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Merger Analysis Using Inverse DEA: the Case of Variable Returns to Scale

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Abstract

In a dynamic economy, mergers and consolidations of economic and finance sectors like banks, etc. becoming more common. One of the applications of the inverse DEA is the merger analysis of a series of production units. Data envelopment analysis (DEA) measures the efficiency score of a decision making unit (DMU) considering its input and output level. On the other side, the inverse DEA approach aims to find required input levels for DMU to produce a presumed output level preserving the efficiency score. In a rather recent paper, (Gattoufi, Amin, & Emrouznejad, 2014) introduced an interesting application of inverse DEA models for merger analysis. The current paper extends this work by developing a generalized inverse DEA model assuming variable returns to scale. In contrast with (Gattoufi et al., 2014), it is shown that proposed models are always feasible and bounded. The idea is illustrated using a methodical argument and two numerical examples. An empirical study of financial institutes shows the strength and the applicability of the proposed methods.

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INTRODUCTION

Risk is Data envelopment analysis (DEA) is a mathematical programming based approach for efficiency analysis of a group of decision making units (DMUs) proposed by (Charnes et al., 1978). An extension for considering variable returns to scale (VRS) was proposed by (Banker et al., 1984) that has been used in many applications. The strength of the former is letting the production space have different types of returns to scale, thus it has rather been more interested of users unless there exists specific prior information about returns to scale of the production. The traditional DEA model considered estimating the relative efficiency of DMUs based on observed input and output data. In a different path, the inverse DEA models keep relative efficiencies unchanged and then try to find (a) required input for a provided output (input oriented) or (b) producible output for a given level of input. It is assumed that the efficiency score is fixed and the aim is to find required input (output) levels for a given perturbed outputs (inputs) level. (Wei, Zhang, & Zhang, 2000) were inspired by inverse optimization and the work of (Zhang & Cui, 1999) to start this new path in DEA literature that yields inverse DEA models. Different extensions of this method have been done in the literature so far. (Jahanshahloo, Vencheh, Foroughi, & Matin, 2004) investigated the input and output estimation when some of the output is undesirable. (Hadi-Vencheh, Foroughi, & Modelling, 2006) uses inverse DEA models for generalized resource allocation problems. (Jahanshahloo, Soleimani-Damaneh, & Ghobadi, 2015) considered the inter-temporal dependence DEA model and proposed an inverse framework for this case. (Lertworasirikul, Charnsethikul, & Fang, 2011) considered the variable returns to scale (VRS) properties like (Banker et al., 1984) for the production technology in the inverse DEA problem. However, there exist some drawbacks in their model that were pointed out and revised by (Ghiyasi, 2015). (Ghiyasi, 2017) dealt with the cost and revenue efficient in the inverse literature. (Eyni, 2017) dealt with cone constraint inverse DEA modeling in the presence of undesirable output. (Ghiyasi & Zhu, 2020) dealt with the negative data in the inverse DEA modeling.

In a dynamic economy, mergers and consolidations of economic and finance sectors like banks, etc. becoming more common. One of the attractive applications of the DEA models is in the banking sector. We refer the readers to an interesting review of DEA models applied in the banking sectors by (Paradi & Zhu, 2013). A few research also used the DEA models for the merger analysis in the banking sectors. See for instance (Wheelock & Wilson, 2000) which studied the characteristics of U.S. individual banks to be acquired. (Sherman & Rupert, 2006) considered the merger issue for bank branches and analyzed the potential of avoiding the operational costs using merger analysis of branches. In another application, bi-level programming models were used by (Wu, Luo, Wang, & Birge, 2016) based on a leader-follower game model for the merger effects of banks. The bootstrap DEA approach was utilized by (Moradi-Motlagh & Babacan, 2015) for analysing the merger's impact on the Australian banks during the period of the financial crisis.

In an interesting application of the inverse DEA approach, (Gattoufi et al., 2014) utilized the inverse DEA models for merger analysis of the world bank. This recent contribution also is extended by different studies. (Amin & Al-Muharrami, 2016) proposed an inverse DEA model for merger analysis capable of dealing with negative data. (Amin & Boamah, 2020) dealt with the cost efficiency concept in the merger and inverse DEA analysis. (Amin, Al-Muharrami, & Toloo, 2019) combined goal programming and inverse DEA model for target setting and merge. (Amin & Boamah) proposed an inverse DEA model for estimating potential merger gain, considering cost efficiency measurement of DMUs. By this, they distinguished the technical and cost efficiency measure in the merger analysis using the inverse DEA based models.

The returns to scale is an important characteristic of the production technology and may affect the level of production for a different level of the input. This issue becomes more important when we deal with the merger problem. Consider the merger of two banks. If we assume the constant returns to scale for the production technology, then we face a merged bank that still operates in constant returns to scale region. However, one of the main issues of the merger analysis in the scale effect of the merger that should be considered in the analysis. This can be done using a production technology with variable returns to scale properties.

In the current paper, we show that we need more care when we deal with the merger analysis in case of the variable returns to scale. The current paper shows the models of (Gattoufi et al., 2014) may not be feasible in some situations. The main source of infeasibility is in fact due to the variable returns to scale properties of their model, not because of the inverse structure of the models. Although they categorized mergers as consequence feasible and infeasible mergers by minor and major consolidation in another recent paper (Amin et al., 2017), but the infeasibility of the merger in the variable returns to scale is still an important issue. In the current paper, problematic issues are described and the source of the obstacle is scrutinized and then some new models are proposed to extend the work of (Gattoufi et al., 2014) and tackle problematic issues. Proposed models are motivated and illustrated using mathematical arguments and two numerical examples. Moreover, we applied the proposed models for the merger analysis of financial institute in Iran. Section 2 reviews relative DEA and inverse DEA. Section 3 describes the existing problems in the inverse merger DEA model and then some new models are developed as an extension of existing models in the literature for overcoming the aforementioned existing problems.

DEA and INVERSE DEA

Suppose there are n DMUs that are using m inputs to produce p outputs. Let $x_{ij} > 0 \;\; 1 \le i \le m$ be i-th input of j-th unit and $y_{rj} > 0 \;\; 1 \le r \le p$ be r-th output of j-th unit, $j \in J = \{1, 2, ..., n\}$. The following DEA model measures the efficiency of DMU_o, that is, the DMU under evaluation

$$\begin{aligned}
&\theta_o = Min\ \theta \\
&s.t \\
&\sum_{J \in J} \lambda_j x_{ij} \le \theta x_{io}, \quad 1 \le i \le m \qquad (1)\\
&\sum_{J \in J} \lambda_j y_{rj} \ge y_{ro}, \quad 1 \le r \le s \\
&\sum_{J \in J} \lambda_j = 1 \\
&\lambda_j \ge 0, \qquad j \in J
\end{aligned}$$

In a different path, the following inverse DEA model finds the required input level Δx_i for producing a perturbed given level of output, preserving the relative efficiency of this DMU.

$$\begin{aligned}
&Min\ (\Delta x_1, \Delta x_2, ..., \Delta x_m) \\
&s.t \\
&\sum_{J \in J} \lambda_j x_{ij} \le \theta x_{io}, \quad 1 \le i \le m \qquad (2)\\
&\sum_{J \in J} \lambda_j y_{rj} \ge y_{ro}, \quad 1 \le r \le s \\
&\sum_{J \in J} \lambda_j = 1 \\
&\lambda_j \ge 0, \qquad j \in J
\end{aligned}$$

An interesting application of the inverse DEA model was proposed by (Gattoufi et al., 2014) for merger analysis. Considering DMU_k and DMU_l to be merged to a new DMU, let us call it DMU_M, the following model finds the minimum required input of merged DMUs for producing the aggregated output of the new DMU_M.

$$
\begin{aligned}
& Min\ \Delta x_{1k} + \Delta x_{1l} + \Delta x_{2k} + \Delta x_{2l} + \Delta x_{mk} + \Delta x_{ml} \\
& s.t. \\
& \sum_{j\in F\subseteq J} \lambda_j x_{ij} \le \bar{\theta}(\Delta x_{ik} + \Delta x_{il}),\ 1\le i\le m, \\
& \sum_{j\in F\subseteq J} \lambda_j y_{rj} \ge y_{rk} + y_{rl}, 1\le r\le s, \\
& \sum_{j\in F\subseteq J} \lambda_j = 1, \qquad (3) \\
& 0\le \Delta x_{ik} \le x_{ik}, \\
& 0\le \Delta x_{il} \le x_{il}, \\
& \lambda_j \ge 0,\ j\in F.
\end{aligned}
$$

, where F is the index set of remaining units in the market that can be either $F=\{j : j\in J,\ j\ne l\}$ or $F=\{j : j\in J,\ j\ne k, l\}$. In the former DMU_k remains in the market but in the latter setting, both DMU_k and DMU_l vanish after merging.

MOTIVATION and EXTENSION of The INVERSE DEA for MERGER ANALYSIS

In this section, we first describe and highlight the shortcoming of the merger model (Gattoufi et al., 2014) by mathematical argument and a numerical example and then propose a new model that tackles existing shortcoming.

Theoretical Motivation

As stated by (Gattoufi et al., 2014) model (3) is feasible in their theorem 1. They yielded this result by saying that (3) is bounded and its dual is feasible as follows:

$$
\begin{aligned}
& Max \sum_{r=1}^{s}(y_{rk}+y_{rl})u_r - \sum_{i=1}^{m}(p_i x_{ik} + q_i x_{il}) \\
& s.t., \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0,\ j\in F\subseteq J, \\
& \sum_{i=1}^{m} v_i x_{io} = 1, \\
& \bar{\theta} v_i - p_i \le 1, 1\le i\le m, \\
& \bar{\theta} v_i - q_i \le 1, 1\le i\le m, \\
& p_i, q_i, v_i \ge 0, 1\le i\le m, \\
& u_r \ge 0, 1\le r\le s.
\end{aligned}
$$

However, having linear programming bounded and its dual is feasible we cannot conclude that the primal is feasible. This ambiguity yields a wrong result in the theorem 1 of (Gattoufi et al., 2014). This problem is illustrated in the following example.

Numerical example 1. Consider four DMUs with one input and two outputs listed in Table 1.

Table 1: Input-output data of numerical example 1

Top of Form

	A	B	C	D
Input	7	6	9	5
First Output	4	8	11	5
Second Output	10	9	5	13

Consider DMU A and suppose DMU B aims to merge with DMU A, thus the aggregated output levels of the first and second output are 12 and 19 respectively. (Gattoufi et al., 2014) considered two cases. In The first case, both DMU_k and DMU_l disappear, that is, both DMU A and DMU B disappear, then the exiting DMUs are $F_1=\{DMU\ C, DMU\ D\}$. In the second case, only DMU_k remains, that is, DMU A remains and then the set of existing DMUs is $F_1=\{DMU\ A, DMU\ C, DMU\ D\}$. Considering the first case we yield the following linear programming

$$
\begin{aligned}
& Min \;\; \alpha_{1A} + \alpha_{1B} \\
& s.t. \\
& 9\lambda_3 + 5\lambda_4 + \bar{\theta}(\alpha_{1A} + \alpha_{1B}) \le 0, \\
& 11\lambda_3 + 5\lambda_4 \ge 12, \\
& 5\lambda_3 + 3\lambda_4 \ge 19, \\
& 0 \le \alpha_{1A} \le 7, \\
& 0 \le \alpha_{1B} \le 6, \\
& \lambda_3 + \lambda_4 = 1, \\
& \lambda_3, \lambda_4 \ge 0.
\end{aligned}
$$

And considering the second case, we get the following linear programming

$$
\begin{aligned}
& Min \;\; \alpha_{1A} + \alpha_{1B} \\
& s.t. \\
& 7\lambda_1 + 9\lambda_3 + 5\lambda_4 + \bar{\theta}(\alpha_{1A} + \alpha_{1B}) \le 0, \\
& 4\lambda_1 + 11\lambda_3 + 5\lambda_4 \ge 12, \\
& 10\lambda_1 + 5\lambda_3 + 3\lambda_4 \ge 19, \\
& 0 \le \alpha_{1A} \le 7, \\
& 0 \le \alpha_{1B} \le 6, \\
& \lambda_1 + \lambda_3 + \lambda_4 = 1, \\
& \lambda_1, \lambda_3, \lambda_4 \ge 0.
\end{aligned}
$$

A generalized inverse DEA model for merger analysis

The source of infeasibility in the model (3) is because even the classical DEA model considering VRS assumption may fail on merger analysis. In other words, classical DEA models, specifically the well-known BCC model with VRS assumption may be infeasible when assessing merged units. See (Ghiyasi, 2016) for more details. However, (Mojtaba Ghiyasi, 2016) proposed a new production set called semi-additive with VRS properties that does not face any problem in terms of infeasibility. The following model gauges the relative efficiency of DMU_o using semi-additive technology

$$
\begin{aligned}
& \theta_o = Min \;\theta \\
& s.t. \\
& \sum_{J \in J'} \lambda_j x_{ij} \le \theta x_{io}, \; 1 \le i \le m, \qquad (4) \\
& \sum_{J \in J'} \lambda_j y_{rj} \ge y_{ro}, \; 1 \le r \le s, \\
& \sum_{J \in J'} \lambda_j = 1, \\
& \lambda_j \ge 0, \; j \in J'.
\end{aligned}
$$

where J' is the index set of all aggregated, but not self-aggregated units. See (Ghiyasi, 2016) for more detail about the characteristics of semi-additive technology.

The above model is capable of measuring the efficiency of any aggregated (merged) unit for the VRS case, without any concern about the infeasibility problem. The following inverse DEA model for merger analysis is developed using semi-additive production technology and we make sure it has no problem in terms of infeasibility.

$$
\begin{aligned}
& Min \; \sum_{i=1}^{m} \Delta x_{ik} + \Delta x_{il} \\
& s.t. \\
& \sum_{j \in F' \subseteq J'} \lambda_j x_{ij} \le \bar{\theta}(\Delta x_{ik} + \Delta x_{il}), \; 1 \le i \le m, \qquad (5) \\
& \sum_{j \in F' \subseteq J'} \lambda_j y_{rj} \ge y_{rk} + y_{rl}, \; 1 \le r \le s, \\
& \sum_{j \in F' \subseteq J'} \lambda_j = 1, \\
& 0 \le \Delta x_{ik} \le x_{ik}, \\
& 0 \le \Delta x_{il} \le x_{il}, \\
& \lambda_j \ge 0, \; j \in F'
\end{aligned}
$$

where F' is the index set of reaming units that can be either $F' = \{j \in J', j \ne k, l\}$ (ignoring both DMU_k and DMU_l)or $F' = \{j \in J', j \ne l\}$ (keeping DMU_k).

Theorem 1. The linear programming model of (5) is always feasible and bounded.

Proof. For merger analysis of DMU_k and DMU_l and regardless of the selection of F' we know that the aggregated unit of DMU_k and DMU_l and its index exist in the index set of F'. Thus, considering $(\lambda, \theta_{kl}, \Delta x_k, \Delta x_l)$ is a feasible solution of model (5) such that $\lambda_j = \begin{cases} 1 & j = kl \\ 0 & OW \end{cases}$, $\begin{matrix} \Delta x_{ik} = x_{ik}, 1 \le i \le m \\ \Delta x_{il} = x_{il}, 1 \le i \le m \end{matrix}$ and θ_{kl} is the efficiency of aggregated DMU_M using model (4). This guarantees the feasibility of the model (5). The objective value of the aforementioned feasible solution, namely, $(\lambda, \theta_{kl}, \Delta x_k, \Delta x_l)$ is , that is, input summation of DMU_k and DMU_l .

Numerical example 2. Considering the same data set as numerical example 3, we take DMU A into consideration and merge DMU B with DMU A. the following model finds the required input level for this merger provided by a predefined efficiency score of $\bar{\theta}$.

$$
\begin{aligned}
& Min \;\; \alpha_{1A} + \alpha_{1B} \\
& s.t., \\
& 9\lambda_3 + 5\lambda_4 + 13\lambda_{12} + 16\lambda_{13} + 12\lambda_{14} + 15\lambda_{23} + 11\lambda_{24} \\
& +14\lambda_{34} + 22\lambda_{123} + 18\lambda_{124} + 20\lambda_{234} + 21\lambda_{134} \\
& +27\lambda_{1234} \; + \bar{\theta}(\alpha_{1A} + \alpha_{1B}) \le 0, \\
& 11\lambda_3 + 5\lambda_4 + 12\lambda_{12} + 15\lambda_{13} + 9\lambda_{14} + 19\lambda_{23} + 13\lambda_{24} \\
& +16\lambda_{34} + 23\lambda_{123} + 17\lambda_{124} + 24\lambda_{234} + 20\lambda_{134} \\
& +28\lambda_{1234} \ge 12, \\
& 5\lambda_3 + 3\lambda_4 + 19\lambda_{12} + 15\lambda_{13} + 23\lambda_{14} + 14\lambda_{23} + 22\lambda_{24} \\
& +18\lambda_{34} + 24\lambda_{123} + 32\lambda_{124} + 28\lambda_{234} + 27\lambda_{134} \\
& +37\lambda_{1234} \ge 19, \\
& 0 \le \alpha_{1A} \le 7, \\
& 0 \le \alpha_{1B} \le 6, \\
& \lambda_3 + \lambda_4 + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{23} + \lambda_{24} + \lambda_{34} + \lambda_{123} \\
& +\lambda_{124} + \lambda_{234} + \lambda_{134} + \lambda_{1234} = 1, \\
& \lambda_3, \lambda_4, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{23}, \lambda_{24}, \lambda_{34}, \lambda_{123}, \lambda_{124}, \lambda_{234}, \\
& \lambda_{134}, \lambda_{1234} \ge 0.
\end{aligned}
$$

The above model ignores both DMU A and DMU B. However, one may consider the case that DMU A stays in the market and for this case, we can use the following model.

$$
\begin{aligned}
& Min \;\; \alpha_{1A} + \alpha_{1B} \\
& s.t., \\
& 7\lambda_1 + 9\lambda_3 + 5\lambda_4 + 13\lambda_{12} + 16\lambda_{13} + 12\lambda_{14}, +15\lambda_{23} \\
& +11\lambda_{24} + 14\lambda_{34} + 22\lambda_{123} + 18\lambda_{124} + 20\lambda_{234} \\
& +21\lambda_{134} + 27\lambda_{1234} \; + \bar{\theta}(\alpha_{1A} + \alpha_{1B}) \le 0, \\
& 4\lambda_1 + 11\lambda_3 + 5\lambda_4 + 12\lambda_{12} + 15\lambda_{13} + 9\lambda_{14} + 19\lambda_{23} \\
& +13\lambda_{24} + 16\lambda_{34} + 23\lambda_{123} + 17\lambda_{124} + 24\lambda_{234} \\
& +20\lambda_{134} + 28\lambda_{1234} \ge 12, \\
& 10\lambda_1 + 5\lambda_3 + 3\lambda_4 + 19\lambda_{12} + 15\lambda_{13} + 23\lambda_{14} + 14\lambda_{23} \\
& +22\lambda_{24} + 18\lambda_{34} + 24\lambda_{123} + 32\lambda_{124} + 28\lambda_{234} + 27\lambda_{134} \\
& +37\lambda_{1234} \ge 19, \\
& 0 \le \alpha_{1A} \le 7, \\
& 0 \le \alpha_{1B} \le 6, \\
& \lambda_1 + \lambda_3 + \lambda_4 + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{23} + \lambda_{24} \\
& +\lambda_{34} + \lambda_{123} + \lambda_{124} + \lambda_{234} + \lambda_{134} + \lambda_{1234} = 1, \\
& \lambda_1, \lambda_3, \lambda_4, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{23}, \lambda_{24}, \lambda_{34}, \lambda_{123}, \\
& \lambda_{124}, \lambda_{234}, \lambda_{134}, \lambda_{1234} \ge 0.
\end{aligned}
$$

In the proposed model of (5), we used the full semi-additive technology for the inverse merger model. However, we can use partial semi-additive technology. Considering DMU_o for the merger analysis, we can only think through those aggregated units that include DMU_o. For this case, we just need to update the index set of J' to include observed DMU and aggregated DMUs that consist of DMU_o, in the proposed model of (5). We prevent rewriting associated models. For the sake of clarification, for the numerical example in this case we have the following models.

$$
\begin{aligned}
& Min \;\; \alpha_{1A} + \alpha_{1B} \\
& s.t., \\
& 9\lambda_3 + 5\lambda_4 + 13\lambda_{12} + 16\lambda_{13} + 12\lambda_{14} + 22\lambda_{123} + 18\lambda_{124} \\
& +21\lambda_{134} + 27\lambda_{1234} \; + \bar{\theta}(\alpha_{1A} + \alpha_{1B}) \le 0, \\
& 11\lambda_3 + 5\lambda_4 + 12\lambda_{12} + 15\lambda_{13} + 9\lambda_{14} + 23\lambda_{123} + 17\lambda_{124} \\
& +20\lambda_{134} + 28\lambda_{1234} \ge 12, \\
& 5\lambda_3 + 3\lambda_4 + 19\lambda_{12} + 15\lambda_{13} + 23\lambda_{14} + 24\lambda_{123} + 32\lambda_{124} \\
& +27\lambda_{134} + 37\lambda_{1234} \ge 19, \\
& 0 \le \alpha_{1A} \le 7, \\
& 0 \le \alpha_{1B} \le 6, \\
& \lambda_3 + \lambda_4 + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{123} + \lambda_{124} + \lambda_{134} + \lambda_{1234} = 1 \\
& \lambda_3, \lambda_4, \lambda_{12}, \lambda_{13}, \lambda_{14}\lambda_{123}, \lambda_{124}, \lambda_{134}, \lambda_{1234} \ge 0.
\end{aligned}
$$

$$
\begin{aligned}
& Min\ \ \alpha_{1A}+\alpha_{1B} \\
& s.t., \\
& 7\lambda_1+9\lambda_3+5\lambda_4+13\lambda_{12}+16\lambda_{13}+12\lambda_{14}+22\lambda_{123} \\
& +18\lambda_{124}+21\lambda_{134}+27\lambda_{1234}\ \ +\bar{\theta}(\alpha_{1A}+\alpha_{1B})\le 0, \\
& 4\lambda_1+11\lambda_3+5\lambda_4\ +12\lambda_{12}+15\lambda_{13}+9\lambda_{14}+23\lambda_{123} \\
& +17\lambda_{124}+20\lambda_{134}+28\lambda_{1234}\ge 12, \\
& 10\lambda_1+5\lambda_3+3\lambda_4\ +19\lambda_{12}+15\lambda_{13}+23\lambda_{14}+24\lambda_{123} \\
& +32\lambda_{124}+27\lambda_{134}+37\lambda_{1234}\ge 19, \\
& 0\le\alpha_{1A}\le 7, \\
& 0\le\alpha_{1B}\le 6, \\
& \lambda_1+\lambda_3+\lambda_4\ +\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{123}+\lambda_{124}+\lambda_{134} \\
& +\lambda_{1234}=1, \\
& \lambda_1,\lambda_3,\lambda_4,\lambda_{12},\lambda_{13},\lambda_{14}\lambda_{123},\lambda_{124},\lambda_{134},\lambda_{1234}\ge 0.
\end{aligned}
$$

Output estimation in the merger analysis

Taking the output oriented inverse DEA model for merger analysis for the case of VRS, we get the following *feasible* model for output estimation.

$$
\begin{aligned}
& Max\ \sum_{r=1}^{s}\Delta y_{rk}+\Delta y_{rl} \\
& s.t. \\
& \sum_{j\in F'\subseteq J'}\lambda_j x_{ij}\le x_{ik}+x_{il},\ 1\le i\le m, \qquad (6) \\
& \sum_{j\in F'\subseteq J'}\lambda_j y_{rj}\ge\bar{\varphi}(\Delta y_{rk}+\Delta y_{rl}),\ 1\le r\le s \\
& \sum_{j\in F'\subseteq J'}\lambda_j=1, \\
& \lambda_j\ge 0,\ j\in F'.
\end{aligned}
$$

where $\bar{\varphi}$ is the desired output efficiency level for the merged unit by DMU_k and DMU_l . The above model finds the maximum output level that can be produced using the sum of input level of DMU_k and DMU_l, that is, $x_{ik}+x_{il}$, $1\le i\le m$. The following model shows the feasibility of model (6) and it also shows that this model is bounded too.

Theorem 2. The linear programming model of (6) is always feasible and bounded.

Proof. It is similar to the proof of theorem 1 with some minor modifications.

An APPLICATION

In this section, we perform a performance assessment and then a merger analysis for 19 branches of a financial institute in Iran. Branches' area (m^2) and total cost (1000 Iranian Rial) are considered as inputs. On the other side, a number of transactions and deposited value (1000 Iranian Rial) are considered as outputs. Table 2 reports the statistical description of the data.

In the first part of the analysis, we gauge the efficiency measure of the branches using the model (4). The results are reported in the second column of Table 3 which shows only three efficient branches. However, the mean efficiency score is about 70 percent, considering all branches. This shows a 30 percent possibility of improvement in the system.

In the next run, we perform a merger analysis of branches. In this analysis, we consider the merger of those branches that are potentially possible in reality. We used the ideas of the institute's top managers at the province in this step. There are possibilities of having ten potential mergers based on the opinion of the managers. Then we performed the merger analysis for these branches using the proposed model (5) and the results are reported in table 4. The required inputs for producing the aggregated level of the merging branches with a given efficiency level are reported in the second and third columns of Table 4. Merging branches of B6 and B17 requires the highest area while merging branches of B11 and B17 suffers the highest cost. However, these poetical mergers switch their places when we look at the second highest cost, and area then we see B17 and B11. Therefore, these two mergers are the most resource-demanding in merger planning. It is important to note that model (3) falls into infeasibility for analysis of some potential mergers like B1-B10.

Table 2: Summary statistics of data

Variable name	Branches' area	Total cost	Number of transactions	Deposited value
Mean	211.7368421	3538.052632	55353.73684	193960.5263
Max	296	7303	93898	285459
Min	103	1209	16003	105069
Standard error	67.98052516	1757.384674	22055.06285	59334.74128

Table 3: Efficiency score of branches

Branches	Efficiency scores
B1	0.906890176
B2	0.456379301
B3	0.481927626
B4	0.69320566
B5	1
B6	0.825405847
B7	0.497660997
B8	0.807301362
B9	1
B10	0.615041543
B11	0.476851852
B12	0.911504425
B13	0.546131971
B14	0.437497773
B15	0.830645161
B16	0.881354412
B17	0.38576779
B18	1
B19	0.556756757

Table 4: The merger analysis

Merging branches	Required 1th input	Required 2th input	Expected efficiency
B1 & B10	180.7641	2523.7959	0.91
B5 & B12	193.6139	3220.2301	0.9036
B6 & B17	538.0289	6451.9006	0.5
B4 & B11	180.5104	3156.2816	0.9700
B7 & B15	328.0361	4361.7827	0.58
B10 & B19	355.5545	4325.99	0.6
B11 & B17	530.0199	6588.7803	0.35
B13 & B2	222.154	4415.0414	0.8

Table 5 reports the input share of each branch involved in the merger process. We see for instance in the merger plan of B1-B10, B4-B11, and B13-B2 that more efficient branches cover the less efficient peer branch. B1 has more efficiency level compared with B10, thus only B1 brings more share of the first input in the merger. We have the same finding for B4-B11 and B13-B2. however, this may not always be the case. In some merger plans, the less efficient branch should put more effort into the efficient ways of using resources. See for instance B7-B15, where B15 with the lower efficiency should bring more resources into the merger. These sort of resources that could have been wasted by B15 should be used more efficiently by merging with a more efficient branch.

Table 5: Input share of merging branches

Merging branches	The first input share of merging branches	The first input share of merging branches	The second input share of merging branches	The second input share of merging branches
B1 & B10	180.7641	0.0000	244.0000	2279.7959
B5 & B12	121.0000	72.6139	113.0000	3107.2301
B6 & B17	142.0000	396.0289	267.0000	6184.9006
B4 & B11	180.5104	0.0000	291.0000	2865.2816
B7 & B15	267.0000	61.0361	124.0000	4237.7827
B10 & B19	244.0000	111.5545	185.0000	4140.9900
B11 & B17	216.0000	314.0199	267.0000	6321.7803
B13 & B2	222.1540	0.0000	288.0000	4127.0414

CONCLUSION

This paper extended the merger inverse DEA models in the case of VRS for the production technology. Some problematic issues in the merger analysis using the inverse DEA models are pointed out. This highlights the importance of using the methodology in real-world problems. Then a generalized inverse DEA model for merger analysis is proposed that considers the VRS and it is capable of dealing with all merger analyses without any concern about infeasibility. Proposed models are illustrated using numerical examples and their applicability is shown in a real word problem.

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