p: The interest rate charged in a cycle

C_s: The cost of obsolescence per unit ($/year)

c: The cost of purchasing per unit ($/year)

P_s: The probability that obsolescence does not occur during the order cycle.

Decision variables

T: Replenishment cycle time

Cost function

The objective function of a period consists of holding cost, obsolescence cost, the interest payable and interest receivable, as follows.

The obsolescence costs of a period are equivalent to:

$$\int_0^T (T * D_0 - t * D_0) * (C_s) * \left(\left(\frac{1}{L}\right) e^{\frac{-2t}{L}}\right) dt = \frac{-D_0\left(L - e^{-\frac{2T}{L}} L - 2T\right) C_s}{4} \tag{2}$$

The holding costs of a period are equal to:

If obsolescence occurs at 0 < t < T, the mean inventory available over a period is $(T * D_0 - \left(\frac{t * D_0}{2}\right)) e^{\frac{-t}{L}}$. Therefore, the holding cost would be equal to:

$$\int_0^T \left(T * D_0 - \left(\frac{t * D_0}{2}\right)\right) * (\mathrm{C} * H * t) * \left(\left(\frac{1}{L}\right) e^{\frac{-2t}{L}}\right) dt = \frac{D_0 e^{-\frac{2T}{L}} H\left(L^2 - 2T^2 - e^{\frac{2T}{L}} L(L - 2T)\right) \mathrm{C}}{8} \tag{3}$$

If obsolescence occurs at t > T with concerning to that the probability of obsolescence will not occur in coming cycle, is an exponential distribution with parameter $(\frac{1}{L})$, the mean inventory over the period is $\frac{Q}{2}$; therefore, the holding cost would be as follows:

$$\int_{T}^{\infty} (\frac{T * D_0}{2}) * (T * \mathrm{C} * H) * ((\frac{1}{L}) e^{\frac{-2t}{L}})\, dt = \frac{D_0 CHT^2 (e^{-\frac{2T}{L}})}{4} \qquad (4)$$

The average interest earned and average interest paid in two cases according to the status of M relative to T are:

First case: M < T

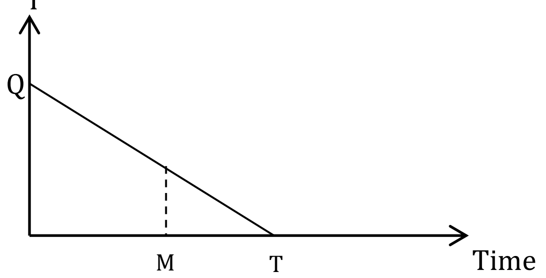


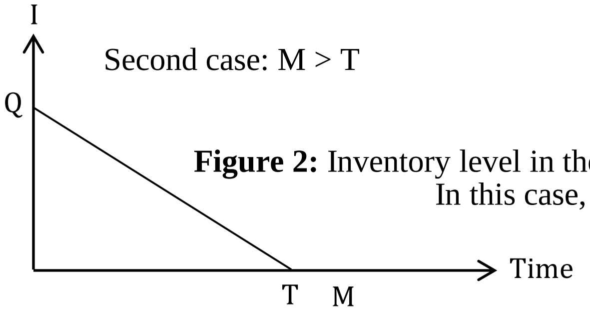
Figure 1: Inventory level in the range [0, T] when M ≤ T

The cost of interest payable of a period in the first case is equal to:

$$\mathrm{CI}_p \int_{M}^{T} (t - M) D_0 (\frac{1}{L}) e^{-\frac{2t}{L}}\, dt = \frac{(e^{-\frac{2T}{L}} L - e^{-\frac{2M}{L}} (L + 2(M - T))) D_0 \mathrm{CI}_p}{4} \qquad (5)$$

The interest receivable revenue of a period by the retailer is equal to:

$$CI_e \int_{0}^{M} t\, D_0 \left(\frac{1}{L}\right) e^{-\frac{2t}{L}} dt = \frac{CD_0 I_e (L - (\mathrm{e})^{-\frac{2M}{L}} (L + 2M))}{4} \qquad (6)$$



Second case: M > T

Figure 2: Inventory level in the range [0, ∞) when M > T

In this case, there is only interest receivable:

$$CI_e \int_0^T tD_0 \left[\left(\frac{1}{L}\right) e^{-\frac{2t}{L}}\right] dt = \frac{CDI_e (L - e^{-\frac{2T}{L}} (L + 2T))}{4}$$

$$CDI_e \int_{T}^{M} (t - T) * (\frac{1}{L}) e^{-\frac{2t}{L}}\, dt = \frac{CDI_e (e^{-\frac{2T}{L}} L - e^{-\frac{2M}{L}} (L + 2(M - T)))}{4} \qquad (7)$$

Therefore, by summing the obsolescence cost, holding cost, and payable costs, as well as the interest receivable revenue, the total cost in cases 1 and 2 are as equations (8) and (9) respectively:

First case:

$$C_{C\,1} = \frac{D_0\left(\frac{1}{2}HCL - C_s\right)\left(L\left(1 - e^{-\frac{2T}{L}}\right) - 2T\right)}{4} + \frac{\left(e^{-\frac{2T}{L}}L - e^{-\frac{2M}{L}}\left(L + 2(M - T)\right)\right)D_0\mathrm{CI}_p}{4} - \frac{CD_0I_e\left(L - (\mathrm{e})^{-\frac{2M}{L}}(L + 2M)\right)}{4} \quad (8)$$

Second case:

$$C_{C\,2} = \frac{-D_0(\frac{1}{2}HCL + C_s)\left(L\left(1 - e^{-\frac{2T}{L}}\right) - 2T\right)}{4} - \frac{\frac{CDI_e(L - e^{-\frac{2T}{L}}(L + 2T))}{4}}{4} - \frac{CI_e(e^{-\frac{2T}{L}}L - e^{-\frac{2M}{L}}(L + 2(M - T)))}{4} \quad (9)$$

By dividing C_{c1} and C_{c2} into $(1 - e^{-\frac{T}{L}})$ [40], the total cost for all periods will be resulted respectively by equations (10) and (11).

$$C_{L1} = \frac{\frac{-D_0\left(\frac{1}{2}HCL + C_s\right)\left(L\left(1 - e^{-\frac{2T}{L}}\right) - 2T\right)}{4} + \frac{\left(e^{-\frac{2T}{L}}L - e^{-\frac{2M}{L}}\left(L + 2(M - T)\right)\right)D_0\mathrm{CI}_p}{4} - \frac{CD_0I_e\left(L - e^{-\frac{2M}{L}}(L + 2M)\right)}{4}}{(1 - e^{-\frac{T}{L}})} \quad (10)$$

$$C_{L2} = \frac{\frac{-D_0(\frac{1}{2}HCL + C_s)\left(L(1 - e^{-\frac{2T}{L}}) - 2T\right)}{4} - \frac{CDI_e\left(L - (L + 2M)e^{-\frac{2M}{L}}\right) + 2TCDI_e(e^{-\frac{2T}{L}} -}{4}}{(1 - e^{-\frac{T}{L}})} \quad (11)$$

The convexity proof of the objective functions at the first and second cases are brought in appendices A and B respectively.

Solution Approach

The meta-heuristic PSO algorithm is an evolutionary computational method based on the population of solutions, which have been used to solve the models in this study. Like other meta-heuristic algorithms, this algorithm is an optimization tool to solve various optimization problems and reach near optimal solutions. This algorithm is one of the most recent meta-heuristic solutions, inspired by the social behavior of groups of migratory birds trying to reach a destination. There are two main reasons for using the PSO method to solve the present models.

The introduced method is based on swarm intelligence, and the particles move in the direction that achieves the best result. In inventory control models, this is fundamentally important specially in the models with time-dependent demands.

The PSO algorithm’s convergence speed is relatively high, and it is not memoryless. In fact, the particles benefit from their past information (i.e. inventory of the previous period), and this advantage cannot be found in other optimization algorithms. For example, the prior information in genetic algorithm will be removed by changing in population.

With these two main properties, while the results are more confident, they are also more suitable for inventory control models such as models in this research.

In the PSO algorithm, the solutions' population is called a group. Each solution is like a bird in a flock of birds called a particle and is similar to the genetic algorithm's chromosome. All particles have a fitness value calculated using the fitness function, and during the implementation of the algorithm the particle fitness function of the all particles will be optimized. The velocity vector of a particle determines the direction of motion of it. Unlike the genetic algorithm, in the algorithm's evolutionary process, new birds are not created from the previous generation. Instead, each bird optimizes its social behavior according to its experiences and other birds' behavior in the group and accordingly improves its movement towards the destination. The PSO algorithm starts with random solution and then seeks the optimal solution by updating the particles per iteration.

The decision variables according to the position of the velocity vector of particles, and each particle's position per iteration of the algorithm are calculated based on the following notations and equations.

i: Particle index

t: Iteration index

V_{it-1}: The new velocity vector of each particle based on the its previous velocity
$pBest_i$: The best position the particle has ever reached
$nBest_i$: The position of the best particle in the neighborhood that has been obtained so far
$gBest$: The position of the best particle among the group
r_1: and r_2: Two random numbers (with an uniform distribution between [0,1])
c_1: and c_2: Learning coefficients that control the effect of pBest and nBest on the search process.

$$V_{it} = V_{it-1} + c_1 . r_1 . (pBest_i - x_{it}) + c_2 . r_2 . (nBest_i - x_{it}) \quad (12)$$

$$-V_{\max} \le V_{it} \le V_{\max} \quad (13)$$

$$S_i = 1 \big/ 1 + e^{-V_{it}} \quad (14)$$

$$x_{it} = \begin{cases} 1 \rightarrow p \le s_i \\ 0 \rightarrow otherwise \end{cases} \quad (15)$$

According to equations (12-15), the new velocity vector of each particle is calculated based on the previous velocity of the particle itself (V_{it-1}), the best position the particle has ever reached ($pBest_i$), and the position of the best particle in its neighborhood ($nBest_i$) that has been obtained so far. If each particle's neighborhood includes all the particles in the group, then $nBest_i$ indicates the position of the best particle in the group, which is shown as gBest. r_1 and r_2 are two random numbers generated as independent of each other. Learning coefficients c_1 and c_2, control the effect of pBest and nBest on the search process. The $V_{\max}$ limits the particle velocity vector. In other word, $V_{\max}$ is a limit that controls the global search capability of the particle group. Using equation (15), the velocity vector of each particle is converted to the change probability vector. In this equation, s_i indicates the probability that x_{it} is equal to 1, which is calculated based on a random value of p (with a uniform distribution between zero and one). Then, using the above vector relation, the position of each particle is updated.

The time-dependent inventory control problem of obsolete items under the assumptions examined in this research, is one of the problems with a high complexity. As a result, in this research, a new improved PSO algorithm has been developed to solve the above problem. According to (10) and (11), the inventory control model of obsolete items with time-dependent demand is a nonlinear problem in which the decision variable is defined as T.

The proposed algorithm consists of two stages, the first of which consists of two individual parts. In the first part, SSize particles are first created using a quasi-random method as primary particles. After that, the random neighborhood with NSize particle is formed around each initial solution. However, to increase the quality of the solutions during the algorithm's steps, it is necessary to exchange information between different neighborhoods of the particles. Accordingly, after updating the nBest values in each iteration of the algorithm, 2 * SSize particles from the nBest particles are selected as parents using the match selection method. Then the SSize new particles (offspring's) are selected randomly. The new particles replace the worst particles in any neighborhood. As shown during the algorithm's steps, the quality and distribution of solutions are improved and a maximum of SSize particles of suitable and non-repetitive solutions is chosen and placed in a set called the "reference set" for use in the next steps of the algorithm. Meanwhile, the particle diversification mechanism is performed to prevent early convergence and at result create initial solutions with good distribution and quality for the next steps. Accordingly, a set of best particles among the reference set is selected and then, the MRate percentage of the group particles is randomly generated (using the uniform distribution function) based on the selected particles. The new particles are replaced with the worst particles in the

group.
At the end of the first part, the reference set of solutions are considered as the initial particles to start the second part. If the number of particles in the reference set is less than the SSize, the remaining required particles are created and added randomly. In the second part, improving the solutions continues with a different mechanism, and good and non-repetitive solutions are placed in the reference set per iteration. At the end of the second part, the reference set of solutions are considered as the initial particles to start the second stage.
The first and second parts are terminated after P1S1 and P1S2 iterations or in the case of reaching to the unchanged best solution after CR11 and CR12 iterations. Besides, the first stage of the algorithm terminates after P1 times sequential repetition of the first and second parts.
In the second stage, first, the optimal results are determined from the model, and the best solution is selected among all solutions. The quality of the chosen solution is then improved by using the local search procedure. Therefore, a new particle is obtained with only one component different from the original one in a way almost similar to mutation operator in genetic algorithm. The new particle's fitness value is calculated using the mathematical model. If the new particle's fitness value is better than the original particle, the new particle replaces the original particle. Otherwise, the new particle is removed, and the original one remains unchanged. The procedure is stopped at the latest after P2 iterations or if the particle's best fitness value is not improved during CR2 consecutive iterations. The second stage of the algorithm ends after P2 iterations or in the absence of any change in the best particle's value after CR2 iterations.

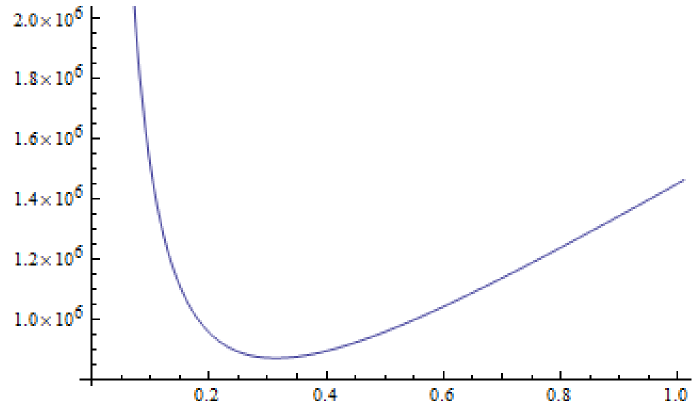
Case study

This study focuses on the mobile phone wholesale industry in Iran. As a historical experiment, with the obsolescence of the A8 series and the Edge model of Samsung mobile phones, the mobile phone wholesalers in the Iranian market suffered huge losses and hence they were seeking to determine the optimal economic ordering quantity in this field. So, it seems that in this industry reviewing the wholesale inventory control model of mobile phones would be essential considering the possibility of obsolescence due to the emergence of new technology.
It is more challenging to decide on replenishment cycle time due to the unclear customer demand caused by obsolescence. Furthermore, these decisions would be more complex due to the unclear supply side by suppliers. Therefore, for the products facing obsolescence, developing optimization models is more essential. All data used in this study are gathered from wholesale mobile phone industries in working period of 2021-2020. To protect data privacy, more information about mobile phones are ignored.

Numerical result

Now after introducing the mathematical model, the solutions approach has been presented in this section. The model was solved on a PC with windows 7, version 2012, and Intel(R) Pentium (R), Core-I5, 2.9 GHz CPU with 2 Giga bytes of RAM. In this study, some numerical examples are presented to evaluate the performance of the proposed model and solution procedure. Accordingly, the convexity of the objective function is demonstrated for the presented example by the following Figures. The exact results have been obtained by Mathematica 6.0 and Figures were plotted using the "Extremum" in GeoGebra. Also, PSO toolbox in MATLAB V9.0 has been used for developing the introduced Meta-heuristic solution approach.
Let A=200, D_0=10000, c=200, C_s=200, H=0.15, I_e=0.13, I_p=0.15, L=4, M=0.04
The objective function of the first case based on the above information is shown in Figure 3:

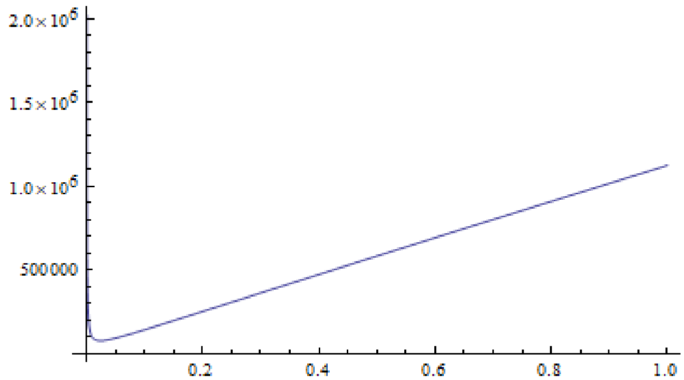
Figure 3: The inventory objective function values in the case $M \leq T$



The optimal value according to the above example in the first objective function (i.e. equation 10) is equal to: $T^*=0.378155$, $C_{L1} = 640230$

The objective function of the second case is also shown in Figure 4:

Figure 4: The inventory objective function values in the case $M > T$



The optimal value in the second objective function (i.e. equation 11) is also equal to:
$T^* = 0.0241189,\ C_{L2} = 79873$

Sensitivity analysis results

Generally, the changes in the values of the model parameters lead to uncertainty in decision-making. In this section, the sensitivity analysis is performed on the presented numerical example by setting different values for the critical parameters of the model. This analysis is based on the obtained solution by replacing the Taylor series approximation and model solving by the PSO algorithm.

Table 1 represents the changes on the decision variable and objective function for different values of the M, C_s and L. Also, Table 2 provides the changes in decision variable and objective function for different values of H and I_e.

In these Tables, the approximate cost resulted from PSO algorithm (C_Lpso) and Taylor series (C_{LTapp}) are respectively presented.

Table 1: Changing inventory costs by parameters L, M and c

M ≤ T					
C		C_s = 200, c =200		C_s =400, c =400	
M	L	2	4	2	4
0 day	T_{pso}	0.01667	0.0214494	0.0117828	0.0151627
	T_{app}	0.0175027	0.0234544	0.0123843	0.0165919
	C_{Lpso}	49352.4	77528.3	71697.8	114326.0
	C_{LTapp}	53747.	85989.4	80530.8	131978.0
60 days	T_{pso}	0.0242833	0.0268244	0.0211952	0.0221003
	T_{app}	0.0241201	0.0313758	0.0207021	0.0266371
	C_{Lpso}	22679.2	51349.4	33682.7	77364.2
	C_{LTapp}	29763.1	56809.8	42831.9	83939.2

Table 2: Changing inventory costs by parameters H, I_e and C

M ≤ T							
Cs =200,= C_p= 200					Cs =400, C_p = 400		
H	I_e	0.07	0.11	0.15	0.07	0.11	0.15
0.3	T_{pso}	0.0386555	0.0312675	0.0214844	0.0355366	0.0273212	0.0151878
	C_{Lpso}	109935.	84200.	50123.	198139.	140907.	56381.6
0.2	T_{pso}	0.0410873	0.0332336	0.0228346	0.0377713	0.0290387	0.0161421
	C_{Lpso}	99545.2	75325.7	43257.4	178639.	124779.	45237.9

Based on Table 1, the optimal replenishment cycle and the ordering quantity increase by increasing the amount of L, since the risk of product obsolescence decreases by increasing the amount of L. In addition, the average product lifecycle of inventories increases by increasing on L, leading to a rise in the holding cost during product lifetime, while the annual average inventory cost (CL / T) demonstrates a reduction.

Therefore, for products with a decreasing obsolescence rate whereas an increasing expected product lifetime, the retailers should increase the order quantity per cycle to avoid the ordering cost and increasing the total inventory cost.

Similarly, the replenishment cycle time (T) and consequently the optimal ordering quantity increase by increasing M, and so the retailers are encouraged for more ordering. Accordingly, the retailers can undergo lower inventory costs and lower obsolete item costs by selecting an appropriate policy and considering a more extended payment time.

The comparison of the objective function results of the Taylor series and the PSO algorithm revealed that PSO results have priority over those of the Taylor series. The results in the case M > T can be extracted similarly.

Furthermore, the optimal replenishment cycle time decreased, whereas the total inventory cost increased by increasing the holding cost (Table 2). Conversely, the optimal replenishment cycle time increased while the total inventory cost represented a decline by raising the receivable interest rate.

Validity of the model

To check the validation of the proposed solution method based on PSO algorithm, it is necessary to compare the solutions obtained from the algorithm with the already existing exact solution of the model. Accordingly, the results of the PSO algorithm solution for the first case of the inflation model, was compared with the solutions which have been solved manually without using metaheuristic algorithms, and the outputs are presented in Table 3. The ratio of the algorithm solution and the exact solution is about 0.007, which indicates the high accuracy of the algorithm in providing the solution near to the optimal solutions.

Table 3: Comparing the exact solution results with the solution of the proposed PSO algorithm

M<T					
c		Cs=Cp=c=20		Cs=Cp=c=200	
M	L	2	4	2	4
15 day	Q_{opt}	119.42	150.41	39.34	49.98
	Q_{pso}	120.4	151.8	32.02	54.8
	C_{Lopt}	38809.92	64518.67	349503.16	593070.67
	C_{Lpso}	39052.7	64973.3	352373.9	597721
	$\frac{C_{Lpso}}{C_{Lopt}}$	1.00625562	1.007046	1.008214	1.007841
30 days	Q_{pso}	102.7	166.8	48.5	73.7
	C_{Lopt}	38662.68	64276.29	349188.85	592689.29
	C_{Lpso}	38987.64	64765.2	351862.9	598368.8
	$\frac{C_{Lpso}}{C_{Lopt}}$	1.008405	1.007606	1.007658	1.009583

Conclusion

In this study, an inventory control model was introduced for obsolete items in which the demand is a decreasing function of time as expected in the real world. It is noteworthy that the suppliers permanently attempt to attract more customers or increase their current customers' loyalty by various incentive policies such as making delays and facilitation of payment terms. Therefore, the trade credit policy has been used along with time-dependent demands in developing this research model as the main contribution to the real world.

In the modeling process, the total inventory cost of the model was determined in both cases of M ≤ T and M > T after extracting the obsolescence cost, the holding cost, and trade credit-related costs. The convexity of both models was proved as well.

However, considering that the proposed model has no closed-form due to its nonlinear complexity, the Taylor series was applied as an approximation. Nonetheless, using only two or three terms of the Taylor series leads to highly different results compared to the optimal solution.

Therefore, the PSO meta-heuristic algorithm was introduced and employed as an alternative solution method, which was evaluated by comparing the algorithm's performance with that of the

Taylor series method in their approximation errors. Numerical results approved the superiority of PSO results over those of the Taylor series approximation.
In addition, according to the numerical results, the costs of the model can be decreased for obsolete items by increasing the expected payment time (M) and the expected lifetime of the product (L).
Overall, the presented model in this research improves the inventory control and ordering policies in businesses dealing with obsolete items such as electronic products (e.g., mobile phones, tablets, laptops, and the like). In future studies, it is possible to add realistic assumptions such as considering partial backlog items and inflation rates to the model. Finally, the proposed model can be solved with other evolutionary algorithms. The results can be compared in terms of time to reach a near-optimal solution and concerning efficiency with the presented solution in this study.

Appendix A

To prove the convexity of the objective function at the first case, the following equations are used to simplify the equations (10) and (11):

$$\alpha = D_0\left(\frac{1}{2}HCL + C_s\right) > 0 \tag{A.1}$$

$$\beta = CD_0I_e\left(L - e^{-\frac{2M}{L}}(L + 2M)\right) > 0 \tag{A.2}$$

Using the Taylor series approximation as $e^x = 1 + x + \frac{x^2}{2}$, we will have :

$$C_{L1} = \frac{\alpha T}{2(1 - \frac{\mathrm{T}}{2L})} + \frac{2(M-T)(\frac{M}{L} - \frac{T}{L} - \frac{2M^2}{L^2})D_0\mathrm{CI}_p}{4(\frac{T}{L} - \frac{T^2}{2L^2})} - \frac{\beta}{4(\frac{T}{L} - \frac{T^2}{2L^2})} \tag{A.3}$$

$$C_{L2} = \frac{\frac{2\alpha T^2}{L}}{4(\frac{T}{L} - \frac{T^2}{2L^2})} - \frac{\gamma}{4\left(\frac{T}{L} - \frac{T^2}{2L^2}\right)} - \frac{2TCDI_e(-\frac{2T}{L} + \frac{2T^2}{L^2} + \frac{2M}{L} - \frac{2M^2}{L^2})}{4(\frac{T}{L} - \frac{T^2}{2L^2})} \tag{A.4}$$

The second-order derivative of the first objective function is equal to:

$$\frac{\partial^2 C_{L1}}{\partial T^2} = \tag{A.5}$$

$$\frac{\alpha}{2\left(L-\frac{T}{2}\right)^2}+\frac{T\alpha}{4\left(L-\frac{T}{2}\right)^3}+\frac{\mathrm{CI}_p D_0}{L\left(\frac{T}{L}-\frac{T^2}{4L^2}\right)}+\frac{1}{L}\left(\frac{1}{L}-\frac{T}{2L^2}\right)(M-T)\mathrm{CI}_p D_0$$
$$+\frac{1}{L}\left(\frac{1}{L}-\frac{T}{2L^2}\right)\mathrm{CI}_p D_0\left[1+\left(\frac{1}{L}-\frac{T}{2L^2}\right)(M-T)\right]\left\{M-T-\frac{2M^2}{L^2}\right\}\left(\frac{1}{\left(\frac{T}{L}-\frac{T^2}{4L^2}\right)^2}\right)$$
$$+\frac{1}{2}(M-T)\left(\frac{M}{L}-\frac{2M^2}{L^2}-\frac{T}{L}\right)\left(\frac{1}{2L^2\left(\frac{T}{L}-\frac{T^2}{4L^2}\right)^2}\right)\mathrm{CI}_p D_0$$
$$+\frac{1}{4}\left(-\frac{2\left(\frac{1}{L}-\frac{T}{2L^2}\right)^2}{\left(\frac{T}{L}-\frac{T^2}{4L^2}\right)^3}-\frac{1}{2L^2\left(\frac{T}{L}-\frac{T^2}{4L^2}\right)^2}\right)\beta$$

In order to simplify the calculations, let’s assume:

$$\frac{1}{\mathrm{L}}-\frac{\mathrm{T}}{2\mathrm{L}^2}=\mathrm{x},\quad \mathrm{M}-\mathrm{T}=\mathrm{y},\qquad \frac{\mathrm{T}}{\mathrm{L}}-\frac{\mathrm{T}^2}{4\mathrm{L}^2}=\mathrm{z}, \tag{A.6}$$

So given that $0<\mathrm{T}<1$ and $\mathrm{L}>\mathrm{T}$, we will have: $1>\mathrm{x}>0, -1<\mathrm{y}<0,\ \ 1>\mathrm{z}>0$,
By replacing x, y and z in (A.5), we will have:

$$\frac{\partial^2 \mathrm{C_{L1}}}{\partial \mathrm{T}^2}=\left\{\frac{\mathrm{T}\alpha}{\mathrm{L}^2\mathrm{z}^2}\left(\frac{3\mathrm{T}^2}{4\mathrm{L}^2}-\frac{\mathrm{T}}{\mathrm{L}}\right)\right\}+\left\{\frac{\alpha\mathrm{T}^2}{2\mathrm{Lz}^2}\left(\frac{2\left(\frac{1}{\mathrm{L}}-\frac{\mathrm{T}}{2\mathrm{L}^2}\right)^2}{\mathrm{z}}+\frac{1}{2\mathrm{L}^2}\right)\right\}$$
$$+\left\{\frac{\mathrm{CI_p D_0}}{\mathrm{Lz}}\left(1+\frac{\mathrm{xy}}{\mathrm{z}}\right)\left(1+\frac{\mathrm{x}\left[\mathrm{y}-\frac{2\mathrm{M}^2}{\mathrm{L}}\right]}{\mathrm{z}}\right)\right\} \tag{A.7}$$
$$+\left\{\frac{1}{2}\mathrm{y}\left(\frac{\mathrm{y}}{\mathrm{L}}-\frac{2\mathrm{M}^2}{\mathrm{L}^2}\right)\left(\frac{1}{2\mathrm{L}^2\mathrm{z}^2}\right)\mathrm{CI_p D_0}\right\}-\left\{\frac{\beta}{4\mathrm{z}^2}\left(\frac{2\left(\frac{1}{\mathrm{L}}-\frac{\mathrm{T}}{2\mathrm{L}^2}\right)^2}{\mathrm{z}}+\frac{1}{2\mathrm{L}^2}\right)\right\}$$

To prove that equation (A.7) is greater than zero, it should be proved that the third term of this equation is positive, as shown in (A.8):

$$z=\frac{T}{L}-\frac{T^2}{4L^2}>\frac{T}{L}-\frac{T^2}{2L^2}=T\left(\frac{1}{L}-\frac{T}{2L^2}\right)>\left|(M-T)\left(\frac{1}{L}-\frac{T}{2L^2}\right)=xy\right|\rightarrow z>|xy|$$
$$\rightarrow 1+\frac{xy}{z}>0\rightarrow \tag{A.8}$$

$$T > \left|\left(M - T - \frac{2M^2}{L}\right)\right| \rightarrow 1 + \frac{x\left[y - \frac{2M^2}{L}\right]}{z} > 0$$

$$\rightarrow \frac{\mathrm{CI}_p D_0}{Lz}\left(1 + \frac{xy}{z}\right)\left(1 + \frac{x\left[y - \frac{2M^2}{L}\right]}{z}\right) > 0$$

It should also be proved that the fourth term of this equation is greater than zero, as shown in (A.9):

$$-1 < y < 0 \rightarrow \left(\frac{y}{L} - \frac{2M^2}{L^2}\right) < 0 \rightarrow \frac{1}{2} y \left(\frac{y}{L} - \frac{2M^2}{L^2}\right)\left(\frac{1}{2L^2z^2}\right) \mathrm{CI}_p D_0 > 0 \tag{A.9}$$

Meanwhile, the second term of the equation (A.7) is positive, but the last term is negative. Therefore, it should be proved that the sum of the first and last terms is higher than zero.

Given that, $\frac{2\alpha T^2}{L}$ in logical conditions is more than β, therefore:

$$\frac{2\alpha T^2}{L} > \beta \rightarrow \frac{\alpha T^2}{2Lz^2}\left(\frac{2\left(\frac{1}{L} - \frac{T}{2L^2}\right)^2}{z} > \frac{\beta}{4z^2}\left(\frac{2\left(\frac{1}{L} - \frac{T}{2L^2}\right)^2}{z}\right)\right.$$

$$\rightarrow \frac{\alpha T^2}{2Lz^2}\left(\frac{2\left(\frac{1}{L} - \frac{T}{2L^2}\right)^2}{z} + \frac{1}{2L^2}\right) - \frac{\beta}{4z^2}\left(\frac{2\left(\frac{1}{L} - \frac{T}{2L^2}\right)^2}{z} + \frac{1}{2L^2}\right) > 0 \tag{A.10}$$

The first term of equation (A.7) is above zero, but the fourth term is less than zero. It is necessary to show that the sum of these two terms is also greater than zero:

Since $\mathrm{T}\alpha > -\frac{1}{4}\mathrm{yCI_pD_0}$, considering $\mathrm{T} > 2\sqrt{\frac{(\mathrm{ML}-2\mathrm{M}^2)}{3}}$, we will have:

$$\left|\frac{4(\mathrm{ML} - 2\mathrm{M}^2)}{3}\right| < |\mathrm{T}^2| \rightarrow \left|\frac{M - T}{\mathrm{L}} - \frac{2\mathrm{M}^2}{\mathrm{L}^2}\right| < \left|\frac{3\mathrm{T}^2}{4\mathrm{L}^2} - \frac{\mathrm{T}}{\mathrm{L}}\right| \rightarrow \left|\frac{y}{\mathrm{L}} - \frac{2\mathrm{M}^2}{\mathrm{L}^2}\right| < \left|\frac{3\mathrm{T}^2}{4\mathrm{L}^2} - \frac{\mathrm{T}}{\mathrm{L}}\right| \tag{A.11}$$

Then:

$$\frac{T\alpha}{L^2z^2}\left(\frac{3T^2}{4L^2} - \frac{T}{L}\right) > -\frac{1}{2} y \left(\frac{y}{L} - \frac{2M^2}{L^2}\right)\left(\frac{1}{2L^2z^2}\right) \mathrm{CI}_p D_0 \tag{A.12}$$

And hence, the objective function of the first case can be considered to be convex.

Appendix B

The second-derivative of the second objective function is equal to:

$$\frac{\partial^2 \mathrm{TC_{L2}}}{\partial \mathrm{T}^2} = -\frac{2\mathrm{T}\left(\frac{1}{\mathrm{L}}-\frac{\mathrm{T}}{2\mathrm{L}^2}\right)\alpha}{\mathrm{L}\left(\frac{\mathrm{T}}{\mathrm{L}}-\frac{\mathrm{T}^2}{4\mathrm{L}^2}\right)^2} + \frac{\alpha}{\mathrm{L}\left(\frac{\mathrm{T}}{\mathrm{L}}-\frac{\mathrm{T}^2}{4\mathrm{L}^2}\right)} + \frac{\mathrm{T}^2\left(\frac{2\left(\frac{1}{\mathrm{L}}-\frac{\mathrm{T}}{2\mathrm{L}^2}\right)^2}{\left(\frac{\mathrm{T}}{\mathrm{L}}-\frac{\mathrm{T}^2}{4\mathrm{L}^2}\right)^3}+\frac{1}{2\mathrm{L}^2\left(\frac{\mathrm{T}}{\mathrm{L}}-\frac{\mathrm{T}^2}{4\mathrm{L}^2}\right)^2}\right)\alpha}{2\mathrm{L}}$$
$$-\frac{\beta}{4}\left(\frac{2\left(\frac{1}{\mathrm{L}}-\frac{\mathrm{T}}{2\mathrm{L}^2}\right)^2}{\left(\frac{\mathrm{T}}{\mathrm{L}}-\frac{\mathrm{T}^2}{4\mathrm{L}^2}\right)^3}+\frac{1}{2\mathrm{L}^2\left(\frac{\mathrm{T}}{\mathrm{L}}-\frac{\mathrm{T}^2}{4\mathrm{L}^2}\right)^2}\right) - \frac{2\mathrm{CTI_eD_0}}{\mathrm{L}^2(\frac{\mathrm{T}}{\mathrm{L}}-\frac{\mathrm{T}^2}{4\mathrm{L}^2})} + 2(-\frac{2}{\mathrm{L}} \tag{B.1}$$
$$+\frac{4\mathrm{T}}{\mathrm{L}^2})(\frac{\mathrm{CT}(\frac{1}{\mathrm{L}}-\frac{\mathrm{T}}{2\mathrm{L}^2})}{2(\frac{\mathrm{T}}{\mathrm{L}}-\frac{\mathrm{T}^2}{4\mathrm{L}^2})^2} - \frac{\mathrm{C}}{2(\frac{\mathrm{T}}{\mathrm{L}}-\frac{\mathrm{T}^2}{4\mathrm{L}^2})})\mathrm{I_eD_0} - \frac{1}{2}\mathrm{C}(\frac{2\mathrm{M}}{\mathrm{L}}-\frac{2\mathrm{M}^2}{\mathrm{L}^2}-\frac{2\mathrm{T}}{\mathrm{L}}$$
$$+\frac{2\mathrm{T}^2}{\mathrm{L}^2})(-\frac{2(\frac{1}{\mathrm{L}}-\frac{\mathrm{T}}{2\mathrm{L}^2})}{(\frac{\mathrm{T}}{\mathrm{L}}-\frac{\mathrm{T}^2}{4\mathrm{L}^2})^2} + \mathrm{T}(\frac{2(\frac{1}{\mathrm{L}}-\frac{\mathrm{T}}{2\mathrm{L}^2})^2}{(\frac{\mathrm{T}}{\mathrm{L}}-\frac{\mathrm{T}^2}{4\mathrm{L}^2})^3} + \frac{1}{2\mathrm{L}^2(\frac{\mathrm{T}}{\mathrm{L}}-\frac{\mathrm{T}^2}{4\mathrm{L}^2})^2}))\mathrm{I_eD_0}$$

In order to simplify the calculations, let's assume:

$$CI_eD_0 = m > 0 \tag{B.2}$$

So we will have:

$$\frac{\boldsymbol{\partial}^2 TC_{L2}}{\boldsymbol{\partial} T^2} = \frac{T\alpha}{L^2z^2}\left(\frac{3T^2}{4L^2}-\frac{T}{L}\right)+\left(\frac{T^2\alpha}{2L}-\frac{\beta}{4}\right)\left(\frac{2x^2}{z^3}+\frac{1}{2L^2z^2}\right)+\frac{2Tm}{z^2L^2}\left(\frac{T^2}{4L^2}+1-3\frac{T}{L}\right)$$
$$+\frac{mT}{2L^2z^2}(\frac{M}{L}-\frac{T}{L})(1-\frac{M}{L}-\frac{T}{L})(\frac{-\frac{T^2}{2L^3}-\frac{T^2}{8L^4}+\frac{T^3}{4L^4}+\frac{T}{2L^3}}{z^2}) \tag{B.3}$$

Let's consider:

$$A = \frac{2Tm}{z^2L^2}(\frac{T^2}{4L^2}+1-3\frac{T}{L}) \tag{B.4}$$

$$B = \left(\frac{T^2\alpha}{2Lz^2}-\frac{\beta}{4z^2}\right)\left(\frac{2x^2}{\mathrm{z}}+\frac{1}{2L^2}\right) \tag{B.5}$$

$$\mathrm{C} = \frac{\mathrm{T}^2\alpha}{\mathrm{L}^3\mathrm{z}^2}\left(\frac{3\mathrm{T}}{4\mathrm{L}}-1\right) \tag{B.6}$$

$$D = \frac{mT^2}{4L^4z^2}\left(\frac{M}{L}-\frac{T}{L}\right)\left(1-\frac{M}{L}-\frac{T}{L}\right)\left(-\frac{T}{L}-\frac{T}{4L^2}+\frac{1}{L}+\frac{T^2}{2L^2}\right) \tag{B.7}$$

About above terms, it should be noted:

(A) is greater than zero.

Considering $\frac{2\alpha\mathrm{T}^2}{\mathrm{L}} > \beta$, (B) is also greater than zero.

(C) is less than zero. But Considering

$$\frac{T^2\alpha}{2Lz^2}-\frac{\mathrm{T}^2\alpha}{\mathrm{L}^3\mathrm{z}^2}-\frac{\beta}{4z^2}>0\rightarrow\frac{T^2\alpha}{Lz^2}\left(\frac{1}{2}-\frac{1}{L^2}\right)-\frac{\beta}{4z^2}>0 \quad \text{(B.8)}$$

And then

$$\frac{2x^2}{\mathrm{z}}+\frac{1}{2L^2}+\left(\frac{3T}{4L}-1\right)>0 \quad \text{(B.9)}$$

B + C would be greater than zero.

It is not possible to decide about D with certainty; but considering M > T in this case, we will have:

$$0<\left(\frac{M}{L}-\frac{T}{L}\right)<1\ \&\ 0<\left(1-\frac{M}{L}-\frac{T}{L}\right)<1 \quad \text{(B.10)}$$

In addition

$$\left(\frac{T^2}{4L^2}+1-3\frac{T}{L}\right)+\frac{T}{8L^2}\left(-\frac{T}{L}-\frac{T}{4L^2}+\frac{1}{L}+\frac{T^2}{2L^2}\right)>0 \quad \text{(B.11)}$$

Hence, A + D would be greater than zero.

Consequently, the objective function of the second case can be considered to be convex

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