Two-stage Stochastic Programming Problem with Fuzzy Random Variables

Suresh Kumar Barik[1]* and Sarbeswar Mohanty[2]

[1*]Shree Jagannath Mahavidylaya, Naugaonhat, Jagatsinghpur-754 113, Odisha, India

[2]School of Economics& Commerce, KIIT Deemed to be University,

Bhubaneswar - 751 024, India

Abstract

The objective of the present work is to develop a fuzzy random two-stage stochastic programming problem by considering some of the right hand side parameters of the constraints as fuzzy random variables with known fuzzy probability distributions along with known parameters as positive triangular fuzzy number. Both randomness and fuzziness are simultaneously considered in the present model. The recommended mathematical programming model cannot be solved directly due to presence of fuzzy-randomness in the model parameters. Therefore, the problem has been first transformed into a crisp equivalent nonlinear programming model by removing the fuzziness and randomness from the proposed model. Therefore, the crisp equivalent model is solved using the standard nonlinear programming technique. A numerical example is presented to demonstrate the usefulness of the suggested methodology.

Keywords: Stochastic programming; Fuzzy stochastic programming; Fuzzy two-stage stochastic non-linear programming; Fuzzy random variables; Triangular fuzzy number.

INTRODUCTION

Due to uncertain nature of the input parameters which are sometimes inexistent, scarce, subject to change and difficult to estimate, it is effortful to certainly notice all information of the in the mathematical programming model. From among several uncertainties, randomness and fuzziness play an important role in the real life decision making problems. In order to model these uncertainties, stochastic programming and fuzzy programming have been used to make decisions under the uncertainty conditions. Different types of stochastic programming and fuzzy programming models have been developed to suit the different purposes of management such as the expectation model (Liu and Liu, 2003), chance constrained programming (Charnes & Cooper, 1959), dependent chance constrained programming (Liu, 2009), fuzzy mathematical programming

*Correspondence E-mail: skb.math@gmail.com

models (Fullér & Zimmermann, 1993), the minimum risk problem (Liu and Liu, 2005), the fuzzy mathematical programming approach and the stochastic programming approach (Inuiguchi and Ramik, 2000). In these models, randomness and fuzziness are studied separately. However, in actual decision-making process, sometimes we make decision under a hybrid uncertain environment where fuzziness and randomness coexist. This type of situations can be handled by using fuzzy random variable introduced by Kwakernaak (1978) which is useful to deal with the two types of uncertainty at the same time.

Several research work have been reported by considering both fuzziness and randomness within a general optimization framework such as Li et al. (2006), Liu and Liu (2005), Luhandjula (1996) and Qiao et al. (1994). A survey of the essential elements, methods and algorithms has been carried out by Luhandjula (2006) for the class of linear programming problems involving fuzziness and randomness along with promising research directions. Nanda et al. (2006) proposed a new solution method for an optimization model involving fuzzy random variables. A mixed interval parameter fuzzy-stochastic robust programming model is developed by Cai et al. (2007) which is applied to the planning of solid waste management systems under uncertainty. Eshghi and Nematian (2008) discussed some mathematical programming models with fuzzy decision variables and fuzzy random coefficients. The applicability of optimization models involving fuzzy random variables is presented by Luhandjula and Gupta (1996). Luhandjula and Joubert (2010) presented some mathematical programming models in a hybrid uncertain environment. Zheng et al. (2010) gave a solution method for the multiobjective mathematical programming problem involving fuzzy random variables. A fuzzy stochastic two-stage programming approach is developed by Guo et al. (2010) for water resources management under uncertainty where the fuzzy random variable expressed as parameters' uncertainties with both stochastic and fuzzy characteristics. Liu (2007) presented a new class of fuzzy random optimization problem called two-stage fuzzy random programming or fuzzy random programming with recourse. Aiche et al. (2013) presented fuzzy stochastic programming problems with a crisp objective function and linear constraints whose coefficients are fuzzy random variables, in particular of type L-R. Chakraborty (2015) studied a posynomial geometric programming problem with the model parameters as fuzzy random variable coefficients. Yuan and Li (2017) proposed a new method for multi-attribute decision making problems with intuitionistic trapezoidal fuzzy random variable based on Mahalanobis-Taguchi Gram-Schmidt and evidence theory. Ke et al. (2017) proposed a hybrid multilevel programming model with uncertain random parameters based on expected value model and dependent-chance programming and solved by an approach integrating uncertain random simulations, Nash equilibrium searching approach and genetic algorithm. Osman et al. (2017) suggested a fuzzy goal programming approach for solving multi-level multi-objective quadratic fractional programming problem with fuzzy parameters in the constraints. Osman et al. (2018) addressed an interactive approach for solving multi-level multi-objective fractional programming problems with fuzzy parameters. Charles et al. (2019) examined a fuzzy goal programming methodology with solution procedures by considering the demand and supply-rooted uncertainty. Alipouri et al. (2020) recommended a project scheduling problem with mixed uncertainty of randomness and fuzziness in the resource constraint. Zhang et al. (2021) considered the allocation model of limited medical reserves during a public health emergency incorporating the uncertain demand and supplies at the health care centers which is formulated as a two-stage stochastic programming model aiming to minimize the total cost. Khalifa et al. (2021) proposed a two- stage stochastic programming for optimizing water resources management problem in pentagonal fuzzy neutrosophic environment. Ranarahu and Dash (2022) presented a mathematical model that combines the mathematical models of two-stage stochastic programming and chance-constrained programming under fuzzy environment. Liu et al. (2022) established a data-driven two-stage fuzzy random mixed integer optimization model by considering the uncertainty of transportation cost and customer demand in a hybrid uncertain environment with both randomness and fuzziness. Li et al. (2023) investigated a comprehensive production planning problem under uncertain demand which is modelled as a two-stage stochastic program assuming a risk-averse decision maker.

SOME PRELIMINARIES

In this section, we discuss some fundamental concept of fuzzy numbers and the concept of fuzzy probability by Buckley (2005) along with different types of fuzzy random variables.

Definition 1. ***(Fuzzy Number (*****Buckley, 2005*****))*** *Fuzzy number $\tilde{A}$ is a fuzzy set of the real line $\mathbb{R}$, with membership function $\mu_A: \mathbb{R} \rightarrow [0; 1]$, satisfying the following conditions:*

- *convex fuzzy set*
- *normalized fuzzy set*
- *its membership function is piecewise continuous*
- *It is defined in the real number*

Definition 2. ***(Triangular Fuzzy Number (*****Buckley, 2005*****))*** *A fuzzy number $\tilde{A} = \left(A^{(m)}/A^{(l)}/A^{(r)}\right)$ is said to be triangular if its membership function is strictly increasing in the interval $\left(A^{(l)}, A^{(m)}\right)$ and strictly decreasing in $\left(A^{(m)}, A^{(r)}\right)$ and $\mu_A\left(A^{(m)}\right) = 1$, where $A^{(m)}$ is core, $A^{(m)} - A^{(l)}$ is left spread and $A^{(r)} - A^{(m)}$ is right spread of the fuzzy number $\tilde{A}$.*

Definition 3. ***(α-cut (*****Buckley, 2005*****))*** *α-cut of the fuzzy number $\tilde{A}$ is the set $\{x|\mu_A(x) \geq \alpha\}$ for $0 < \alpha < 1$ and denoted by $\tilde{A}[\alpha]$. The α-cut of the fuzzy number is a closed and bounded interval generally denoted by $\tilde{A}[\alpha] = [A_*(\alpha), A^*(\alpha)]$ where $A_*(\alpha)$, $A^*(\alpha)$ are the increasing and decreasing functions of α respectively and $A_*(1) \leq A^*(1)$.*

For an example, in triangular fuzzy number $\tilde{A}$, $A_*(\alpha)$ is a continuous monotonically strictly increasing function of α and $A^*(\alpha)$ is a continuous monotonically strictly decreasing function of α satisfying $A_*(1) = A^*(1)$.

If $\tilde{A}$ =$(A_1/A_2/A_3)$ is a linear triangular fuzzy number then α-cut of $\tilde{A}$ can be expressed as $\tilde{A}[\alpha]$ = $[A_*(\alpha), A^*(\alpha)]$ = $[A_1 + (A_2 - A_1)\alpha, A_3 - (A_3 - A_2)\alpha]$.

Definition 4. ***(Positive Fuzzy Number (**Buckley, 2005**))*** *A fuzzy number $\tilde{A}$ is said to be positive if it's membership function $\mu_A(x)$=0; $\forall x \leq 0$. It may be stated as follow:*

Let $\tilde{A}[\alpha] = [A_*, A^*]$ be the α-cut of the fuzzy number $\tilde{A}$ for $0 < \alpha < 1$. $\tilde{A}$ is said to be positive if $A_* > 0$.

Definition 5. ***(Inequalities [(**Buckley, 2005**), (Nanda and Kar, 1992)])*** *Let $\tilde{A}$=$\left(A^{(m)}/A^{(l)}/A^{(r)}\right)$ and $\tilde{B}$ =$\left(B^{(m)}/B^{(p)}/B^{(o)}\right)$ be two fuzzy numbers with α-cut $\tilde{A}[\alpha] = [A_*, A^*]$ and $\tilde{B}[\alpha] =$* $[B_*, B^*]$, *respectively. then $\tilde{A} \preccurlyeq \tilde{B}$ iff $A^* \leq B_*$.*

Definition 6. ***(Fuzzy Random Variable [(**Buckley, 2005**), (**Buckley & Eslami, 2004**)])*** *A fuzzy random variable is a random variable whose parameter is a fuzzy number. Let $\tilde{X}$ be continuous random variable with fuzzy parameter $\tilde{\theta}$ and $\tilde{P}$ as fuzzy probability, then $\tilde{X}$ is said to be a continuous fuzzy random variable with density function $f\left(x, \tilde{\theta}\right)$.*

$\tilde{P}\left(\tilde{X} \leq x\right)$ =$\left\{\int_{-\infty}^{x} f\,(x, \theta)dx \mid \theta \in \tilde{\theta}[\alpha]\right\}$ =$\tilde{\beta}$, *where* $0 \leq \tilde{\beta} \leq 1, \tilde{\beta}$=$\left(\beta^{(m)}/\beta^{(l)}/\beta^{(r)}\right)$, $\beta^{(l)} \geq 0$ *and* $\beta^{(r)} \leq 1$

Definition 7. ***(Continuous Fuzzy Random Variable [(**Buckley, 2005**),9])*** *Let E= $[c, d]$ be an event. Then the probability of the event E of continuous fuzzy random variable $\tilde{X}$ is a fuzzy number whose α-cut is*

$$\tilde{P}\left(c \leq \tilde{X} \leq d\right)[\alpha] = \left(\min:\left\{\int_c^d f\,(x, \theta)dx \,\middle|\, \theta \in \tilde{\theta}[\alpha], \int_{-\infty}^{\infty} f\,(x, \theta)dx = 1\right\}, \max:\left\{\int_c^d f\,(x, \theta)dx \,\middle|\, \theta \in \tilde{\theta}[\alpha], \int_{-\infty}^{\infty} f\,(x, \theta)dx = 1\right\}\right) = (\beta_*[\alpha], \beta^*[\alpha])$$

Definition 8. ***(Mean and Variance of Continuous Fuzzy Random Variable (**Buckley, 2005**))*** *Mean and variance of a continuous fuzzy random variable $\tilde{X}$ with density function $f\left(x, \tilde{\theta}\right)$ are fuzzy numbers denoted by $M_x\left(\tilde{\theta}\right)$ and $V_x^2\left(\tilde{\theta}\right)$ whose α-cuts are given below:*

$$M_x\left(\tilde{\theta}\right)[\alpha]=\left\{\int_{-\infty}^{\infty} x\, f(x, \theta)dx \mid \theta \in \tilde{\theta}[\alpha], \int_{-\infty}^{\infty} f\,(x, \theta)dx = 1\right\}$$

$$V_x^2\left(\tilde{\theta}\right)[\alpha]=\left\{\int_{-\infty}^{\infty} \left(x - M_x(\theta)\right)^2 f(x, \theta)dx \mid \theta \in \tilde{\theta}[\alpha], M_x(\theta) \in M_x\left(\tilde{\theta}\right)[\alpha], \int_{-\infty}^{\infty} f\,(x, \theta)dx = 1\right\}$$

Variance of any fuzzy random variable is a positive fuzzy number i.e. $0 \notin V_x^2\left(\tilde{\theta}\right)[\alpha]$ *for* $\alpha \in$ $[0,1]$.

Definition 9. ***(Fuzzy uniform distribution (**Buckley & Eslami, 2004**))** Let X be an uniformly distributed random variable on $[a, b]$ with probability density function as:*

$$f(x; a, b) = \begin{cases} \frac{1}{b-a}, & a \le x \le b; \\ 0 & otherwise \end{cases}$$

If the random variable X takes the values in a domain where the end points are uncertain in nature, in particular if a and b are fuzzy numbers and denoted by $\tilde{a}$ and $\tilde{b}$, then the random variable X is said to be an uniformly distributed fuzzy random variable. It is denoted by $\tilde{X}$. Now the fuzzy probability of the uniform fuzzy random variable $\tilde{X}$ on the interval $[c, d]$ is a fuzzy number whose α-cut is

$\tilde{P}\left(c \le \tilde{X} \le d\right)[\alpha]$=$\left\{\int_c^d \frac{1}{b-a} dx \mid a \in \tilde{a}[\alpha], b \in \tilde{b}[\alpha], a < b\right\}$

Now the α-cut of mean and variance of the uniform fuzzy random variable are defined as:

$\widetilde{M}[\alpha]$=$\left\{\int_c^d x \frac{1}{b-a} dx \mid a \in \tilde{a}[\alpha], b \in \tilde{b}[\alpha], a < b\right\}$ *for all* $\alpha \in [0,1]$.

$\tilde{V}^2[\alpha]$=$\left\{\int_c^d (x - M)^2 \frac{1}{b-a} dx \mid M \in \widetilde{M}[\alpha], a \in \tilde{a}[\alpha], b \in \tilde{b}[\alpha], a < b\right\}$ *for all* $\alpha \in [0,1]$.

Definition 10. ***(Fuzzy exponential distribution (**Buckley & Eslami, 2004**))** Let X be exponentially distributed random variable with probability density function as:*

$$f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x}, & x > 0; \\ 0 & otherwise \end{cases}$$

If the parameter λ is a fuzzy number denoted by $\tilde{\lambda}$, then the random variable X is called exponential fuzzy random variable denoted by $\tilde{X}$. Now the fuzzy probability of the exponential fuzzy random variable $\tilde{X}$ on the interval $[c, d]$ with $c > 0$ is a fuzzy number whose α-cut is

$\tilde{P}\left(c \le \tilde{X} \le d\right)[\alpha]$=$\left\{\int_c^d \lambda e^{-\lambda x} dx \mid \lambda \in \tilde{\lambda}[\alpha]\right\}$ *for all* $\alpha \in [0,1]$.

Now the α-cut of mean and variance of the exponential fuzzy random variable are defined as:

$\widetilde{M}[\alpha]$=$\left\{\int_0^\infty x \frac{1}{b-a} dx \;\middle|\; \lambda e^{-\lambda x} dx \;\middle|\; \lambda \in \tilde{\lambda}[\alpha]\right\}$ *for all* $\alpha \in [0,1]$.

$\tilde{V}^2[\alpha]$=$\{\int_0^\infty (x - M)^2 \lambda e^{-\lambda x} dx \mid M \in \widetilde{M}[\alpha], \lambda \in \tilde{\lambda}[\alpha]\}$ *for all* $\alpha \in [0,1]$.

Definition 11. ***(Fuzzy normal distribution (**Buckley & Eslami, 2004**))** Let $\tilde{X}$= $N\left(\widetilde{M}, \tilde{V}^2\right)$ be a normal fuzzy random variable with fuzzy mean $\widetilde{M}$ and fuzzy variance $\tilde{V}^2$ as fuzzy parameters. Now crisp normal random variable X= $N(M, V^2)$ with mean M and variance V^2 with probability density function $f(x, M, V^2)$ be*

$$f(x, M, V^2) = \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{1}{2}\left(\frac{x-M}{V}\right)^2}, -\infty < x < \infty, -\infty < M < \infty, V^2 > 0$$

Now the fuzzy probability of $\tilde{X} = N\left(\tilde{M}, \tilde{V}^2\right)$ *defined on the interval* $[c, d]$ *is a fuzzy number with* α*-cut as:*

$$\tilde{P}\left(c \leq \tilde{X} \leq d\right)[\alpha] = \left\{\frac{1}{\sqrt{2\pi V^2}} \int_c^d e^{-\frac{1}{2}\left(\frac{x-M}{V}\right)^2} dx \mid M \in \tilde{M}[\alpha], V^2 \in \widetilde{V^2}[\alpha], \frac{1}{\sqrt{2\pi V^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-M}{V}\right)^2} dx = 1\right\} \text{ for all } \alpha \in [0,1].$$

$$= \left\{\frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{z^2}{2}} dz \mid M \in \tilde{M}[\alpha], V^2 \in \widetilde{V^2}[\alpha], \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dx = 1\right\}$$

where $z = \frac{x-M}{V}$, $z_1 = \frac{c-M}{V}$, *and* $z_1 = \frac{d-M}{V}$ *for all* $\alpha \in [0,1]$.

Now the α*-cut of mean and variance of the normal fuzzy random variable are defined as:*

$$\tilde{M}[\alpha] = \left\{\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{1}{2}\left(\frac{x-M}{V}\right)^2} dx \mid M \in \tilde{M}[\alpha], V^2 \in \widetilde{V^2}[\alpha]\right\} \text{ for all } \alpha \in [0,1].$$

$$\tilde{V}^2[\alpha] = \left\{\int_{-\infty}^{\infty} (x-M)^2 \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{1}{2}\left(\frac{x-M}{V}\right)^2} dx \mid M \in \tilde{M}[\alpha], V^2 \in \widetilde{V^2}[\alpha]\right\} \text{ for all } \alpha \in [0,1].$$

FUZZY STOCHASTIC PROGRAMMING PROBLEM

The occurrence of randomness in the model parameters can be formulated as stochastic programming (SP) within a general optimization framework. Due to the robustness, SP is widely used in many real-world decision making problems of management science, engineering, and technology. Also, it has been applied to various areas such as, manufacturing product and capacity planning, electrical generation capacity planning, financial planning and control, supply chain management, airline planning (fleet assignment), water resource modeling, forestry planning, dairy farm expansion planning, macroeconomic modeling and planning, portfolio selection, traffic management, asset liability management, etc.

Mathematically, a stochastic programming problem can be stated as (Birge and Louveaux ,1997):

$$\min: z = \sum_{j=1}^{n} c_j x_j \quad (1)$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1,2, \ldots, m \quad (2)$$

$$\sum_{j=1}^{n} r_{sj}\, x_j \geq h_s, s = 1,2, \dots, l \quad (3)$$

$$x_j \geq 0, j = 1,2, \dots, n \quad (4)$$

where $x_j, j = 1,2, \dots, n$ are the decision variables, $a_{ij}, i = 1,2, \dots, m, j = 1,2, \dots, n$ are the coefficients of the technological matrix, $c_j, j = 1,2, \dots, n$ are the coefficients associated with the objective function. Only the right hand side parameters $b_i, i = 1,2, \dots, m$ are considered as random variables with known mean and variance.

When some of the model input parameters are considered as fuzzy random variables with known fuzzy parameters, then the above model (1)-(4) is known as fuzzy stochastic programming problem which can be stated as:

$$\text{min}: z = \sum_{j=1}^{n} c_j\, x_j \quad (5)$$

subject to

$$\sum_{j=1}^{n} a_{ij}\, x_j \leq \tilde{b}_i, i = 1,2, \dots, m \quad (6)$$

$$\sum_{j=1}^{n} r_{sj}\, x_j \geq h_s, s = 1,2, \dots, l \quad (7)$$

$$x_j \geq 0, j = 1,2, \dots, n \quad (8)$$

where the right hand side parameters $\tilde{b}_i, i = 1,2, \dots, m$ are considered as fuzzy random variables with known fuzzy parameters. All other parameters are deterministic in the problem.

Fuzzy Two-stage Stochastic Programming Problem

Two-stage stochastic programming (TSP) is an efficient method for solving stochastic programming problems with recourse. In the standard TSP paradigm, the decision variables of an optimization problem under uncertainty are partitioned into two sets. The first stage decision variables are those that have to be decided before the actual realization of the uncertain parameters. Afterward, once the random events have exhibited themselves, further decision can be made by selecting, at a certain cost, the values of the second-stage, or recourse, variables i.e. a second-stage decision variable can be made to minimize "penalties" that may appear due to any infeasibility [(Sahinidis, 2004), (Walkup and Wets, 1967)]. The formulation of two-stage stochastic programming problems was first introduced by Dantzig (1955). Further it was developed by Beale (1955) and Dantzig and Madansky (1961). Barik et al. (2012, 2013, 2014) developed several methodologies in this directions.

Mathematically, a two-stage stochastic programming problem with simple recourse can be stated as:

$$\text{min}: \bar{z} = \sum_{j=1}^{n} c_j\, x_j + E\left(\sum_{i=1}^{m} p_i\, |y_i|\right) \quad (9)$$

subject to

$$y_i = b_i - \sum_{j=1}^{n} a_{ij}\, x_j, i = 1,2, \dots, m \quad (10)$$

$$\sum_{j=1}^{n} r_{sj}\, x_j \geq h_s, s = 1,2, \dots, l \quad (11)$$

$$x_j \geq 0, i = 1,2, \dots, m, j = 1,2, \dots, n \quad (12)$$

where it is assumed that the first stage decision variables x_j, $j = 1,2, \dots, n$ and second stage decision variables y_i, $i = 1,2, \dots, m$ are deterministic in the problem, p_i , $i = 1,2, \dots, m$ are the penalty cost associated with the discrepancy between $\sum_{j=1}^{n} a_{ij}$ and b_i and E is used to represent the expected value associated with the random variables $b_i, i = 1,2, \dots, m$.

When some of the input parameters of the model are considered as fuzzy random variables with known fuzzy parameters, then the model can be formulated as fuzzy two-stage stochastic programming model. It can be stated as:

$$\min: \bar{z} = \sum_{j=1}^{n} c_j\, x_j + \tilde{E}\left(\sum_{i=1}^{m} p_i\, |\tilde{y}_i|\right) \quad (13)$$

subject to

$$\tilde{y}_i = \tilde{b}_i - \sum_{j=1}^{n} a_{ij}\, x_j, i = 1,2, \dots, m \quad (14)$$

$$\sum_{j=1}^{n} r_{sj}\, x_j \geq h_s, s = 1,2, \dots, l \quad (15)$$

$$x_j \geq 0, i = 1,2, \dots, m, j = 1,2, \dots, n \quad (16)$$

where $\tilde{b}_i, i = 1,2, \dots, m$ are considered as fuzzy random variables with known fuzzy parameters. All other parameters are considered as deterministic in the problem.

Now, the equivalent deterministic models of the fuzzy two-stage stochastic programming problem when the right hand side parameter b_i follows different fuzzy distribution can be established as follows:

Case-I: b_i Follows Fuzzy Uniform Distribution

Assume that $\tilde{b}_i$, $i = 1,2, \dots, m$ are fuzzy uniform random variables defined on $[\tilde{L}_i, \tilde{U}_i]$ where $\tilde{L}_i$ and $\tilde{U}_i$ are positive fuzzy numbers with α-cuts are given as:

$\tilde{L}_i[\alpha]$=$[L_{i*}(\alpha), L_i^*(\alpha)]$ and $\tilde{U}_i[\alpha]$=$[U_{i*}(\alpha), U_i^*(\alpha)]$

where $L_{i*}(\alpha)$ is the minimum value and $L_i^*(\alpha)$ is the maximum value of $\tilde{L}_i[\alpha]$. Similarly, $U_{i*}(\alpha)$ is the minimum value and $U_i^*(\alpha)$ is the maximum value of $\tilde{U}_i[\alpha]$.

Now, to compute $\tilde{E}(p_i|\tilde{y}_i|)$ = $p_i\tilde{E}\left(\left|\tilde{b}_i - g_i\right|\right), i = 1,2, \dots, m$ where $g_i = \sum_{j=1}^{n} a_{ij}\, x_j, i = 1,2, \dots, m$ and $g_i \geq 0$, we have find the α-cut of $E\left(\left|\tilde{b}_i - g_i\right|\right)$ as:

$$\tilde{E}\left(\left|\tilde{b}_i - g_i\right|\right)[\alpha] = \left\{\int_{L_i}^{U_i} |b_i - g_i| \frac{1}{U_i - L_i} db_i \mid L_i \in \tilde{L}_i[\alpha], U_i \in \widetilde{U}_i[\alpha], i = 1,2,\dots,m\right\} \qquad (17)$$

Integrating (3.17), we get

$$\tilde{E}\left(\left|\tilde{b}_i - g_i\right|\right)[\alpha]$$

$$= \left\{\frac{1}{U_i - L_i}\left[\frac{U_i^2 + L_i^2}{2} + g_i^2 - g_i(U_i + L_i)\right] \mid L_i \in \tilde{L}_i[\alpha], U_i \in \widetilde{U}_i[\alpha], i = 1,2,\dots,m\right\}$$

$$= (T_{i*}[\alpha], T_i^*[\alpha]), i = 1,2,\dots,m \qquad (18)$$

where

$$T_{i*}[\alpha] = \min\left\{\frac{1}{U_i - L_i}\left[\frac{U_i^2 + L_i^2}{2} + g_i^2 - g_i(U_i + L_i)\right] \mid L_i \in \tilde{L}_i[\alpha], U_i \in \widetilde{U}_i[\alpha], i = 1,2,\dots,m\right\}$$

and

$$T_i^*[\alpha] = \max\left\{\frac{1}{U_i - L_i}\left[\frac{U_i^2 + L_i^2}{2} + g_i^2 - g_i(U_i + L_i)\right] \mid L_i \in \tilde{L}_i[\alpha], U_i \in \widetilde{U}_i[\alpha], i = 1,2,\dots,m\right\}$$

for all $\alpha \in [0,1]$.

The function $\frac{1}{U_i - L_i}\left[\frac{U_i^2 + L_i^2}{2} + g_i^2 - g_i(U_i + L_i)\right]$ is an increasing function. So the minimum value occurs at $U_i = U_{i*}(\alpha)$ and $L_i = L_{i*}(\alpha)$. Then the minimum value of the function is

$$\frac{1}{U_{i*}(\alpha) - L_{i*}(\alpha)}\left[\frac{U_{i*}(\alpha)^2 + L_{i*}(\alpha)^2}{2} + g_i^2 - g_i\left(U_{i*}(\alpha) + L_{i*}(\alpha)\right)\right]$$

Hence,

$$\tilde{E}\left(\sum_{i=1}^{m} p_i\, |\tilde{y}_i|\right)[\alpha] = \sum_{i=1}^{m} p_i\, (T_{i*}[\alpha], T_i^*[\alpha])$$

$$= \Big(\sum_{i=1}^{m} p_i \frac{1}{U_{i*}(\alpha) - L_{i*}(\alpha)}\left[\frac{U_{i*}(\alpha)^2 + L_{i*}(\alpha)^2}{2} + g_i^2 - g_i\left(U_{i*}(\alpha) + L_{i*}(\alpha)\right)\right],$$
$$\frac{1}{U_i^*(\alpha) - L_i^*(\alpha)}\left[\frac{U_i^*(\alpha)^2 + L_i^*(\alpha)^2}{2} + g_i^2 - g_i\left(U_i^*(\alpha) + L_i^*(\alpha)\right)\right][\alpha]\Big) \qquad (19)$$

Using (19) in the fuzzy two-stage stochastic programming model (13) -(16), we establish the deterministic crisp model as:

$$\min: \bar{z} = \sum_{j=1}^{n} c_j\, x_j + \sum_{i=1}^{m} p_i \frac{1}{U_{i*}(\alpha) - L_{i*}(\alpha)}\left[\frac{U_{i*}(\alpha)^2 + L_{i*}(\alpha)^2}{2} + g_i^2 - g_i\left(U_{i*}(\alpha) + L_{i*}(\alpha)\right)\right] \ \forall\, \alpha \in [0,1] \qquad (20)$$

subject to

$\sum_{j=1}^{n} r_{sj}\, x_j \geq h_s, s = 1,2, \ldots, l$ (21)

$x_j \geq 0, j = 1,2, \ldots, n$ (22)

where $g_i = \sum_{j=1}^{n} a_{ij}\, x_j, i = 1,2, \ldots, m$

Case-II: b_i Follows Fuzzy Exponential Distribution

Assume that $\tilde{b}_i$, $i = 1,2, \ldots, m$ are fuzzy exponential random variables with fuzzy parameters $\tilde{\lambda}_i$ positive fuzzy numbers with α-cuts are given as:

$\tilde{\lambda}_i[\alpha]$=$[\lambda_{i*}(\alpha), \lambda_i^*(\alpha)]$

where $\lambda_{i*}(\alpha)$ is the minimum value and $\lambda_i^*(\alpha)$ is the maximum value of $\tilde{L}_i[\alpha]$.

Now, to compute $\tilde{E}(p_i|\tilde{y}_i|) = p_i\tilde{E}\left(\left|\tilde{b}_i - g_i\right|\right), i = 1,2, \ldots, m$ where $g_i = \sum_{j=1}^{n} a_{ij}\, x_j, i = 1,2, \ldots, m$ and $g_i \geq 0$, we have find the α-cut of $E\left(\left|\tilde{b}_i - g_i\right|\right)$ as:

$E\left(\left|\tilde{b}_i - g_i\right|\right)[\alpha] = \left\{\int_0^\infty |b_i - g_i|\, \lambda_i e^{-\lambda_i b_i} db_i \mid \lambda_i \in \tilde{\lambda}_i[\alpha], i = 1,2, \ldots, m\right\}$ (23)

Integrating (3.23), we get

$$E\left(\left|\tilde{b}_i - g_i\right|\right)[\alpha] = \left\{\left(\frac{2}{\lambda_i}\right) e^{-\lambda_i g_i} + g_i - \frac{1}{\lambda_i} \mid \lambda_i \in \tilde{\lambda}_i[\alpha], i = 1,2, \ldots, m\right\}$$

$= (T_{i*}[\alpha], T_i^*[\alpha]), i = 1,2, \ldots, m$ (24)

where

$$T_{i*}[\alpha] = \min\left\{\left(\frac{2}{\lambda_i}\right) e^{-\lambda_i g_i} + g_i - \frac{1}{\lambda_i} \mid \lambda_i \in \tilde{\lambda}_i[\alpha], i = 1,2, \ldots, m\right\}$$

and

$$T_i^*[\alpha] = \max\left\{\left(\frac{2}{\lambda_i}\right) e^{-\lambda_i g_i} + g_i - \frac{1}{\lambda_i} \mid \lambda_i \in \tilde{\lambda}_i[\alpha], i = 1,2, \ldots, m\right\}$$

for all $\alpha \in [0,1]$.

The function $\left(\frac{2}{\lambda_i}\right) e^{-\lambda_i g_i} + g_i - \frac{1}{\lambda_i}$ is a decreasing function. So the minimum value occurs at $\lambda_i = \lambda_i^*(\alpha)$. Then the minimum value of the function is

$$\left(\frac{2}{\lambda_i^*(\alpha)}\right) e^{-\lambda_i^*(\alpha) g_i} + g_i - \frac{1}{\lambda_i^*(\alpha)}$$

Hence,

$$E(\sum_{i=1}^{m} p_i\,|y_i|) = \left[\left(\frac{2}{\lambda_i^*(\alpha)}\right) e^{-\lambda_i^*(\alpha) g_i} + g_i - \frac{1}{\lambda_i^*(\alpha)}, \left(\frac{2}{\lambda_{i*}(\alpha)}\right) e^{-\lambda_{i*}(\alpha) g_i} + g_i - \frac{1}{\lambda_{i*}(\alpha)}\right] \tag{25}$$

Using (25) in the fuzzy two-stage stochastic programming model (13) -(16), we establish the deterministic model as:

$$\min: \bar{z} = \sum_{j=1}^{n} c_j\, x_j + \sum_{i=1}^{m} p_i \left(\frac{2}{\lambda_i^*(\alpha)}\right) e^{-\lambda_i^*(\alpha) g_i} + g_i - \frac{1}{\lambda_i^*(\alpha)} \tag{26}$$

subject to

$$\sum_{j=1}^{n} r_{sj}\, x_j \geq h_s, s = 1,2, \dots, l \tag{27}$$

$$x_j \geq 0, j = 1,2, \dots, n \tag{28}$$

where $g_i = \sum_{j=1}^{n} a_{ij}\, x_j, i = 1,2, \dots, m$

Case-III: b_i Follows Fuzzy Normal Distribution

Assumed that b_i, $i = 1,2, \dots, m$ are fuzzy normal random variables with fuzzy mean and fuzzy variance as

$\tilde{E}(\tilde{b}_i) = \tilde{\mu}_i$ and $\widetilde{Var}(\tilde{b}_i) = \tilde{\sigma}_i^2$, $i = 1,2, \dots, m$

Now, to compute $\tilde{E}(p_i|\tilde{y}_i|) = p_i\tilde{E}(|\tilde{b}_i - g_i|), i = 1,2, \dots, m$ where $g_i = \sum_{j=1}^{n} a_{ij}\, x_j, i = 1,2, \dots, m$ and $g_i \geq 0$, we have find the α-cut of $E(|\tilde{b}_i - g_i|)$ as:

$$E(|\tilde{b}_i - g_i|)[\alpha]$$

$$= \left\{ \int_{-\infty}^{\infty} |b_i - g_i| \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{1}{2}\left(\frac{b_i - \mu_i}{\sigma_i}\right)^2} db_i \mid \mu_i \in \tilde{\mu}_i[\alpha], \sigma_i \in \tilde{\sigma}_i[\alpha], i = 1,2, \dots, m \right\} \tag{29}$$

Integrating (3.29), we get

$$E(|\tilde{b}_i - g_i|)[\alpha]$$

$$= \left\{ \mu_i - g_i + 2(g_i - \mu_i)\Phi\left(\frac{g_i - \mu_i}{\sigma_i}\right) + \frac{2\sigma_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{g_i - \mu_i}{\sigma_i}\right)^2} \mid \mu_i \in \tilde{\mu}_i[\alpha], \sigma_i \in \tilde{\sigma}_i[\alpha], i = 1,2, \dots, m \right\}$$

$$= (T_{i*}[\alpha], T_i^*[\alpha]), i = 1,2, \dots, m \tag{30}$$

where

$$T_{i*}[\alpha] = \min\left\{\mu_i - g_i + 2(g_i - \mu_i)\Phi\left(\frac{g_i - \mu_i}{\sigma_i}\right) + \frac{2\sigma_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{g_i-\mu_i}{\sigma_i}\right)^2} \mid \mu_i \in \tilde{\mu}_i[\alpha], \sigma_i \in \tilde{\sigma}_i[\alpha], i = 1,2,\dots,m\right\}$$

and

$$T_i^*[\alpha] = \max\left\{\mu_i - g_i + 2(g_i - \mu_i)\Phi\left(\frac{g_i - \mu_i}{\sigma_i}\right) + \frac{2\sigma_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{g_i-\mu_i}{\sigma_i}\right)^2} \mid \mu_i \in \tilde{\mu}_i[\alpha], \sigma_i \in \tilde{\sigma}_i[\alpha], i = 1,2,\dots,m\right\}$$

for all $\alpha \in [0,1]$.

The function $\mu_i - g_i + 2(g_i - \mu_i)\Phi\left(\frac{g_i-\mu_i}{\sigma_i}\right) + \frac{2\sigma_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{g_i-\mu_i}{\sigma_i}\right)^2}$ is an increasing function. So the minimum value occurs at $\mu_i = \mu_{i*}(\alpha)$ and $\sigma_i^2 \in \sigma_{i*}^2[\alpha]$. Then the minimum value of the function is

$$\mu_{i*}(\alpha) - g_i + 2\big(g_i - \mu_{i*}(\alpha)\big)\Phi\left(\frac{g_i - \mu_{i*}(\alpha)}{\sigma_{i*}(\alpha)}\right) + \frac{2\sigma_{i*}(\alpha)}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{g_i-\mu_{i*}(\alpha)}{\sigma_{i*}(\alpha)}\right)^2}$$

Hence,

$$\tilde{E}\left(\sum_{i=1}^m p_i\,|\tilde{y}_i|\right) = \sum_{i=1}^m p_i\left[\mu_{i*}(\alpha) - g_i + 2\big(g_i - \mu_{i*}(\alpha)\big)\Phi\left(\frac{g_i-\mu_{i*}(\alpha)}{\sigma_{i*}(\alpha)}\right) + \frac{2\sigma_{i*}^2[\alpha]}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{g_i-\mu_{i*}(\alpha)}{\sigma_{i*}(\alpha)}\right)^2}\right] \tag{31}$$

Using (31) in the two-stage stochastic programming model (13) -(16), we establish the deterministic crisp model as:

$$\min: \bar{z} = \sum_{j=1}^n c_j\,x_j + \sum_{i=1}^m p_i\left[\mu_{i*}(\alpha) - g_i + 2\big(g_i - \mu_{i*}(\alpha)\big)\Phi\left(\frac{g_i-\mu_{i*}(\alpha)}{\sigma_{i*}[\alpha]}\right) + \frac{2\sigma_{i*}^2[\alpha]}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{g_i-\mu_{i*}(\alpha)}{\sigma_{i*}(\alpha)}\right)^2}\right] \tag{32}$$

subject to

$$\sum_{j=1}^n r_{sj}\,x_j \ge h_s, s = 1,2,\dots,l \tag{33}$$

$$x_j \geq 0, j = 1,2, \dots, n \quad (34)$$

where $g_i = \sum_{j=1}^{n} a_{ij}\, x_j, i = 1,2, \dots, m$

NUMERICAL EXAMPLE

For the numerical example, consider the fuzzy stochastic programming model as:

$$\min: \bar{z} = 12x_1 + 15x_2 + 14x_3 \quad (35)$$

Subject to

$$3x_1 + 5x_2 + 5x_3 \leq \tilde{b}_1 \quad (36)$$

$$7x_1 + 3x_2 + 4x_3 \leq \tilde{b}_2 \quad (37)$$

$$5x_1 + 7x_2 + 6x_3 \leq \tilde{b}_3 \quad (38)$$

$$9x_1 + 3x_2 + 7x_3 \leq 25 \quad (39)$$

$$7x_1 + 5x_2 + 3x_3 \geq 18 \quad (40)$$

$$x_1, x_2, x_3 \geq 0 \quad (41)$$

where $\tilde{b}_1$, $\tilde{b}_2$, and $\tilde{b}_3$ are fuzzy uniform, fuzzy exponential, and fuzzy normal random variables, respectively with parameters as positive triangular fuzzy number.

Further, using the above model (35)-(41) a fuzzy two-stage stochastic programming model with simple unit recourse cost (i.e. p_i=1, $\forall i$) is stated as:

$$\min: \bar{z} = 12x_1 + 15x_2 + 14x_3 + \tilde{E}(|\tilde{y}_1|) + E(|\tilde{y}_2|) + E(|\tilde{y}_3|) \quad (42)$$

Subject to

$$\tilde{y}_1 = \tilde{b}_1 - (3x_1 + 5x_2 + 5x_3) \quad (43)$$

$$\tilde{y}_2 = \tilde{b}_2 - (7x_1 + 3x_2 + 4x_3) \quad (44)$$

$$\tilde{y}_3 = \tilde{b}_3 - (5x_1 + 7x_2 + 6x_3) \quad (45)$$

$$9x_1 + 3x_2 + 7x_3 \leq 25 \quad (46)$$

$$7x_1 + 5x_2 + 3x_3 \geq 18 \quad (47)$$

$$x_1, x_2, x_3, \geq 0 \quad (48)$$

where $\tilde{b}_1$, $\tilde{b}_2$, and $\tilde{b}_3$ are fuzzy uniform, fuzzy exponential, and fuzzy normal random variables, respectively with parameters as positive triangular fuzzy number given as:

$[\tilde{L}_i, \tilde{U}_i]$=$[\widetilde{16}, \widetilde{20}]$, $\tilde{\lambda}_i$=$\widetilde{0.5}$, and $\tilde{\mu}_i$=$\widetilde{15}$, $\tilde{\sigma}_i$=$\tilde{4}$

i.e. $\widetilde{16}$=(14/16/18), $\widetilde{20}$=(17/20/22), $\widetilde{0.5}$=(0.2/0.5/0.6), $\widetilde{10}$=(13/15/18),$\tilde{4}$=(3/4/6).

Thus, the α-cuts of the above triangular fuzzy numbers are calculated as:

$\tilde{L}_i[\alpha]$=$(L_{i*}[\alpha], L_i{}^*[\alpha])$

i.e. $\widetilde{16}[\alpha]$= $(16_*[\alpha], 16^*[\alpha])$=$(14 + 2\alpha, 18 - 2\alpha)$, $\forall \alpha \in [0,1]$

$\tilde{U}_i[\alpha]$=$(U_{i*}[\alpha], U_i{}^*[\alpha])$

i.e. $\widetilde{20}[\alpha]$= $(20_*[\alpha], 20^*[\alpha])$=$(17 + 3\alpha, 22 - 2\alpha)$, $\forall \alpha \in [0,1]$

$\tilde{\lambda}_i[\alpha]$=$(\lambda_{i*}[\alpha], \lambda_i{}^*[\alpha])$

i.e. $\widetilde{0.5}[\alpha]$= $(0.5_*[\alpha], 0.5^*[\alpha])$=$(0.2 + 0.3\alpha, 0.6 - 0.1\alpha)$, $\forall \alpha \in [0,1]$

$\tilde{\mu}_i[\alpha]$= $(\mu_{i*}[\alpha], \mu_i{}^*[\alpha])$

i.e. $\widetilde{15}[\alpha]$=$(15_*[\alpha], 15^*[\alpha])$=$(13 + 2\alpha, 18 - 3\alpha)$, $\forall \alpha \in [0,1]$

$\tilde{\sigma}_i[\alpha]$= $(\sigma_{i*}[\alpha], \sigma_i{}^*[\alpha])$

i.e. $\tilde{4}[\alpha]$=$(4_*[\alpha], 4^*[\alpha])$=$(3 + \alpha, 6 - 2\alpha)$, $\forall \alpha \in [0,1]$

On simplification, the above model (4.8) -(4.14) can be written as:

$$\min: \bar{z} = 12x_1 + 15x_2 + 14x_3 + \tilde{E}\left(\left|\tilde{b}_1 - (3x_1 + 5x_2 + 5x_3)\right|\right)$$

$$+\tilde{E}\left(\left|\tilde{b}_2 - (7x_1 + 3x_2 + 4x_3)\right|\right) + \tilde{E}\left(\left|\tilde{b}_3 - (5x_1 + 7x_2 + 6x_3)\right|\right) \quad (49)$$

Subject to

$$9x_1 + 3x_2 + 7x_3 \le 25 \quad (50)$$

$$7x_1 + 5x_2 + 3x_3 \ge 18 \quad (51)$$

$$x_1, x_2, x_3 \ge 0 \quad (52)$$

Further, using the α-cuts values associated with the respective fuzzy random variables, the above model (49)-(52) can be transformed into an equivalent deterministic crisp non-linear programming model as:

$$\min: \check{z} = 12x_1 + 15x_2 + 14x_3 + \frac{1}{(3+\alpha)}\left[\frac{13\alpha^2 + 158\alpha + 485}{2} + g_1^2 - g_1(31 + 5\alpha)\right]$$

$$+\left[\frac{2}{(0.6 - 0.1\alpha)}e^{-(0.6-0.1\alpha)g_2} + g_2 - \frac{1}{0.6 - 0.1\alpha}\right]$$

$$+\left[(13+2\alpha)-g_3+2(g_3-13-2\alpha)\Phi\left(\frac{g_3-13-2\alpha}{(3+\alpha)}\right)+\frac{2(3+\alpha)}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{g_3-13-2\alpha}{3+\alpha}\right)^2}\right] \quad (53)$$

Subject to

$$9x_1 + 3x_2 + 7x_3 \leq 25 \quad (54)$$

$$7x_1 + 5x_2 + 3x_3 \geq 18 \quad (55)$$

$$0 \leq \alpha \leq 1 \quad (56)$$

$$x_1, x_2, x_3 \geq 0 \quad (57)$$

where g_1=$3x_1 + 5x_2 + 5x_3$, g_2=$7x_1 + 3x_2 + 4x_3$, and g_3=$5x_1 + 7x_2 + 6x_3$.

The above crisp nonlinear programming model (53) -(57) is solved by using LINGO (Language for Interactive General Optimization) Version 11.0 (Schrage, 2006). The optimal solutions are given in Table 1.

Table 1: Optimal solutions of different α values

Different Values α	Optimal decision variables	Value of the objective function
0	$x_1 = 1.597426, x_2 = 1.363604, x_3 = 0.0$	$\check{z} = 63.72682$
0.1	$x_1 = 1.601794, x_2 = 1.357489, x_3 = 0.0$	$\check{z} = 65.56349$
0.2	$x_1 = 1.604056, x_2 = 1.354322, x_3 = 0.0$	$\check{z} = 67.32546$
0.3	$x_1 = 1.604439, x_2 = 1.353785, x_3 = 0.0$	$\check{z} = 69.02110$
0.4	$x_1 = 1.603170, x_2 = 1.355562, x_3 = 0.0$	$\check{z} = 70.65756$
0.5	$x_1 = 1.600462, x_2 = 1.359354, x_3 = 0.0$	$\check{z} = 72.24095$
0.6	$x_1 = 1.596510, x_2 = 1.364886, x_3 = 0.0$	$\check{z} = 73.77650$
0.7	$x_1 = 1.591489, x_2 = 1.371916, x_3 = 0.0$	$\check{z} = 75.26872$
0.8	$x_1 = 1.585551, x_2 = 1.380229, x_3 = 0.0$	$\check{z} = 76.72151$
0.9	$x_1 = 1.578829, x_2 = 1.389639, x_3 = 0.0$	$\check{z} = 78.13825$
1	$x_1 = 1.571440, x_2 = 1.399983, x_3 = 0.0$	$\check{z} = 79.52190$

CONCLUSIONS

A fuzzy two-stage stochastic programming problem involving fuzzy random variables has some practical applications in management science and technology. Three fuzzy random variables such as fuzzy uniform, fuzzy exponential and fuzzy normal are considered for the right hand side parameter of the model. The crisp equivalent models have been established by removing fuzziness and randomness to make the proposed problem solvable. Since the crisp equivalent models are nonlinear programming problems, we use LINGO software tool to solve the model. A suited numerical example is presented to verify the proposed methodology and solution procedures. From the result Table 1, we can see that the obtained values for the optimal decision variables are different for different α values and the values of the objective functions also improve accordingly.

REFERENCES

[1] Aiche, F., Abbas, M. and Dubois, D. (2013), Chance-constrained programming with fuzzystochastic coefficients, *Fuzzy Optimization and Decision Making, 12*, 125-152.

[2] Alipouri Y., Sebt M.H., Ardeshir A., & Zarandi M.H.F. (2020). A mixed-integer linear programming model for solving fuzzy stochastic resource constrained project scheduling problem, *Operational Research, 20*(1), 197-217.

Barik, S. K., Biswal, M. P. & Chakravarty, D. (2012) Multiobjective Two-Stage Stochastic Programming Problems with Interval Discrete Random Variables, *Advances in Operations Research, 2012*, Article ID 279181, 1-21.

Barik, S. K., Biswal, M. P. & Chakravarty, D. (2013). Two-Stage Stochastic Programming Problems Involving Some Continuous Random Variables, *Journal of Uncertain Systems 7* (.4), 277-288.

Barik, S. K., Biswal, M. P. & Chakravarty, D. (2014). Two-stage stochastic programming problems involving multi-choice parameters, *Applied Mathematics and Computation, 240*, 109-114.

Beale E. M. L. (1955) On minimizing a convex function subject to linear inequalities. *Journal of the Royal Statistical Society, 17B*, 173-184.

Birge, J. R. and Louveaux, F. (1997). Introduction to Stochastic Programming. Springer-Verlag, New York.

Buckley, J. J. (2005). Fuzzy probabilities: New approach and applications: Physica-Verlag Heidelberg.

Buckley, J.J., and E. Eslami. (2004). Uncertain probabilities-II, Soft computing 8, 193-199.

Cai Y., Huang G.H., Nie X.H., Li Y.P. & Tan Q. (2007) Municipal solid waste management under uncertainty: A Mixed interval parameter fuzzy-stochastic robust programming approach, *Environmental Engineering Science. 24*(3), 338-352.

Charnes A. & Cooper W. W. (1959) Chance-constraint programming. *Management Science 6*, 73-79.

Charles V., Gupta S. & Ali I. (2019), A Fuzzy Goal Programming Approach for Solving Multi-Objective Supply Chain Network Problems with Pareto-Distributed Random Variables, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 27* (4), 559-593.

Dantzig, G. B. (1955) Linear programming under uncertainty, *Management Science, 1*: 197-206.

Dantzig, G.B. & Madansky, A. (1961) On the solution of two-stage linear programs under uncertainty. In: 4th Berkeley Symposium on Statistics and Probability, Vol. 1. Berkeley, CA: University California Press, 165-176.

Chakraborty, D. (2015), Solving geometric programming problems with fuzzy random variable coefficients, *Journal of Intelligent & Fuzzy Systems, 28* (6), 2493-2499.

Eshghi K. & Nematian J. (2008), Special classes of mathematical programming models with fuzzy random variables, *Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology, 19* (2), 131-140.

Fullér R. & Zimmermann H-J. (1993) Fuzzy reasoning for solving fuzzy mathematical programming problems, *Fuzzy Sets and Systems 60*(2), 121-133.

Guo P., Huang G. H., Zhu H. & Wang X.L. (2010), A two-stage programming approach for water resources management under randomness and fuzziness, *Environmental Modelling & Software, 25*(12), 1573-1581.

Ke H., Ma J. & Tian G. (2017) Hybrid multilevel programming with uncertain random parameters, *Journal of Intelligent Manufacturing, 28*(3), 589-596.

Inuiguchi M. & Ramik J. (2000) Possibilistic linear programming: A brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem, *Fuzzy Sets and Systems, 111*(1) 3-28.

Kwakernaak H. (1978), Fuzzy random variables: definitions and theorems, *Information Science 15* (1), 1-29.

Khalifa H.A., Alodhaibi S.S. & Kumar P. (2021), An application of two- Stage Stochastic Programming for water resources management problem in pentagonal fuzzy neutrosophic environment, *Water, Energy, Food and Environment Journal, 2*(1), 31-40.

Li, J., Xu, J. & Gen, M. (2006), A class of multiobjective linear programming model with fuzzy random coefficients, *Mathematical and Computer Modelling, 44* (11-12) 1097-1113.

Liu, Y. K. & Liu, B. (2003), A class of fuzzy random optimization: Expected value models, *Information Sciences 155* (1-2), 89-102.

Liu B. (2009)., Theory and Practice of Uncertain Programming, 3rd Edition, UTLAB, Beijing, China.

Liu, Y. K. & Liu, B. (2005), On minimum-risk problems in fuzzy random decision systems, *Computers & Operations Research, 32* (2) 257-283.

Liu, Y. K. & Liu, B. (2005), Fuzzy random programming with equilibrium chance constraints, *Information Sciences 170* (2-4) 363-395.

Liu, Y. K. (2007), The approximation method for two-stage fuzzy random programming with recourse, *IEEE Transactions On Fuzzy Systems, 15* (6), 1197-1208.

Liu Z., Huang R. & Shao, S. (2022), Data-driven two-stage fuzzy random mixed integer optimization model for facility location problems under uncertain environment. *AIMS Mathematics, 7*(7), 13292-13312.

Li Y., Saldanha-da-Gama F., Liu M., Yang Z. (2023), A risk-averse two-stage stochastic programming model for a joint multi-item capacitated line balancing and lot-sizing problem, *European Journal of Operational Research, 304*(1), 353-365.

Luhandjula M. K. & Gupta M. M. (1996), On fuzzy stochastic optimization, *Fuzzy Sets and Systems, 81*(1), 47-55.

Luhandjula, M. K. (1996), Fuzziness and randomness in an optimization framework, *Fuzzy Sets and Systems, 77* (3), 291-297.

Luhandjula, M.K. (2006), Fuzzy stochastic linear programming: Survey and future research directions, *European Journal of Operational Research 174*(3),1353-1367.

Luhandjula, M. K. & Joubert, J. W. (2010), On some optimisation models in a fuzzy-stochastic environment, *European Journal of Operational Research, 207*(3), 1433-1441.

Nanda S., Panda G. & Dash J. K. (2006), A new solution method for fuzzy chance constrained programming problem, *Fuzzy Optimization and Decision Making, 5*, 355-370.

Nanda, S. & Kar, K. (1992). Convex fuzzy mapping, *Fuzzy Sets and Systems 48*(1), 129-132.

Osman M., Emam O.E. & Sayed M.A. El. (2017)., FGP approach for solving multi-level multi-objective quadratic fractional programming problem with fuzzy parameters, Journal of *Abstract and Computational Mathematics*, *2*(1), 56-70.

Osman M., Emam O.E. & Sayed M.A. El (2018), Interactive Approach for Multi-Level Multi-Objective Fractional Programming Problems with Fuzzy Parameters, Beni-Suef University, *Journal of Basic and Applied Sciences, 7*(1), 139-149.

Qiao Z., Zhang Y. & Wang G. (1994), On fuzzy random linear programming, *Fuzzy Sets and Systems 65*(1) 31- 49.

Ranarahu, N. & Dash, J.K. (2022). Computation of multi-objective two-stage fuzzy probabilistic programming problem, *Soft Computing, 26*, 271-282.

Schrage, L. (2006) Optimization Modeling with LINGO (Sixth Edition). LINDO Systems Inc, Chicago.

Sahinidis, N.V. (2004). Optimization under uncertainty: State-of-the-art and opportunities, *Computers and chemical engineering, 28*(6-7), 971-983.

Walkup, D. W. and Wets, R. J. B. (1967), Stochastic programs with recourse, *SIAM Journal on Applied Mathematics, 15*(5), 1299-1314.

Yuan, J. & Li, C. (2017) A New Method for Multi-Attribute Decision Making with Intuitionistic Trapezoidal Fuzzy Random Variable, *International Journal of Fuzzy Systems, 19*, 1-12.

Zheng M., Li, B. & Kou G. (2010), A solution method for random fuzzy multiobjective programming, 2010 Seventh International Conference on Fuzzy Systems and Knowledge Discovery (FSKD 2010), pp- 76-80.

Zhang, Y., Li, Z. & Jiao, P. (2021), Two-stage stochastic programming approach for limited medical reserves allocation under uncertainties. *Complex & Intelligent Systems, 7*, 3003-3013.