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Efficiency Evaluation of Economic Enterprise in Presence of Interval Undesirable and Negative Data

Mahnaz Maghbouli [1*], Mahdi Eini[2], Farhad Taher[3] and Fatemeh Ghomanjani[4]
[1]Department of Mathematics, Arac,Islamic Azad University, Jolfa, Iran.
[2]Department of Mathematics, Payame Noor University, Tehran, Iran.
[3]Department of Mathematics, Education of West Azerbaijan Province, Khoy, Iran.
[4]Kashmar higher education institute, kashmar,Iran.

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Abstract

Data envelopment analysis (DEA) as a non-parametric method has covered a wide range of applications in measuring comparative efficiency of decision making units (DMUs) with multiple incommensurate inputs and outputs. The standard DEA method requires that all input and output variables be known as semi positive. In many real situations, the presence of undesirable and even negative data are inevitable. In DEA literature there have been various approaches to enable DEA to deal with negative data. On the other hand, the structure of interval data has recently attracted considerable attention among DEA researchers. According to importance of interval data, this paper proposes a radial measure which permits the presence of undesirable and negative data with interval structure. The proposed model can evaluate the efficiency of all DMUs and leads to improve the inefficient unit with interval negative and undesirable data. To elucidate the details of the proposed method an illustrative example of a private bank in IRAN explores the applicability of the proposed method.

Keywords:
DEA
Efficiency
Negative Data
Undesirable factors
Interval Data

***Correspondence E-mail**: mmaghbouli@gmail.com

INTRODUCTION

DEA is concerned with comparative assessment of the efficiency of decision making units (DMUs). In the classical DEA models, the efficiency of a DMU is obtained by maximizing the ratio of the weighted sum of its outputs to the weighted sum of its inputs, subject to the condition that this ratio does not exceed one for any DMU. Although the first DEA models were formulated in (1) Charnes et al. (1978), under the assumption of constant returns to scale, and (2) Banker et al. (1984) under variable returns to scale technology. In conventional DEA models, given a set of assumption, the efficiency frontier is specified with desirable input and output. However, in some real occasions, both desirable and undesirable outputs may be generated such as noise pollution and wastage. Since the last decade, DEA as a non-parametric technique has played an important role in analyzing undesirable factors. The challenge of modeling undesirable factors causes considerable attention in two different aspects. The first is for measuring efficiency and productivity and the second, points out to estimating pollution abatement factor. The approaches for reducing undesirable factors such as noise pollution or wastage has been investigated by Fare and Grosskopf (1989, 2003), Jahanshahloo et al. (2005), Lotfi et al. (2007), Tao et al. (2016), Halkos et al. (2019), Yousefi et al. (2018) as well as Shirazi and Mohammadi (2020). Recently, Zhu et al. (2019) changed the undesirable outputs to be positive desirable outputs by an exponential function and employed the transformed data in evaluating the efficiency of thirty-eight industrial zones in China. The concept of output “weak – disposability” was employed in the paper of Kuosmanen and Kazemi Matin (2011). Piao et al. (2019) joined the output weak and strong disposability axioms with Malmquist Productivity Index (MPI) and evaluated environmental efficiency. Since the researchers have made some contributions to deal with undesirable outputs into DEA models. They are sometimes interested in estimating the efficiency in presence of undesirable inputs. The most important example of undesirable inputs is green water which returns to recycled system process. As another instance, the used ironwork can be considered as undesirable inputs which need to be reconstituted and re-entered to production process (EINI et.al (2017)). With reference to the interval structure of data set, Jahanshahloo et.al (2009) proposed an approach to evaluate efficiency in presence of interval data. Khalili Damghani et al. (2015) employed the interval data in presence of undesirable factors to assess the efficiency of units. Recently, Taher et.al (2019) suggested an approach for estimating the undesirable outputs and desirable inputs in presence of interval data. In real world occasions, the existence of negative data is indisputable. For example, if the revenue of the given enterprise has exceeded its income in a specific time period, the profit of that enterprise will meet the negative data. Or the temperature of a subject can meet both negative and positive quantities. The pioneer model for dealing with negative data was introduced by Seiford and Zhu (2007). Portela et.al (2004) introduced another technique named as RDM model. The most employed model in dealing with negative data proposed by Sharp (2007) named as MSMB. The readers may refer to Asmild et.al (2010), Emrouznejad et.al (2010) and Sahoo et al. (2016) to follow different techniques for dealing with negative data. All in all, employing negative data can affect the performance of a unit. That is to say, employing negative data may challenge a unit for assessing the efficiency. On the other hand, the joint production of desirable and undesirable output is an interest field for discussion. The most important example for joint employing of negative and undesirable input trace back to wastewater recycling process. Sewage effluent can be taken as negative data in the recycling process. A bank can be taken as a good economic example. If a bank branch cannot collect all the claims, it is called as detriment branch. The quantity of this loss is taken as negative output in bank efficiency assessment. In this paper we aim to search for an approach to determine the efficiency of DMUs in presence of both undesirable and negative data. The main contribution of this study is employing the interval data set, which was not discussed in the literature. Employing the interval structure of the data set, the paper develops a linear programming for evaluating efficiency using negative, desirable and undesirable inputs to generate negative, desirable and undesirable output. The paper is organized as follows. The following section reviews the basic concepts. The proposed approach is introduced in the third section. A real example of a bank in IRAN is illustrated in Section 4. Paper will end with conclusion.

PRELIMINARIES

Evaluating efficiency in presence of interval data has attracted considerable attention among researchers. To describe the DEA efficiency measurement, assume there are n DMUs and the performance of each DMU is characterized by a production process of m inputs $X_{ij}(i=1,...,m, j=1,...,n)$ to yield s outputs $Y_{rj}(r=1,...,s, j=1,...,n)$. Assume that the inputs and outputs are described as interval which demonstrated as $x_{ij}=\left[x_{ij}^{l},x_{ij}^{u}\right]$, and $y_{rj}=\left[y_{rj}^{l},y_{rj}^{u}\right]$. In this presentation, x_{ij}^{l} is the lower of inputs and x_{ij}^{u} depicts the upper bound of inputs. Similarly, y_{rj}^{l} presents the lower bound of outputs and y_{rj}^{u} shows the upper bound of outputs. Despotis and Esmirlis (2002) proposed DEA-based models to evaluate the interval efficiency $\left[\theta^{l},\theta^{u}\right]$ as follows:

$$
\begin{aligned}
&\text{Min } \theta^{l} \\
&s.t \\
&\sum_{j=1,\, j\neq p}^{n} \lambda_j x_{ij}^{l} + \lambda_p x_{ip}^{u} \le \theta^{l} x_{ip}^{u} \;, i=1,\ldots,m, \\
&\sum_{j=1,\, j\neq p}^{n} \lambda_j y_{rj}^{u} + \lambda_p y_{rp}^{l} \ge y_{rp}^{l}, r=1,\ldots,s, \\
&\sum_{j=1}^{n} \lambda_j = 1 \;, \\
&\lambda_j \ge 0, j=1,\ldots,n.
\end{aligned}
\tag{1}
$$

The above model (1) evaluate the lower bound of efficiency in presence of interval data. As the model (1) describes the upper bound of inputs and the lower bound of outputs for the under evaluated unit (DMU_p) is employed in the model (1). Other units employs their lower bound of inputs and upper bound of outputs for evaluating the lower bound θ^{l}. For calculating the upper bound of efficiency, the following model is proposed:

$$
\begin{aligned}
&\text{Min } \theta^{u} \\
&s.t\;. \\
&\sum_{j=1,\, j\neq p}^{n} \lambda_j x_{ij}^{u} + \lambda_p x_{ip}^{l} \le \theta^{u} x_{ip}^{l}, i=1,\ldots,m, \\
&\sum_{j=1,\, j\neq p}^{n} \lambda_j y_{rj}^{l} + \lambda_p y_{rp}^{u} \ge y_{rp}^{u}, r=1,\ldots,s, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \ge 0 \;, j=1,\ldots,n.
\end{aligned}
\tag{2}
$$

The upper bound of efficiency, θ^{u} , is evaluated by employing the lower bound of inputs and upper bound of outputs for under evaluated unit (DMU_p). Other units employs their lower bound of outputs and upper bound of inputs for evaluating the upper bound θ^{u} . In presence of negative data, Emrouznejad et al. (2010) proposed a model named as semi-oriented radial model (SORM). The authors assumed that there are negative data in some inputs and outputs. Without loss of generality, assume that there are n DMUs and J indicate the number of units $J=\{1,2,...,n\}$. The number of inputs and outputs are represented as $I=\{1,2,...,m\}$ and $R=\{1,2,...,s\}$, respectively. Assume the set of indices for inputs with nonnegative values is indicated by $I'=\{i \in I : \forall j \in J, x_{ij} \geq 0\}$ while I'' denotes the set of index of inputs which have negative value in at least one DMU $I''=\{i \in I : \exists j \in J, x_{ij} < 0\}$. Similarly, $R'=\{r \in R : \forall j \in J, y_{rj} \geq 0\}$ is the set of outputs with nonnegative value and $R''=\{r \in R : \exists j \in J, y_{rj} < 0\}$ denotes the set of index of outputs which have negative value in at least one DMU. Obviously, $I' \bigcup I'' = I$, $I' \bigcap I'' = \phi$ and $R' \bigcup R'' = R$, $R' \bigcap R'' = \phi$. Emrouznejad et al. (2010) defined $x_{ij}=x^1_{ij}-x^2_{ij}$, $i \in I''$ where $x^1_{ij} \geq 0$ and $x^2_{ij} \geq 0$ in which

$$x^1_{ij}=\begin{cases} x_{ij} & x_{ij} \geq 0 \\ 0 & x_{ij} < 0 \end{cases}$$

$$x^2_{ij}=\begin{cases} 0 & x_{ij} \geq 0 \\ -x_{ij} & x_{ij} < 0 \end{cases}$$

Similarly, for each $j \in R''$ the non - negative value y^1_{rj}, y^2_{rj} are used, where

$$y^1_{rj}=\begin{cases} y_{rj} & y_{rj} \geq 0 \\ 0 & y_{rj} < 0 \end{cases}$$

$$y^2_{rj}=\begin{cases} 0 & y_{rj} \geq 0 \\ -y_{rj} & y_{rj} < 0 \end{cases}$$

To assess the efficiency of DMU_o: $o \in J=\{1,...,n\}$, Emrouznejad et al. (2010) proposed input oriented variable returns to scale SORM model when DMUs have positive and negative values in certain input and output variables simultaneously

$$\begin{aligned}
&\text{Min } \theta \\
&s.t. \\
&\sum_{j=1}^{n}\lambda_j x_{ij} \leq \theta x_{io}, \quad i \in I' \\
&\sum_{j=1}^{n}\lambda_j x^1_{ij} \leq \theta x^1_{io}, \quad i \in I'' \\
&\sum_{j=1}^{n}\lambda_j x^2_{ij} \geq \theta x^2_{io}, \quad i \in I'' \\
&\sum_{j=1}^{n}\lambda_j y_{rj} \geq y_{ro}, \quad r \in R' \\
&\sum_{j=1}^{n}\lambda_j y^1_{rj} \geq y^1_{ro}, \quad r \in R'' \\
&\sum_{j=1}^{n}\lambda_j y^2_{rj} \leq y^2_{ro}, \quad r \in R'' \\
&\sum_{j=1}^{n}\lambda_j = 1, \\
&\lambda_j \geq 0, \quad j=1,\ldots,n.
\end{aligned} \tag{3}$$

As the model (3) claims all negative and positive data are introduced with the same intensity variable $\lambda_j (j \in J=\{1,...,n\})$. Due to the orientation of model (3), input oriented, the objective function of the model (3) toys around the reduction of all positive and negative inputs. The model (3) is always feasible and bounded. What's more, the constraint $\sum_{j=1}^{n}\lambda_j = 1$ implies the variable return to scale structure in the above model (3).

PROPOSED APPROACH

These days, modeling undesirable inputs such as plastic waste, grey water or rotten fruits has attracted considerable attention among researchers. With reference to increasing pollutant and decreasing the natural resource, demanding for cleaner sources seems a big challenge. With reference to deficiency of natural resources and the importance of preserving the sources with the attitude of employing renewable resources, the concept of undesirable inputs discovers its role in recent studies. In order to improve the efficiency and productivity in presence of undesirable inputs and outputs, the aim is decreasing undesirable outputs and increasing undesirable inputs, simultaneously. The generation of negative outputs can be addressed with subtraction of two non-negative variables. One of the main examples of negative outputs is the concept of profit and loss. In the presence of both negative and positive data set, development of existing model is required. Assume that the desirable and undesirable inputs take the interval values as $x_{ij}^{g} = [x_{ij}^{\mathrm{l}g}, x_{ij}^{ug}]$, and $x_{i'j}^{b} = [x_{i'j}^{\mathrm{l}b}, x_{i'j}^{ub}]$, respectively. Similarly, the desirable and undesirable outputs can take interval values as $y_{rj}^{g} = [y_{rj}^{\mathrm{l}g}, y_{rj}^{ug}]$, and $y_{r'j}^{b} = [y_{r'j}^{\mathrm{l}b}, y_{r'j}^{ub}]$. One of the key concerns when we have a variable that takes positive values for some and negative values for other DMUs is that its absolute value should rise or fall for the DMU to improve its performance depending on whether the DMU concerned has a positive or negative value on that variable. In the light of mentioned arguments, the simultaneous reduction of desirable inputs and increasing of undesirable inputs are also expected with the interval structures. The proposed model (4) has the following format:

$$
\begin{aligned}
&\text{Min} \quad \theta \\
&s.t \\
&\sum_{j=1}^{n} \lambda_j [x_{ij}^{lg}, x_{ij}^{Ug}] \le \theta [x_{ip}^{lg}, x_{ip}^{ug}], i = 1, \dots, m, \\
&\sum_{j=1}^{n} \lambda_j [x_{i'j}^{lb}, x_{i'j}^{ub}] \ge [x_{i'p}^{lb}, x_{i'p}^{ub}], i' = 1, \dots, \mathrm{m}', \\
&\sum_{j=1}^{n} \lambda_j [x_{ij}^{1l}, x_{ij}^{1u}] \le \theta \ [x_{ip}^{1l}, x_{ip}^{1u}], i \in I'', \\
&\sum_{j=1}^{n} \lambda_j [x_{ij}^{2l}, x_{ij}^{2u}] \ge \theta [x_{ip}^{2l}, x_{ip}^{2u}], i \in I'', \\
&\sum_{j=1}^{n} \lambda_j [y_{rj}^{lg}, y_{rj}^{ug}] \ge [y_{rp}^{lg}, y_{rp}^{ug}], r = 1, \dots, s, \\
&\sum_{j=1}^{n} \lambda_j [y_{r'j}^{lb}, y_{r'j}^{ub}] \le \theta [y_{r'p}^{lb}, y_{r'p}^{ub}], \mathrm{r}' = 1, \dots, \mathrm{s}', \\
&\sum_{j=1}^{n} \lambda_j [y_{rj}^{1l}, y_{rj}^{1u}] \ge [y_{rp}^{1l}, y_{rp}^{1u}], r \in R'', \qquad (4) \\
&\sum_{j=1}^{n} \lambda_j [y_{rj}^{2l}, y_{rj}^{2u}] \le [y_{rp}^{2l}, y_{rp}^{2u}], r \in R'', \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \ge 0, j = 1, \dots, n.
\end{aligned}
$$

In the above model (4), the indexes i and i' are specified to reflect desirable and undesirable inputs, respectively. Similarly, the indexes r and r' indicate the desirable and undesirable outputs in the models. As before index j admits the number of DMUs. One the face of variables that take at least negative values for some inputs and negative for some outputs, the symbols I'' and R'' are specified in the model(4). To be more precise, in real occasions, there are not any examples of negative undesirable inputs and outputs, hence there are no constraints related to these measures in model (4). As a result, the model (4) is proposed for the case of interval –based positive/negative desirable and positive undesirable inputs and outputs. Clearly, model (4) can be modified to handle two different models (5) and (6) to address the lower and upper bound of efficiency for under estimated units. With reference to the interval efficiency concept, the lower bound of efficiency expounded in the respect of model (5) as follows:

$$
\begin{aligned}
&\text{Min } \theta^l \\
&s.t \\
&\sum_{j=1\,,\, j\neq p}^{n} \lambda_j x_{ij}^{lg} + \lambda_p x_{ip}^{ug} \le \theta^l x_{ip}^{ug},\ i = 1,\dots,m, \\
&\sum_{j=1\,,\, j\neq p}^{n} \lambda_j x_{i'j}^{ub} + \lambda_p x_{i'p}^{lb} = x_{i'p}^{lb}\ ,\ i' = 1,\dots,m', \\
&\sum_{j=1, j\neq p}^{n} \lambda_j x_{ij}^{1l} + \lambda_p x_{ip}^{1u} \le \theta^l x_{ip}^{1u},\ \ i \in I'', \\
&\sum_{j=1, j\neq p}^{n} \lambda_j x_{ij}^{2l} + \lambda_p x_{ip}^{2u} \ge \theta^l x_{ip}^{2u},\ \ i \in I'' \qquad (5) \\
&\sum_{j=1\,,\, j\neq p}^{n} \lambda_j y_{rj}^{ug} + \lambda_p y_{rp}^{lg} \ge y_{rp}^{lg}, r = 1,\dots,s, \\
&\sum_{j=1\,,\, j\neq p}^{n} \lambda_j y_{r'j}^{lb} + \lambda_p y_{r'p}^{ub} \le \theta^l y_{r'p}^{ub}\ \ ,\ r' = 1,\dots,s', \\
&\sum_{j=1, j\neq p}^{n} \lambda_j y_{rj}^{1u} + \lambda_p y_{rp}^{1l} \ge y_{rp}^{1l}\ ,\ \ r \in R'', \\
&\sum_{j=1, j\neq p}^{n} \lambda_j y_{rj}^{2u} + \lambda_p y_{rp}^{2l} \le y_{rp}^{2l}\ ,\ r \in R'', \\
&\sum_{j=1}^{n} \lambda_j = 1\ , \\
&\lambda_j \ge 0,\ j = 1,\dots,n.
\end{aligned}
$$

Like model (4), in the above model (5), the indexes i and i' are specified to reflect desirable and undesirable inputs, respectively. Similarly, the indexes r and r' indicate the desirable and undesirable outputs in the models. The symbols I'' and R'' are specified the variables with at least negative values for some inputs and negative for some outputs. In model (5) the upper bound of desirable inputs and undesirable outputs and desirable negative outputs along with the lower bound of undesirable input, negative desirable inputs and desirable output are used for under assessment DMU. Model (5) evaluates the lower bound of efficiency from pessimistic point of view. On the other hand, there are relevant constraints for both positive and negative values in inputs and outputs. The following theorem shows the feasibility and bounded of model (5).

Theorem 1. Model (5) is feasible and the optimal value of the model is bounded.

Proof.

Assume that $\theta^l = 1$, $\lambda_p = 1$, $\lambda_j = 0$ $(j = 1,\dots,n, j \neq p)$. It is easily shown that this solution satisfies in all constraints of model (5), hence it is feasible solution for model (5). Since the inputs and outputs are independent, as a result, there is always a feasible solution for model (5). With reference to objective function of the model (5) with the aim of minimizing the factor θ^l, it is concluded that $\theta^{*l} \le 1$. So, the model (5) is bounded in optimality.

To derive the upper bound of efficiency, the optimistic point of view is considered. That is to say, the lower bound of desirable inputs, undesirable outputs and negative outputs and upper bound of undesirable inputs, negative inputs and desirable outputs are employed. The proposed model (6) for evaluating upper bound of efficiency, for each input that takes positive and negative values the model creates two constraints, one for negative values and one for positive values. Note that this also happens for outputs. Model (6) has the following format:

$$
\begin{aligned}
&\text{Min } \theta^u \\
&s.t \\
&\sum_{j=1\,,\, j\neq p}^{n} \lambda_j x_{ij}^{ug} + \lambda_p x_{ip}^{lg} \le \theta^u x_{ip}^{lg}, i = 1,\dots,m, \\
&\sum_{j=1\,,\, j\neq p}^{n} \lambda_j x_{i'j}^{lb} + \lambda_p x_{i'p}^{ub} \ge x_{i'p}^{ub}\ , i' = 1,\dots,m'\ , \\
&\sum_{j=1, j\neq p}^{n} \lambda_j x_{ij}^{1u} + \lambda_p x_{ip}^{1l} \le \theta^u x_{ip}^{1l},\ i \in I'', \\
&\sum_{j=1, j\neq p}^{n} \lambda_j x_{ij}^{2l} + \lambda_p x_{ip}^{2u} \ge \theta^u x_{ip}^{2u},\ \ i \in I'' \qquad (6)
\end{aligned}
$$

$$\sum_{j=1\ ,\ j\neq p}^{n} \lambda_j y_{rj}^{lg} + \lambda_p y_{rp}^{ug} \geq y_{rp}^{ug} \quad , r = 1,\ldots,s,$$

$$\sum_{j=1\ ,\ j\neq p}^{n} \lambda_j y_{r'j}^{ub} + \lambda_p y_{r'p}^{lb} \leq \theta^u y_{r'p}^{lb}, r' = 1,\ldots,s',$$

$$\sum_{j=1, j\neq p}^{n} \lambda_j y_{rj}^{1l} + \lambda_p y_{rp}^{1u} \geq y_{rp}^{1u} \ , r \in R'',$$

$$\sum_{j=1, j\neq p}^{n} \lambda_j y_{rj}^{2l} + \lambda_p y_{rp}^{2u} \leq y_{rp}^{2u} \ , r \in R'',$$

$$\sum_{j=1}^{n} \lambda_j = 1 ,$$

$$\lambda_j \geq 0 \ , j = 1,\ldots,n.$$

Similar to model (5), model (6) is also feasible. One of the key concerns of the model (6) is its optimal solution which may exceed unity. Hence, the following definition can be used to assess the efficient units in evaluating with model (5) and model (6).

Definition1. the under evaluated unit DMU_p in model (5) and model (6) is called efficient if and only if $\theta^{*l} = \theta^{*u} = 1$.

It is worth to note that in two proposed models, the feasible region of model (5) and model (6) are same as the feasible region of radial BCC model. Hence, the efficiency measure yielded by model (5) and model (6) may lead to similar results yielded by radial classical DEA models for desirable and non-negative data. So, this is acceptable, solving model (5) yield the lower bound of efficiency measure θ^{*l} and model(6) leads to upper bound of efficiency , i.e., θ^{*u}. So, the interval efficiency $\theta^* = [\theta^{*l}, \theta^{*u}]$ can identify a measure foe each unit.

ILLUSTRATIVE EXAMPLE

In order to shed a light on applicability of the proposed approach, a real example of a private bank in IRAN is illustrated. Twenty branches of this bank is on our scope. The data set consist of three inputs and two outputs. It is worth to note that the data set are represented in interval format. The desirable and undesirable inputs are cost of personnel and administration (in billions of Rials) and long-term deposits, respectively. The profit and loss from deposits presents the third part of inputs and regarded as negative inputs. Outputs consists of desirable and undesirable outputs. Desirable outputs includes facilities and undesirable outputs depicts overdue claims. Both outputs are reported in billion Rials. Table 1 reports the data set.

Table 1: Data Set of Twenty Bank Branches

Branch	cost of personnel and administration	long-term deposits	profit and loss from deposits	facilities	overdue claims
1	[9139,10314]	[47018,48256]	[45,54]	[101986,111562]	[8,12]
2	[9211,9295]	[47008,47455]	[8,12]	[62755,69821]	[5,7]
3	[5344,5957]	[30179,32478]	[19,21]	[40128,43875]	[3,7]
4	[8001,8239]	[27598,28236]	[27,42]	[41589,42576]	[8,10]
5	[6014,6297]	[61489,62024]	[-23, -18]	[97584,106957]	[5,11]
6	[6329,7547]	[12098,12485]	[4,6]	[51534,5774]	[4,8]
7	[5198,5898]	[19386,21258]	[45,60]	[55741,65234]	[7,13]
8	[5299,6097]	[23567,26389]	[18,22]	[57788,63545]	[4,6]
9	[8485,9325]	[6835,7098]	[-6, -4]	[71324,73623]	[3,5]
10	[1401,2289]	[6284,6439]	[22,28]	[28798,33279]	[3,6]
11	[7658,7912]	[33981,35478]	[-46, -37]	[46623,49876]	[5,10]
12	[7289,7789]	[20963,22565]	[67,77]	[75984,85746]	[5,8]
13	[7821,8293]	[11326,12986]	[-70, -57]	[29378,32576]	[4,6]
14	[2598,3056]	[7169,7489]	[6,10]	[42371,46656]	[3,4]
15	[4120,4412]	[9896,10978]	[-17, -13]	[64594,65769]	[5,7]

16	[5002,5568]	[14855,16784]	[-11, -9]	[38665,40089]	[4,5]
17	[6999,7112]	[42757,43912]	[20,26]	[40138,45977]	[5,9]
18	[4709,4998]	[14564,15497]	[-40, -20]	[70674,81659]	[5,8]
19	[6298,6598]	[17456,18947]	[-30, -21]	[42003,44785]	[3,6]
20	[9989,10856]	[56675,57975]	[45,75]	[50123,60145]	[7,10]

As Table 1 shows the third input, profit and loss from deposits, in some branches has taken positive value and in other branches takes negative values. In order to evaluate the units from pessimistic point of view, consider unit#4 as an example. The values of desirable/ undesirable and negative inputs and desirable/undesirable outputs of this unit are employed as $x_{14}=8239,$ $x_{2'4}=27598,$ $x_{34}=42,$ $y_{14}=41589,$ $y_{2'4}=10$ in model (5). From optimistic point of view, the values for unit #4 are employed as $x_{14}=8001,$ $x_{2'4}=28236,$ $x_{34}=27,$ $y_{14}=42579,$ $y_{2'4}=8$ in model (6). Employing models (5) and (6) on the data set of Table 1, the results are reported in Table 2.

Table 2: The results of Model (5) and Model (6)

branch	θ^l	λ^*	θ^u	Reference set
١	0.٥٥٢٩	$\lambda_5^*=0.9325, \lambda_{10}^*=0.0675$	١	$\lambda_1^*=1$
٢	0.٥٩٠٣	$\lambda_3^*=0.3454, \lambda_5^*=0.5660$ $\lambda_{14}^*=0.0868, \lambda_{19}^*=0.0018$	١	$\lambda_2^*=1$
٣	0.٥٧٢٨	$\lambda_5^*=0.4264, \lambda_{10}^*=0.5367, \lambda_{14}^*=0.0369$	١	$\lambda_3^*=1$
٤	0.٣٨٣٢	$\lambda_5^*=0.3807, \lambda_{10}^*=0.6193$	0.٧٣٢٥	$\lambda_2^*=0.4007, \lambda_5^*=0.0939$ $\lambda_{14}^*=0.5053$
٥	١	$\lambda_5^*=1$	١	$\lambda_5^*=1$
٦	0.٤٨٠٢	$\lambda_5^*=0.0323, \lambda_{14}^*=0.4802$ $\lambda_{15}^*=0.2899, \lambda_{16}^*=0.1975$	١	$\lambda_{٦}^*=١$
٧	0.٤٧٦٠0	$\lambda_5^*=0.3049, \lambda_{10}^*=0.6951$	0.٩٤٣٣	$\lambda_5^*=0.3561, \lambda_9^*=0.1106,$ $\lambda_{14}^*=0.5333$
٨	0.٥٧٩٠	$\lambda_3^*=0.1352, \lambda_5^*=0.2369,$ $\lambda_{10}^*=0.2075, \lambda_{14}^*=0.4204$	١	$\lambda_8^*=1$
٩	0.٧٧٠٧	$\lambda_5^*=0.4269, \lambda_{19}^*=0.5731$	١	$\lambda_9^*=1$
١٠	١	$\lambda_{10}^*=1$	١	$\lambda_{10}^*=1$
١١	0.٧١٢٦	$\lambda_5^*=0.4042, \lambda_{13}^*=0.1291, \lambda_{18}^*=0.4667$	١	$\lambda_{11}^*=1$
١٢	0.٥٢٣١	$\lambda_5^*=0.5796, \lambda_{10}^*=0.4204$	١	$\lambda_{12}^*=1$
١٣	١	$\lambda_{13}^*=1$	١	$\lambda_{13}^*=1$

١٤	١	$\lambda_{14}^{*}=1$	١	$\lambda_{14}^{*}=1$
١٥	١	$\lambda_{15}^{*}=1$	١	$\lambda_{15}^{*}=1$
١٦	0.٨٧٣١	$\lambda_{5}^{*}=0.0264, \lambda_{15}^{*}=0.6562, \lambda_{19}^{*}=0.3173$	١	$\lambda_{16}^{*}=1$
١٧	0.٦١٢٢	$\lambda_{5}^{*}=0.6534, \lambda_{10}^{*}=0.3466$	١	$\lambda_{17}^{*}=1$
١٨	0.٩١٠٨	$\lambda_{5}^{*}=0.1509, \lambda_{13}^{*}=0.0395, \lambda_{15}^{*}=0.8095$	١	$\lambda_{١٨}^{*}=١$
١٩	0.٧٣٨٤	$\lambda_{5}^{*}=0.0344, \lambda_{15}^{*}=0.0826,$ $\lambda_{16}^{*}=0.5695, \lambda_{18}^{*}=0.3135$	١	$\lambda_{19}^{*}=1$
٢٠	0.٥١٣١	$\lambda_{5}^{*}=0.9038, \lambda_{10}^{*}=0.0962$	١	$\lambda_{20}^{*}=1$

As Table (2) reports the units3, 5, 10, 13, 14 and 15are evaluated as efficient in both pessimistic and optimistic point of view. In optimistic perspective, with reported by θ^{u} the inefficient units are recorded as units7 and 14. According to definition1, the rest of unit are regarded as efficient. The third and last column of Table (2) also depicts the reference set. As the axioms of DEA claims the reference set are represented with the optimal value of intensity variable λ^{*} in model (5) and model (6). In presence of negative input, units#5, 13 and 15 are evaluated as efficient. Unit#4 are evaluated as inefficient in pessimistic point of view. In order to improve the efficiency and according to third column of Table (2), that is to say, employing the optimal values of intensity variables, $\lambda_{5}^{*}=0.3807, \lambda_{10}^{*}=0.6193$ the inefficient unit #4 can be stated as the linear combination of these two efficient units and the results are presented as follows:

$$x_{14}=3243.1905,\ x_{2'4}=27600.2095,\ x_{34}=6.772,$$
$$y_{14}=61328.2146,\ y_{2'4}=3.7614.$$

On the other hand, from the optimistic point of view and regarding to the last column of Table(2), unit#4 can improve it efficiency if it can be presented as the linear combination of efficient units#2, 5 and 14. That is to say, the linear combination of the optimal values of $\lambda_{2}^{*}=0.4007$, $\lambda_{5}^{*}=0.0939$, $\lambda_{14}^{*}=0.5053$ may lead to the following quantities:

$$x_{14}=5859.9916,\ x_{2'4}=28232.4184,$$
$$x_{34}=7.7017,\ y_{14}=55719.1324,\ y_{2'4}=5.859$$

REFERENCES

Amirtiemoori, A., & Kordrostami, S. (2013). An alternative clustering approach: a DEA-based procedure. Optimization, 62(2), 227-240.

Asmild, M., & Rastor, J. T. (2010). Slack free MEA and RDM with comprehensive efficiency measures. Omega, 38(6), 475-483.

Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. Management science, 30(9), 1078-1092.

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. European journal of operational research, 2(6), 429-444.

Despotis, D. K., & Smirlis, Y. G. (2002). Data envelopment analysis with imprecise data. European Journal of Operational Research, 140(1), 24-36.

Emrouznejad, A., Anouz, A. L., & Thanassoulis, E. (2010). A semi-oriented radial measure for measuring the efficiency of decision making units with negative data, using DEA. European Journal of Operational Research, 200(1), 297-304.

Esmaeilzadeh, A., & Hadi-Vencheh, A. (2015). A new method for complete ranking of DMUs. Optimization, 64(5), 1177-1193.

Eyni, M., Tohidi, G., & Mehrabeian, S. (2017). Applying inverse DEA and cone constraint to sensitivity analysis of DMUs with undesirable inputs and outputs. Journal of the Operational Research Society, 68(1), 34-40.

Fare, R., & Grosskopf, S. (2003). Nonparametric productivity analysis with undesirable outputs: comment. American Journal of Agricultural Economics, 85(4), 1070-1074.

Fare, R., Grosskopf, S., Lovell, C. K., & Pasurka, C. (1989). Multilateral productivity comparisons when some outputs are undesirable: a nonparametric approach. The review of economics and statistics, 90-98.

Halkos, G., & Petrou, K. N. (2019). Treating undesirable outputs in DEA: A critical review. Economic Analysis and Policy, 62, 97-104.

Jahanshahloo, G. R., Lotfi, F. H., Malkhalifeh, M. R., & Namin, M. A. (2009). A generalized model for data envelopment analysis with interval data. Applied Mathematical Modelling, 33(7), 3237-3244.

Jahanshahloo, G. R., LOTFI, F. H., SHOJA, N., TOHIDI, G., & RAZAVYAN, S. (2005). Undesirable inputs and outputs in DEA models. Applied Mathematics and Computation, 169(2), 917-925.

Khalili-Damghani, K., Tavana, M., & Haji-Saami, E. (2015). A data envelopment analysis model with interval data and undesirable output for combined cycle power plant performance assessment. Expert Systems with Applications, 42(2), 760-773.

Kuosmanen, T., & Matin, R. K. (2011). Duality of weakly disposable technology. Omega, 39(5), 504-512.

Lotfi, F. H., Jahanshahloo, G. R., Namin, M. A., & Irvani, Z. (2007). A generalized model in presence undesirable factors. In International Mathematical Forum (Vol. 2, No. 9, pp. 403-412).

Piao, S. R., LI, J., & Ting, C. J. (2019). Assessing regional environmental efficiency in China with distinguishing weak and strong disposability of undesirable outputs. Journal of Cleaner Production, 227, 748-759.

Portela, M. C. A. S., Thanassoulis, E., & Simpson, G. (2004). A directional distance approach to deal with negative data in DEA: An application to bank branches. Journal of Operational Research Society, 55(10), 1111-1121.

Sahoo, B. K., Khoveyn, M., Eslami, R., & Chaudhury, P. (2016). Returns to scale and most productive scale size in DEA with negative data. European Journal of Operational Research, 255(2), 545-558.

Seiford, L. M., & Zhu, J. (2002). Modeling undesirable factors in efficiency evaluation. European journal of operational research, 142(1), 16-20.

Sharp, J. A., Meng, W., & Liu, W. (2007). A modified slacks-based measure model for data envelopment analysis with 'natural 'negative outputs and inputs. Journal of the Operational Research Society, 58(12), 1672-1677.

Shirazi, F., & Mohammadi, E. (2019). Evaluating efficiency of airlines: A new robust DEA approach with undesirable output. Research in Transportation Business & Management, 33, 100467.

Taher F, Daneshiyan B, Tohidi G, Lotfi F. H, Modarres F (2019). Estimating Undesirable Outputs and Desirable Inputs for Decision Making Units with Interval Data in Data Envelopment Analysis. Accept in Journal of the Operations Research Society of China (JORC).

Tao, X., Wang, P., & Zhu, B. (2016). Provincial green economic efficiency of China: A non-separable input–output SBM approach. Applied energy, 171, 58-66.

Yousefi, S., Alizadeh, A., Hayati, J., & Baghery, M. (2018). HSE risk prioritization using robust DEA-FMEA approach with undesirable outputs: a study of automotive parts industry in Iran. Safety science, 102, 144-158.

Zhou, Z., Xu, G., Wang, C., & Wu, J. (2019). Modeling undesirable output with a DEA approach based on an exponential transformation: An application to measure the energy efficiency of Chinese industry. Journal of Cleaner Production, 236, 117717.