Noname manuscript No.
(will be inserted by the editor)

Sequential Optimality Conditions for Bilevel Multiobjective Fractional Programming Problems with Extremal Value Function

Mohamed Laghdir

Received: date / Accepted: date

Abstract In this paper, we consider a bilevel multiobjective fractional programming problem $(\mathcal{BMFP})$ with an extremal value function. We provide necessary and sufficient optimality conditions characterizing (properly, weakly) efficient solutions of the considered problem. These optimality conditions are obtained in terms of sequences and based on sequential calculus rules for the Brøndsted-Rockafellar subdifferential of the sum and the multi-composition of convex functions, without constraint qualifications.

Keywords Sequential optimality conditions · Brøndsted-Rockafellar subdifferential · Bilevel programming · Multiobjective fractional programming

1 Introduction

Bilevel programming problems are considered as a class of optimization problems for which the feasible set and/or the objective function of the so-called leader's problem depend on the set of solutions or the optimal value function of another optimization problem called the follower's problem. This type of mathematical problems appears in many practical problems dealing for instance with transportation planning and management problems [1, 2], medical engineering [3] and optimal allocation of water resources [4]. For more applications and details about bilevel programming problems, one can see for example [5–10].

In the bilevel programming framework, when the leader's problem contains the optimal value function of the follower's problem in its objective and/or constraint functions then it is called bilevel programming problem with extremal

Mohamed Laghdir
Department of Mathematics, Faculty of Sciences, B.P 20, El-Jadida, Morocco
E-mail: laghdirm@gmail.com

value function. Shimizu and Ishizuka [11] studied bilevel programming problems with extremal value functions and derived necessary conditions by means of the directional derivatives. Aboussoror and Adly [12] considered a bilevel nonlinear optimization problem with an extremal value function and obtained necessary and sufficient optimality conditions under constraint qualifications and via the Fenchel-Lagrange duality approach. Recently, Wang and Zhang [13] have introduced and studied a bilevel multiobjective programming problem with an extremal value function. For obtaining the optimality conditions of the latter problem, the authors have extended the approach in [12] by applying the duality scheme described in [14] under a generalized Slater-type constraint qualification.

Fractional programming was investigated extensively in the literature due to its importance in modelling numerous problems with applications for example in economic, management science, information theory, stochastic programming, electric power system, etc (see [15, 16] and the references therein). To the best of our knowledge, there is no paper integrates fractional programming with the class of bilevel programming problems with extremal value function. Therefore, the aim of this paper is to consider a bilevel programming problem more general than those in [12, 13] which is a bilevel multiobjective fractional programming problem

$$(\mathcal{BMFP}) \;\; \text{v-}\min\Big\{\Big(\frac{f_1(x,v(x))}{g_1(x,v(x))},...,\frac{f_p(x,v(x))}{g_p(x,v(x))}\Big) : x\in\mathcal{A}\Big\}$$

where $\mathcal{A} := \{x \in A : h(x, v(x)) \in -K_s\} \neq \emptyset$, $v(x)$ is the optimal value function of the following problem parametrized by x

$$(\mathcal{FP}_x) \;\; \min\{f(x,y) : y \in B\}.$$

Herein, A is a nonempty subset of $\mathbb{R}^m$ closed and convex , B is a nonempty subset of $\mathbb{R}^d$ compact and convex , K_s is a nonempty closed convex cone of $\mathbb{R}^s$, $f : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$, $h : \mathbb{R}^{m+1} \to \mathbb{R}^s \cup \{+\infty_{\mathbb{R}^s}\}$ and $f_i, g_i : \mathbb{R}^{m+1} \to \mathbb{R}$, $i = 1, ..., p$. Besides, by adopting an approach completely different to that in [12, 13], the optimality conditions characterizing (properly, weakly) efficient solutions of $(\mathcal{BMFP})$ will be obtained without constraint qualifications and in terms of sequences in exact subdifferentials at some nearby points. More precisely, these optimality conditions will be established via sequential calculus rules for the Brøndsted-Rockafellar subdifferential of the sum and the multi-composition of convex functions. It is worth noting that these sequential calculus rules were initiated and developed by Thibault [17, 18] for the Brøndsted-Rockafellar subdifferential of the sum and the composition of two convex functions in order to overcome the drawbacks of constraint qualifications.

The paper is organized as follows. In Section 2, we present some basic definitions, notations and results which will be used throughout the paper. In Section 3, we provide without constraint qualifications a sequential formula for the subdifferential of finite sums involving composed and multi-composed functions under convexity and lower semicontinuity hypotheses. In Section

4, we derive sequential optimality conditions for (properly, weakly) efficient solutions of the problem $(\mathcal{BMFP})$ without constraint qualifications.

2 Preliminaries

In this section, we recall some basic definitions and present some preliminary results which are needed in succeeding sections. We denote by $\mathbb{R}^m_+$ the nonnegative orthant of $\mathbb{R}^m$ the m-dimensional Euclidean space. For $x := (x_1, ..., x_m)$ and $y := (y_1, ..., y_m)$ in $\mathbb{R}^m$, the inner product of x and y is denoted by $\langle x, y\rangle := \sum_{i=1}^m x_i y_i$, while the norm of x is given by $\|x\|_{\mathbb{R}^m} := \sqrt{\langle x, x\rangle}$. Further, we understand by $x_n \xrightarrow[n\to+\infty]{\|.\|_{\mathbb{R}^m}} x$ that the sequence $\{x_n := (x_{1,n}, ..., x_{m,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m$ converges to $x := (x_1, ..., x_m) \in \mathbb{R}^m$ in $(\mathbb{R}^m, \|.\|_{\mathbb{R}^m})$. For a nonempty subset $A \subseteq \mathbb{R}^m$, by $\mathrm{int}(A)$ we will denote the topological interior of A. Let $K_m \subseteq \mathbb{R}^m$ be a nonempty convex cone with $0 \in K_m$, then the dual cone of K_m is given by $K_m^* := \{x^* \in \mathbb{R}^m : \langle x^*, x\rangle \geq 0, \ \forall x \in K_m\}$. On $\mathbb{R}^m$, we consider the partial order " $\leqq_{K_m}$ " induced by the convex cone K_m which is defined by

$$x \leqq_{K_m} y \iff y - x \in K_m, \ x, y \in \mathbb{R}^m.$$

With respect to " $\leqq_{K_m}$ ", the augmented set $\mathbb{R}^m \cup \{+\infty_{\mathbb{R}^m}\}$ is considered where $+\infty_{\mathbb{R}^m}$ is an abstract element verifying the following operations and conventions: $x \leqq_{K_m} +\infty_{\mathbb{R}^m}$, $x + (+\infty_{\mathbb{R}^m}) := (+\infty_{\mathbb{R}^m}) + x := +\infty_{\mathbb{R}^m}$, $\langle x^*, +\infty_{\mathbb{R}^m}\rangle := +\infty$ and $\alpha.(+\infty_{\mathbb{R}^m}) := +\infty_{\mathbb{R}^m}$ for all $(x^*, x) \in K_m^* \times (\mathbb{R}^m \cup \{+\infty_{\mathbb{R}^m}\})$ and all $\alpha \in \mathbb{R}_+$.

Let $f : \mathbb{R}^m \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be a real valued function. Then f is called proper if its effective domain $\mathrm{dom} f := \{x \in \mathbb{R}^m : f(x) \in \mathbb{R}\} \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^m$ and it is called convex if $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$ for all $x, y \in \mathbb{R}^m$ and $t \in [0, 1]$. The function f is called lower semicontinuous if its epigraph $\mathrm{epi} f := \{(x, r) \in \mathbb{R}^m \times \mathbb{R} : \ f(x) \leq r\}$ is a closed subset of $\mathbb{R}^m \times \mathbb{R}$. Furthermore, the function $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ is called K_m-nondecreasing if for all $x, y \in \mathbb{R}^m$

$$x \leqq_{K_m} y \implies f(x) \leq f(y).$$

The function $f^* : \mathbb{R}^m \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in \mathbb{R}^m\}, \ x^* \in \mathbb{R}^m,$$

is called the conjugate function of f. We have the so-called Young-Fenchel inequality

$$f^*(x^*) + f(x) \geq \langle x^*, x\rangle, \ \forall (x, x^*) \in \mathbb{R}^m \times \mathbb{R}^m.$$

The subdifferential of f at $\overline{x} \in \mathrm{dom} f$ is defined by

$$\partial f(\overline{x}) := \{x^* \in \mathbb{R}^m : f(x) \geq f(\overline{x}) + \langle x^*, x - \overline{x}\rangle, \ \forall x \in \mathbb{R}^m\}.$$

It is easy to prove that

$$\partial f(\overline{x}) = \{x^* \in \mathbb{R}^m : f^*(x^*) + f(\overline{x}) = \langle x^*, \overline{x}\rangle\}.$$

Let A be a nonempty subset of $\mathbb{R}^m$, then the indicator function $\delta_A : \mathbb{R}^m \to \overline{\mathbb{R}}$ is defined in the following way

$$\delta_A(x) := \begin{cases} 0, \text{ if } x \in A, \\ \\ +\infty, \text{ otherwise,} \end{cases}$$

while the normal cone $N_A(\overline{x})$ of A at $\overline{x} \in A$ is defined as the subdifferential of δ_A at $\overline{x}$, i.e.

$$N_A(\overline{x}) := \{x^* \in \mathbb{R}^m : \langle x^*, x - \overline{x}\rangle \leq 0, \ \forall x \in A\}.$$

Let $K_q \subseteq \mathbb{R}^q$ and $K_s \subseteq \mathbb{R}^s$ be two nonempty convex cones and let $g : \mathbb{R}^q \to \mathbb{R}^s \cup \{+\infty_{\mathbb{R}^s}\}$ be a given vector valued function. The function g is called proper if its effective domain $\mathrm{dom}g := \{x \in \mathbb{R}^q : g(x) \in \mathbb{R}^s\} \neq \emptyset$ and it is called K_s-epi closed if its epigraph $\mathrm{epi}g := \{(x, y) \in \mathbb{R}^q \times \mathbb{R}^s : g(x) \leqq_{K_s} y\}$ is a closed subset of $\mathbb{R}^q \times \mathbb{R}^s$. The function g is called K_s-convex if for all $x, y \in \mathbb{R}^q$ and all $t \in [0, 1]$ we have

$$g(tx + (1-t)y) \leqq_{K_s} tg(x) + (1-t)g(y).$$

Furthermore, g is called (K_q, K_s)-nondecreasing if for all $x, y \in \mathbb{R}^q$

$$x \leqq_{K_q} y \implies g(x) \leqq_{K_s} g(y).$$

For $y^* \in K_s^*$, we define the function $y^* \circ g : \mathbb{R}^q \to \overline{\mathbb{R}}$ by $(y^* \circ g)(x) := \langle y^*, g(x)\rangle, x \in \mathbb{R}^q$. Let $h : \mathbb{R}^m \to \mathbb{R}^q \cup \{+\infty_{\mathbb{R}^q}\}$ be another vector valued function, then the composed function $g \circ h : \mathbb{R}^m \longrightarrow \mathbb{R}^s \cup \{+\infty_{\mathbb{R}^s}\}$ is defined by

$$(g \circ h)(x) := \begin{cases} g(h(x)), \text{ if } x \in \mathrm{dom}h, \\ \\ +\infty_{\mathbb{R}^s}, \text{ otherwise.} \end{cases}$$

One can prove that if $g : \mathbb{R}^q \to \mathbb{R}^s \cup \{+\infty_{\mathbb{R}^s}\}$ is K_s-convex and (K_q, K_s)-nondecreasing on $\mathrm{dom}g$ and $h : \mathbb{R}^m \to \mathbb{R}^q \cup \{+\infty_{\mathbb{R}^q}\}$ is K_q-convex with $h(\mathrm{dom}h) \subseteq \mathrm{dom}g$, then $g \circ h$ is K_s-convex.

Let us consider the following multiobjective optimization problem

$$(\mathcal{MOP}) \ \text{v-}\min\Big\{\mathcal{F}(x) := (\mathcal{F}_1(x), ..., \mathcal{F}_p(x)) : x \in \mathcal{A}\Big\}$$

where $\mathcal{A}$ is a nonempty subset of $\mathbb{R}^m$, $\mathcal{F} : \mathbb{R}^m \to \mathbb{R}^p \cup \{+\infty_{\mathbb{R}^p}\}$ is a proper vector valued function and $\mathbb{R}^p$ is partially ordered by $\mathbb{R}^p_+$.

The following definitions can be found in [19].

Definition 1 A point $\overline{x} \in \mathcal{A} \cap \mathrm{dom}\mathcal{F}$ is said to be

– efficient solution of $(\mathcal{MOP})$ if there is no $x \in \mathcal{A}$ such that

$$\mathcal{F}_i(x) \leq \mathcal{F}_i(\overline{x}) \text{ for all } i \in \{1, ..., p\}$$

and

$$\mathcal{F}_j(x) < \mathcal{F}_j(\overline{x}) \text{ for some } j \in \{1, ..., p\};$$

– weakly efficient solution of $(\mathcal{MOP})$ if there is no $x \in \mathcal{A}$ such that

$$\mathcal{F}_i(x) < \mathcal{F}_i(\overline{x}), \text{ for all } i \in \{1,...,p\};$$

– properly efficient solution of $(\mathcal{MOP})$ (in the sense of Geoffrion) if it is efficient and there exists $\alpha > 0$ such that for all $i \in \{1,...,p\}$ and all $x \in \mathcal{A}$ satisfying $\mathcal{F}_i(x) < \mathcal{F}_i(\overline{x})$, there exists at least one $j \in \{1,...,p\}$ such that $\mathcal{F}_j(\overline{x}) < \mathcal{F}_j(x)$ and

$$\frac{\mathcal{F}_i(\overline{x}) - \mathcal{F}_i(x)}{\mathcal{F}_j(x) - \mathcal{F}_j(\overline{x})} \leq \alpha.$$

Definition 2 A point $\overline{x} \in \mathcal{A} \cap \mathrm{dom}\mathcal{F}$ is said to be

– weakly efficient solution of $(\mathcal{MOP})$ in linear scalarization's sense if there exists $(\lambda_1,...,\lambda_p) \in \mathbb{R}^p_+ \setminus \{0\}$ such that

$$\sum_{i=1}^{p} \lambda_i \mathcal{F}_i(\overline{x}) \leq \sum_{i=1}^{p} \lambda_i \mathcal{F}_i(x), \forall x \in \mathcal{A};$$

– properly efficient solution of $(\mathcal{MOP})$ in linear scalarization's sense if there exists $(\lambda_1,...,\lambda_p) \in \mathrm{int}(\mathbb{R}^p_+)$ such that

$$\sum_{i=1}^{p} \lambda_i \mathcal{F}_i(\overline{x}) \leq \sum_{i=1}^{p} \lambda_i \mathcal{F}_i(x), \forall x \in \mathcal{A}.$$

Below, the following proposition resumes some relations between the above definitions.

Proposition 1 *([19, Proposition 2.4.18]) Let $\overline{x} \in \mathcal{A} \cap \mathrm{dom}\mathcal{F}$ and assume that $\mathcal{A}$ is convex and $\mathcal{F} : \mathbb{R}^m \to \mathbb{R}^p \cup \{+\infty_{\mathbb{R}^p}\}$ is proper and $\mathbb{R}^p_+$-convex. Then, $\overline{x}$ is a weakly (properly) efficient solution of $(\mathcal{MOP})$ if and only if $\overline{x}$ is a weakly (properly) efficient solution of $(\mathcal{MOP})$ in linear scalarization's sense.*

3 Sequential Subdifferential Calculus Involving Composed and Multi-composed Functions

Let $K_q \subseteq \mathbb{R}^q$ and $K_s \subseteq \mathbb{R}^s$ be two nonempty convex cones. The aim of this section is to derive without qualification assumptions a sequential formula for the subdifferential of the following function $\sum_{i=1}^p f_i \circ \varphi + \sum_{i=1}^p g_i \circ \varphi + l \circ h \circ \varphi + \psi$ where

– $f_i, g_i : \mathbb{R}^q \to \overline{\mathbb{R}}$ are proper, convex, lower semicontinuous, K_q-nondecreasing and $f_i(+\infty_{\mathbb{R}^q}) = g_i(+\infty_{\mathbb{R}^q}) = +\infty$, $i = 1,...,p$,
– $h : \mathbb{R}^q \to \mathbb{R}^s \cup \{+\infty_{\mathbb{R}^s}\}$ is proper, K_s-convex, K_s-epi closed and (K_q, K_s)-nondecreasing with $h(+\infty_{\mathbb{R}^q}) = +\infty_{\mathbb{R}^s}$,
– $l : \mathbb{R}^s \to \overline{\mathbb{R}}$ is proper, convex, lower semicontinuous and K_s-nondecreasing with $l(+\infty_{\mathbb{R}^s}) = +\infty$,

– $\varphi : \mathbb{R}^m \to \mathbb{R}^q \cup \{+\infty_{\mathbb{R}^q}\}$ is proper, K_q-convex and K_q-epi closed with $\varphi(\mathrm{dom}\varphi) \subseteq \mathrm{dom}h$,
– $\psi : \mathbb{R}^m \to \overline{\mathbb{R}}$ is proper, convex and lower semicontinuous,
– $\cap_{i=1}^p \varphi^{-1}(\mathrm{dom}f_i \cap \mathrm{dom}g_i) \cap (\varphi^{-1} \circ h^{-1})(\mathrm{dom}l) \cap \mathrm{dom}\varphi \cap \mathrm{dom}\psi \neq \emptyset$.

Consider now the following functions

$$\begin{array}{llll} F_i : \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s \to \overline{\mathbb{R}} & , & G_i : \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s \to \overline{\mathbb{R}} & (i = 1, ..., p), \\ \quad (x, y, z) \mapsto f_i(y) & & \quad (x, y, z) \mapsto g_i(y) & \end{array}$$

$$\begin{array}{lll} H : \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s \to \overline{\mathbb{R}} & , & L : \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s \to \overline{\mathbb{R}} \\ \quad (x, y, z) \mapsto \delta_{\mathrm{epi}h}(y, z) & & \quad (x, y, z) \mapsto l(z) \end{array}$$

and

$$\begin{array}{lll} \Phi : \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s \to \overline{\mathbb{R}} & , & \Psi : \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s \to \overline{\mathbb{R}} \\ \quad (x, y, z) \mapsto \delta_{\mathrm{epi}\varphi}(x, y) & & \quad (x, y, z) \mapsto \psi(x). \end{array}$$

Remark 1 Let us note that the functions $F_1, ..., F_p$, $G_1, ..., G_p$, H, L, Φ and Ψ are proper, convex and lower semicontinuous.

In the sequel, we will need the following lemmas.

Lemma 1 *Let $\overline{x} \in \cap_{i=1}^p \varphi^{-1}(\mathrm{dom}f_i \cap \mathrm{dom}g_i) \cap (\varphi^{-1} \circ h^{-1})(\mathrm{dom}l) \cap \mathrm{dom}\varphi \cap \mathrm{dom}\psi$, $\overline{y} := \varphi(\overline{x})$ and $\overline{z} := (h \circ \varphi)(\overline{x})$. Then,*

$$x^* \in \partial\Big(\sum_{i=1}^p f_i \circ \varphi + \sum_{i=1}^p g_i \circ \varphi + l \circ h \circ \varphi + \psi\Big)(\overline{x})$$

if and only if

$$(x^*, 0, 0) \in \partial(F_1 + ... + F_p + G_1 + ... + G_p + L + H + \Phi + \Psi)(\overline{x}, \overline{y}, \overline{z}).$$

Proof ($\Rightarrow$) We proceed by contradiction. So, let

$$x^* \in \partial\Big(\sum_{i=1}^p f_i \circ \varphi + \sum_{i=1}^p g_i \circ \varphi + l \circ h \circ \varphi + \psi\Big)(\overline{x})$$

and assume that

$$(x^*, 0, 0) \notin \partial(F_1 + ... + F_p + G_1 + ... + G_p + L + H + \Phi + \Psi)(\overline{x}, \overline{y}, \overline{z}). \quad (1)$$

From (1), it follows that there exist $(x, y, z) \in \cap_{i=1}^p(\mathrm{dom}F_i \cap \mathrm{dom}G_i) \cap \mathrm{dom}L \cap \mathrm{dom}H \cap \mathrm{dom}\Phi \cap \mathrm{dom}\Psi$, such that

$$\begin{aligned} &(F_1 + ... + F_p + G_1 + ... + G_p + L + H + \Phi + \Psi)(x, y, z) \\ &< (F_1 + ... + F_p + G_1 + ... + G_p + L + H + \Phi + \Psi)(\overline{x}, \overline{y}, \overline{z}) + \langle x^*, x - \overline{x}\rangle. \end{aligned}$$

This implies that

$$\begin{aligned} &f_1(y) + ... + f_p(y) + g_1(y) + ... + g_p(y) + l(z) + \psi(x) \\ &< (f_1 \circ \varphi)(\overline{x}) + ... + (f_p \circ \varphi)(\overline{x}) + (g_1 \circ \varphi)(\overline{x}) + ... + (g_p \circ \varphi)(\overline{x}) \\ &\quad + (l \circ h \circ \varphi)(\overline{x}) + \psi(\overline{x}) + \langle x^*, x - \overline{x}\rangle \end{aligned} \quad (2)$$

with

$$\begin{cases} x \in \mathrm{dom}\psi,\ y \in \cap_{i=1}^{p}(\mathrm{dom} f_i \cap \mathrm{dom} g_i),\ z \in \mathrm{dom} l, \\ (x, y) \in \mathrm{epi}\varphi,\ (y, z) \in \mathrm{epi} h. \end{cases} \tag{3}$$

By taking in the account the monotonicity of $f_1, ..., f_p$, $g_1, ..., g_p$, h and l, it follows from (3) that

$$\begin{aligned} &(f_1 \circ \varphi)(x) + ... + (f_p \circ \varphi)(x) + (g_1 \circ \varphi)(x) + ... + (g_p \circ \varphi)(x) + (l \circ h \circ \varphi)(x) \\ &\quad + \psi(x) \leq f_1(y) + ... + f_p(y) + g_1(y) + ... + g_p(y) + l(z) + \psi(x). \end{aligned} \tag{4}$$

Hence, from (2) and (4) we get

$$\begin{aligned} &(f_1 \circ \varphi)(x) + ... + (f_p \circ \varphi)(x) + (g_1 \circ \varphi)(x) + ... + (g_p \circ \varphi)(x) + (l \circ h \circ \varphi)(x) \\ &\quad + \psi(x) < (f_1 \circ \varphi)(\overline{x}) + ... + (f_p \circ \varphi)(\overline{x}) + (g_1 \circ \varphi)(\overline{x}) + ... + (g_p \circ \varphi)(\overline{x}) \\ &\qquad + (l \circ h \circ \varphi)(\overline{x}) + \psi(\overline{x}) + \langle x^*, x - \overline{x} \rangle, \end{aligned}$$

which contradicts $x^* \in \partial\Big(\sum_{i=1}^{p} f_i \circ \varphi + \sum_{i=1}^{p} g_i \circ \varphi + l \circ h \circ \varphi + \psi\Big)(\overline{x})$.

($\Leftarrow$) Follows easily by contradiction too. □

Lemma 2 *Let* $(x, y, z) \in \cap_{i=1}^{p}(\mathrm{dom} F_i \cap \mathrm{dom} G_i) \cap \mathrm{dom} L \cap \mathrm{dom} H \cap \mathrm{dom}\Phi \cap \mathrm{dom}\Psi$*, then*

(a)

$$(x^*, y^*, z^*) \in \partial\Phi(x, y, z) \iff \begin{cases} x^* \in \partial(-y^* \circ \varphi)(x),\ -y^* \in K_q^*, \\ \langle -y^*, y - \varphi(x) \rangle = 0,\ z^* = 0; \end{cases}$$

(b)

$$(x^*, y^*, z^*) \in \partial H(x, y, z) \iff \begin{cases} x^* = 0,\ y^* \in \partial(-z^* \circ h)(y), \\ -z^* \in K_s^*,\ \langle -z^*, z - h(y) \rangle = 0; \end{cases}$$

(c)

$$\begin{cases} \partial F_i(x, y, z) = \{0\} \times \partial f_i(y) \times \{0\}\ (i = 1, ..., p), \\ \partial G_i(x, y, z) = \{0\} \times \partial g_i(y) \times \{0\}\ (i = 1, ..., p), \\ \partial L(x, y, z) = \{0\} \times \{0\} \times \partial l(z), \\ \partial\Psi(x, y, z) = \partial\psi(x) \times \{0\} \times \{0\}. \end{cases}$$

Proof (a) Let $(x, y, z) \in \cap_{i=1}^{p}(\mathrm{dom} F_i \cap \mathrm{dom} G_i) \cap \mathrm{dom} L \cap \mathrm{dom} H \cap \mathrm{dom}\Phi$. It is easily to check that for any $(x^*, y^*, z^*) \in \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s$

$$\Phi^*(x^*, y^*, z^*) = \begin{cases} (-y^* \circ \varphi)^*(x^*),\ \text{if } -y^* \in K_q^*,\ z^* = 0, \\ +\infty,\ \text{otherwise}. \end{cases}$$

Thus

$$(x^*, y^*, z^*) \in \partial\Phi(x, y, z)$$

$\Longleftrightarrow$

$$\Phi^*(x^*, y^*, z^*) + \Phi(x, y, z) = \langle x^*, x\rangle + \langle y^*, y\rangle + \langle z^*, z\rangle$$

$\Longleftrightarrow$

$$\begin{cases} (-y^* \circ \varphi)^*(x^*) - \langle x^*, x\rangle - \langle y^*, y\rangle = 0, \\ -y^* \in K_q^*,\ z^* = 0, \end{cases}$$

$\Longleftrightarrow$

$$\begin{cases} ((-y^* \circ \varphi)^*(x^*) + (-y^* \circ \varphi)(x) - \langle x^*, x\rangle) + \langle -y^*, y - \varphi(x)\rangle = 0, \\ -y^* \in K_q^*,\ z^* = 0. \end{cases} \tag{5}$$

Since $(x, y) \in \mathrm{epi}\varphi$ and according to the Young-Fenchel inequality, (5) becomes

$$\begin{cases} x^* \in \partial(-y^* \circ \varphi)(x),\ \langle -y^*, y - \varphi(x)\rangle = 0, \\ -y^* \in K_q^*,\ z^* = 0, \end{cases}$$

and hence the proof is complete.

For (b) and (c) we apply the same arguments as in (a). □

Before stating the main result of this section, we recall an interesting result established by Laghdir et al. [20] in the setting of Banach spaces which provides a sequential formula for the subdifferential of the sums of proper, convex and lower semicontinuous functions, without constraint qualifications.

Theorem 1 *([20, Theorem 3.2]) Let $(X, \|.\|_X)$ be a Banach space and $(X^*, w(X^*, X))$ its topological dual space paired in duality by $\langle ., .\rangle$ where $w(X^*, X)$ denotes the weak-star topology on X^*. Let $h_1, ..., h_k : X \to \overline{\mathbb{R}}$ be k proper, convex and lower semicontinuous functions. Assume that $\overline{x} \in \cap_{i=1}^k \mathrm{dom}h_i$, then $x^* \in \partial(h_1 + ... + h_k)(\overline{x})$ if and only if there exist nets $\{x_j^i\}_{j\in J} \subseteq \mathrm{dom}h_i$ and $\{x_j^{i*}\}_{j\in J} \subseteq X^*$, $i = 1, ..., k$, such that*

$$x_j^{i*} \in \partial h_i(x_j^i),\ x_j^i \xrightarrow[j\in J]{\|.\|_X} \overline{x},\ x_j^{1*} + ... + x_j^{k*} \xrightarrow[j\in J]{w(X^*,X)} x^*$$

and

$$h_i(x_j^i) - h_i(\overline{x}) - \langle x_j^{i*}, x_j^i - \overline{x}\rangle \xrightarrow[j\in J]{} 0.$$

Here is the main result of this section.

Theorem 2 *Let $\overline{x} \in \cap_{i=1}^p \varphi^{-1}(\mathrm{dom}f_i \cap \mathrm{dom}g_i) \cap (\varphi^{-1} \circ h^{-1})(\mathrm{dom}l) \cap \mathrm{dom}\varphi \cap \mathrm{dom}\psi$, $\overline{y} := \varphi(\overline{x})$ and $\overline{z} := (h \circ \varphi)(\overline{x})$. Then, $x^* \in \partial(\sum_{i=1}^p f_i \circ \varphi + \sum_{i=1}^p g_i \circ \varphi + l \circ h \circ \varphi + \psi)(\overline{x})$ if and only if there exist sequences $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq \mathrm{epi}\varphi$, $\{(x_n^*, y_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m \times \mathbb{R}^q$, $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}l$, $\{z_n^*\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^s$, $\{(r_n, t_n)\}_{n\in\mathbb{N}} \subseteq$*

$\text{epi}h$, $\{(r_n^*, t_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^q \times \mathbb{R}^s$, $\{w_n\}_{n\in\mathbb{N}} \subseteq \text{dom}\psi$, $\{w_n^*\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m$, $\{(u_n^i, w_n^i)\}_{n\in\mathbb{N}} \subseteq \text{dom}f_i \times \text{dom}g_i$, $\{(u_n^{i*}, w_n^{i*})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^q \times \mathbb{R}^q$, $i = 1, ..., p$, *satisfying*

$$\begin{cases} w_n^* \in \partial\psi(w_n),\ u_n^{i*} \in \partial f_i(u_n^i),\ w_n^{i*} \in \partial g_i(w_n^i)\ (i = 1, ..., p), \\ x_n^* \in \partial(-y_n^* \circ \varphi)(x_n),\ -y_n^* \in K_q^*,\ \langle -y_n^*, y_n - \varphi(x_n)\rangle = 0, \\ z_n^* \in \partial l(z_n),\ r_n^* \in \partial(-t_n^* \circ h)(r_n),\ -t_n^* \in K_s^*,\ \langle -t_n^*, t_n - h(r_n)\rangle = 0, \end{cases}$$

$$\begin{cases} w_n^* + x_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} x^*,\ \sum_{i=1}^{p} u_n^{i*} + \sum_{i=1}^{p} w_n^{i*} + r_n^* + y_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} 0, \\ t_n^* + z_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} 0, \end{cases}$$

$$\begin{cases} w_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ x_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ y_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} \overline{y}, \\ r_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} \overline{y},\ u_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} \overline{y},\ w_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} \overline{y}\ (i = 1, ..., p), \\ t_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z},\ z_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z}, \end{cases}$$

and

$$\begin{cases} f_i(u_n^i) - f_i(\overline{y}) - \langle u_n^{i*}, u_n^i - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \\ g_i(w_n^i) - g_i(\overline{y}) - \langle w_n^{i*}, w_n^i - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \\ -\langle x_n^*, x_n - \overline{x}\rangle - \langle y_n^*, y_n - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0, \\ l(z_n) - l(\overline{z}) - \langle z_n^*, z_n - \overline{z}\rangle \xrightarrow[n\to+\infty]{} 0, \\ -\langle r_n^*, r_n - \overline{y}\rangle - \langle t_n^*, t_n - \overline{z}\rangle \xrightarrow[n\to+\infty]{} 0, \\ \psi(w_n) - \psi(\overline{x}) - \langle w_n^*, w_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0. \end{cases}$$

Proof Let $\overline{x} \in \cap_{i=1}^p \varphi^{-1}(\text{dom}f_i \cap \text{dom}g_i) \cap (\varphi^{-1} \circ h^{-1})(\text{dom}l) \cap \text{dom}\varphi \cap \text{dom}\psi$, $\overline{y} := \varphi(\overline{x})$ and $\overline{z} := (h \circ \varphi)(\overline{x})$. From Lemma 1, it is clear that

$$x^* \in \partial\Big(\sum_{i=1}^{p} f_i \circ \varphi + \sum_{i=1}^{p} g_i \circ \varphi + l \circ h \circ \varphi + \psi\Big)(\overline{x})$$

if and only if

$$(x^*, 0, 0) \in \partial(F_1 + ... + F_p + G_1 + ... + G_p + L + H + \Phi + \Psi)(\overline{x}, \overline{y}, \overline{z}).$$

According to Remark 1, one can see that the functions $F_1, ..., F_p$, $G_1, ..., G_p$, H, L, Φ and Ψ verify all the conditions of Theorem 1. Hence by applying Theorem 1, it follows that there exist sequences $\{(\alpha_n^i, u_n^i, \alpha_n'^i)\}_{n\in\mathbb{N}} \subseteq \text{dom}F_i = \mathbb{R}^m \times \text{dom}f_i \times \mathbb{R}^s$, $\{(\alpha_n^{i*}, u_n^{i*}, \alpha_n'^{i*})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s$, $\{(\beta_n^i, w_n^i, \beta_n'^i)\}_{n\in\mathbb{N}} \subseteq$

$\mathrm{dom}G_i = \mathbb{R}^m \times \mathrm{dom}g_i \times \mathbb{R}^s$, $\{(\beta_n^{i*}, w_n^{i*}, \beta_n'^{i*})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s$, $\{(x_n, y_n, \gamma_n)\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}\Phi = \mathrm{epi}\varphi \times \mathbb{R}^s$, $\{(x_n^*, y_n^*, \gamma_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s$, $\{(b_n, c_n, z_n)\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}L = \mathbb{R}^m \times \mathbb{R}^q \times \mathrm{dom}l$, $\{(b_n^*, c_n^*, z_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s$, $\{(e_n, r_n, t_n)\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}H = \mathbb{R}^m \times \mathrm{epi}h$, $\{(e_n^*, r_n^*, t_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s$, $\{(w_n, \lambda_n, \mu_n)\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}\Psi = \mathrm{dom}\psi \times \mathbb{R}^q \times \mathbb{R}^s$ and $\{(w_n^*, \lambda_n^*, \mu_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s$, such that

$$\begin{cases} (\alpha_n^{i*}, u_n^{i*}, \alpha_n'^{i*}) \in \partial F_i(\alpha_n^i, u_n^i, \alpha_n'^i)\ (i = 1, ..., p), \\ (\beta_n^{i*}, w_n^{i*}, \beta_n'^{i*}) \in \partial G_i(\beta_n^i, w_n^i, \beta_n'^i)\ (i = 1, ..., p), \\ (x_n^*, y_n^*, \gamma_n^*) \in \partial\Phi(x_n, y_n, \gamma_n),\ (b_n^*, c_n^*, z_n^*) \in \partial L(b_n, c_n, z_n), \\ (e_n^*, r_n^*, t_n^*) \in \partial H(e_n, r_n, t_n),\ (w_n^*, \lambda_n^*, \mu_n^*) \in \partial\Psi(w_n, \lambda_n, \mu_n) \end{cases} \tag{6}$$

$$\begin{cases} \displaystyle\sum_{i=1}^p \alpha_n^{i*} + \sum_{i=1}^p \beta_n^{i*} + x_n^* + b_n^* + e_n^* + w_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} x^*, \\ \displaystyle\sum_{i=1}^p u_n^{i*} + \sum_{i=1}^p w_n^{i*} + y_n^* + c_n^* + r_n^* + \lambda_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} 0, \\ \displaystyle\sum_{i=1}^p \alpha_n'^{i*} + \sum_{i=1}^p \beta_n'^{i*} + \gamma_n^* + z_n^* + t_n^* + \mu_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} 0, \end{cases} \tag{7}$$

$$\begin{cases} \alpha_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ u_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} \overline{y},\ \alpha_n'^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z}\ (i = 1, ..., p), \\ \beta_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ w_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} \overline{y},\ \beta_n'^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z}\ (i = 1, ..., p), \\ x_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ y_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} \overline{y},\ \gamma_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z}, \\ b_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ c_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} \overline{y},\ z_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z}, \\ e_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ r_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} \overline{y},\ t_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z}, \\ w_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ \lambda_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} \overline{y},\ \mu_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z}, \end{cases} \tag{8}$$

and

$$\begin{aligned} F_i(\alpha_n^i, u_n^i, \alpha_n'^i) - F_i(\overline{x}, \overline{y}, \overline{z}) - \langle \alpha_n^{i*}, \alpha_n^i - \overline{x}\rangle \\ - \langle u_n^{i*}, u_n^i - \overline{y}\rangle - \langle \alpha_n'^{i*}, \alpha_n'^i - \overline{z}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \end{aligned} \tag{9a}$$

$$\begin{aligned} G_i(\beta_n^i, w_n^i, \beta_n'^i) - G_i(\overline{x}, \overline{y}, \overline{z}) - \langle \beta_n^{i*}, \beta_n^i - \overline{x}\rangle \\ - \langle w_n^{i*}, w_n^i - \overline{y}\rangle - \langle \beta_n'^{i*}, \beta_n'^i - \overline{z}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \end{aligned} \tag{9b}$$

$$\begin{aligned}\Phi(x_n, y_n, \gamma_n) - \Phi(\overline{x}, \overline{y}, \overline{z}) - \langle x_n^*, x_n - \overline{x}\rangle \\ - \langle y_n^*, y_n - \overline{y}\rangle - \langle \gamma_n^*, \gamma_n - \overline{z}\rangle \xrightarrow[n\to+\infty]{} 0,\end{aligned} \tag{9c}$$

$$\begin{aligned}L(b_n, c_n, z_n) - L(\overline{x}, \overline{y}, \overline{z}) - \langle b_n^*, b_n - \overline{x}\rangle \\ - \langle c_n^*, c_n - \overline{y}\rangle - \langle z_n^*, z_n - \overline{z}\rangle \xrightarrow[n\to+\infty]{} 0,\end{aligned} \tag{9d}$$

$$\begin{aligned}H(e_n, r_n, t_n) - H(\overline{x}, \overline{y}, \overline{z}) - \langle e_n^*, e_n - \overline{x}\rangle \\ - \langle r_n^*, r_n - \overline{y}\rangle - \langle t_n^*, t_n - \overline{z}\rangle \xrightarrow[n\to+\infty]{} 0,\end{aligned} \tag{9e}$$

$$\begin{aligned}\Psi(w_n, \lambda_n, \mu_n) - \Psi(\overline{x}, \overline{y}, \overline{z}) - \langle w_n^*, w_n - \overline{x}\rangle \\ - \langle \lambda_n^*, \lambda_n - \overline{y}\rangle - \langle \mu_n^*, \mu_n - \overline{z}\rangle \xrightarrow[n\to+\infty]{} 0.\end{aligned} \tag{9f}$$

By Lemma 2, (6) is equivalent to

$$\begin{cases} w_n^* \in \partial\psi(w_n),\ u_n^{i*} \in \partial f_i(u_n^i),\ w_n^{i*} \in \partial g_i(w_n^i)\ (i = 1, ..., p), \\ x_n^* \in \partial(-y_n^* \circ \varphi)(x_n),\ -y_n^* \in K_q^*,\ \langle -y_n^*, y_n - \varphi(x_n)\rangle = 0, \\ z_n^* \in \partial l(z_n),\ r_n^* \in \partial(-t_n^* \circ h)(r_n),\ -t_n^* \in K_s^*,\ \langle -t_n^*, t_n - h(r_n)\rangle = 0, \end{cases}$$

with

$$\begin{cases} b_n^* = 0,\ c_n^* = 0,\ e_n^* = 0,\ \gamma_n^* = 0,\ \lambda_n^* = 0,\ \mu_n^* = 0, \\ \alpha_n^{i*} = 0,\ \alpha_n'^{i*} = 0,\ \beta_n^{i*} = 0,\ \beta_n'^{i*} = 0\ (i = 1, ..., p). \end{cases} \tag{10}$$

By (10), we have

$$(7) \iff \begin{cases} w_n^* + x_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} x^*, \\ \sum_{i=1}^p u_n^{i*} + \sum_{i=1}^p w_n^{i*} + r_n^* + y_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} 0, \\ t_n^* + z_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} 0, \end{cases}$$

and

$$(9a) - (9f) \iff \begin{cases} f_i(u_n^i) - f_i(\overline{y}) - \langle u_n^{i*}, u_n^i - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \\ g_i(w_n^i) - g_i(\overline{y}) - \langle w_n^{i*}, w_n^i - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \\ -\langle x_n^*, x_n - \overline{x}\rangle - \langle y_n^*, y_n - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0, \\ l(z_n) - l(\overline{z}) - \langle z_n^*, z_n - \overline{z}\rangle \xrightarrow[n\to+\infty]{} 0, \\ -\langle r_n^*, r_n - \overline{y}\rangle - \langle t_n^*, t_n - \overline{z}\rangle \xrightarrow[n\to+\infty]{} 0. \end{cases}$$

Hence, the proof is complete since in (8) the sequences $\{b_n\}_{n\in\mathbb{N}}$, $\{c_n\}_{n\in\mathbb{N}}$, $\{e_n\}_{n\in\mathbb{N}}$, $\{\gamma_n\}_{n\in\mathbb{N}}$, $\{\lambda_n\}_{n\in\mathbb{N}}$, $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{(\alpha_n^i, \alpha_n'^i)\}_{n\in\mathbb{N}}$, $\{(\beta_n^i, \beta_n'^i)\}_{n\in\mathbb{N}}$, $i = 1, ..., p$, are superfluous. □

4 Sequential Optimality Conditions for $(\mathcal{BMFP})$

In this section, we consider the following bilevel multiobjective fractional programming problem (the leader's problem)

$$(\mathcal{BMFP}) \ \text{v-}\min\Big\{\Big(\frac{f_1(x,v(x))}{g_1(x,v(x))}, ..., \frac{f_p(x,v(x))}{g_p(x,v(x))}\Big) : x \in \mathcal{A}\Big\}$$

where $\mathcal{A} := \{x \in A : h(x, v(x)) \in -K_s\} \neq \emptyset$ and $v(x)$ is the optimal value function of the following problem parametrized by x (the follower's problem)

$$(\mathcal{FP}_x) \ \min\{f(x,y) : y \in B\}.$$

Herein, A is a nonempty, closed and convex subset of $\mathbb{R}^m$, B is a nonempty, compact and convex subset of $\mathbb{R}^d$, K_s is a nonempty closed convex cone of $\mathbb{R}^s$, $f : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is a convex function, $f_i, -g_i : \mathbb{R}^{m+1} \to \mathbb{R}$ are convex and $\mathbb{R}_+^{m+1}$-nondecreasing functions, $i = 1, ..., p$, and $h : \mathbb{R}^{m+1} \to \mathbb{R}^s \cup \{+\infty_{\mathbb{R}^s}\}$ is a proper, K_s-convex, K_s-epi closed and $(\mathbb{R}_+^{m+1}, K_s)$-nondecreasing function with $h(+\infty_{\mathbb{R}^{m+1}}) = +\infty_{\mathbb{R}^s}$. Furthermore, we assume that

$$f_i(x, v(x)) \geq 0 \text{ and } g_i(x, v(x)) > 0, \ i = 1, ..., p, \ \forall x \in \mathcal{A}.$$

We mention that the functions $f_1, ..., f_p$ and $g_1, ..., g_p$ are all continuous since $\text{int}(\text{dom} f_i) = \text{int}(\text{dom} g_i) = \mathbb{R}^{m+1}$, $i = 1, ..., p$. Moreover, one can see that the function $v : \mathbb{R}^m \to \mathbb{R}$ is finite, convex, continuous and for each $x \in \mathbb{R}^m$, there exists $y \in B$ such that $v(x) = f(x, y)$.

Now, our aim is to derive sequential optimality conditions characterizing (properly, weakly) efficient solutions of the problem $(\mathcal{BMFP})$. For this, we begin by formulating scalar convex optimization problems by using the parametric approach due to Dinkelbach [21]. So, for a given $\eta \in \mathbb{R}_+^p$, we consider below a multiobjective optimization problem (associated to $(\mathcal{BMFP})$) denoted by $(\mathcal{P}_\eta)$

$$\text{v-}\min\Big\{\Big(f_1(x,v(x)) - \eta_1 g_1(x,v(x)), ..., f_p(x,v(x)) - \eta_p g_p(x,v(x))\Big) : x \in \mathcal{A}\Big\}.$$

Remark 2 Let us note that by using Dinkelbach's transformation, the (weakly) efficient solutions of $(\mathcal{BMFP})$ and $(\mathcal{P}_\eta)$ coincide. For the case of proper efficiency, one needs the following additional assumption

$$\exists a, b > 0, \ 0 < a \leq g_i(x, v(x)) \leq b, \ \text{for all } i \in \{1, ..., p\} \text{ and all } x \in \mathcal{A}.$$

Proposition 2 *Let $\overline{x} \in \mathcal{A}$ and $\eta \in \mathbb{R}^p_+$ with $\eta_i := \frac{f_i(\overline{x},v(\overline{x}))}{g_i(\overline{x},v(\overline{x}))} \geq 0,\ i = 1,...,p$. Then $\overline{x}$ is an efficient solution of $(\mathcal{BMFP})$ if and only if $\overline{x}$ is an optimal solution of the following scalar convex optimization problem*

$$(\mathcal{EP}_\eta)\ \ \min\Big\{\sum_{i=1}^{p}\Big(f_i(x,v(x)) - \eta_i g_i(x,v(x))\Big) : x \in \mathcal{A}'\Big\}$$

where $\mathcal{A}' := \{x \in \mathcal{A} : f_i(x,v(x)) - \eta_i g_i(x,v(x)) \leq 0,\ i = 1,...,p\}$.

Proof $(\Rightarrow)$ Assume that $\overline{x}$ is an efficient solution of $(\mathcal{BMFP})$, then by Definition 1 one can see easily that there exist no $x \in \mathcal{A}$ such that $f_i(x,v(x)) - \eta_i g_i(x,v(x)) \leq 0$ for all $i \in \{1,...,p\}$ and $f_j(x,v(x)) - \eta_j g_j(x,v(x)) < 0$ for some $j \in \{1,...,p\}$. Therefore, we have for all $x \in \mathcal{A}'$

$$f_i(x,v(x)) - \eta_i g_i(x,v(x)) = 0,\ i = 1,...,p.$$

This implies that

$$\sum_{i=1}^{p}\Big(f_i(x,v(x)) - \eta_i g_i(x,v(x))\Big) = 0,\ \forall x \in \mathcal{A}'.$$

On other hand, it is clear that $\overline{x} \in \mathcal{A}'$ and

$$\sum_{i=1}^{p}\Big(f_i(\overline{x},v(\overline{x})) - \eta_i g_i(\overline{x},v(\overline{x}))\Big) = 0.$$

Hence, it follows that

$$\sum_{i=1}^{p}\Big(f_i(\overline{x},v(\overline{x})) - \eta_i g_i(\overline{x},v(\overline{x}))\Big) = \sum_{i=1}^{p}\Big(f_i(x,v(x)) - \eta_i g_i(x,v(x))\Big),\ \forall x \in \mathcal{A}',$$

and thus the right implication is proved.

$(\Leftarrow)$ For the reciprocal implication, we proceed by contradiction. Assume that $\overline{x}$ is an optimal solution of the problem $(\mathcal{EP}_\eta)$. If $\overline{x}$ is not an efficient solution of $(\mathcal{BMFP})$, then there exists $x \in \mathcal{A}$ such that $f_i(x,v(x)) - \eta_i g_i(x,v(x)) \leq 0$ for all $i \in \{1,...,p\}$ and $f_j(x,v(x)) - \eta_j g_j(x,v(x)) < 0$ for some $j \in \{1,...,p\}$. Thus, it follows that $x \in \mathcal{A}'$ and

$$\sum_{i=1}^{p}\Big(f_i(x,v(x)) - \eta_i g_i(x,v(x))\Big) < 0 = \sum_{i=1}^{p}\Big(f_i(\overline{x},v(\overline{x})) - \eta_i g_i(\overline{x},v(\overline{x}))\Big)$$

and this contradicts $\overline{x}$ is an optimal solution of the problem $(\mathcal{EP}_\eta)$. □

Proposition 3 *Let $\overline{x} \in \mathcal{A}$ and $\eta \in \mathbb{R}^p_+$ with $\eta_i := \frac{f_i(\overline{x},v(\overline{x}))}{g_i(\overline{x},v(\overline{x}))} \geq 0,\ i = 1,...,p$. Then $\overline{x}$ is a weakly efficient solution of $(\mathcal{BMFP})$ if and only if there exists $(\lambda_1,...,\lambda_p) \in \mathbb{R}^p_+ \setminus \{0\}$ such that $\overline{x}$ is an optimal solution of the following scalar convex optimization problem*

$$(\mathcal{WP}_\eta)\ \ \min\Big\{\sum_{i=1}^{p}\lambda_i\Big(f_i(x,v(x)) - \eta_i g_i(x,v(x))\Big) : x \in \mathcal{A}\Big\}.$$

Proof According to Remark 2, $\overline{x}$ is a weakly efficient solution of $(\mathcal{BMFP})$ if and only if it is a weakly efficient solution of $(\mathcal{P}_\eta)$. Since $\mathcal{A}$ is convex and the functions $f_i(.,v(.)) - \eta_i g_i(.,v(.)) : \mathbb{R}^m \to \mathbb{R}$ are convex, $i = 1, ..., p$, it follows by applying Proposition 1 that $\overline{x}$ is a weakly efficient solution of $(\mathcal{P}_\eta)$ in linear scalarization's sense. Hence the proof is complete. □

Proposition 4 *Let $\overline{x} \in \mathcal{A}$ and $\eta \in \mathbb{R}^p_+$ with $\eta_i := \frac{f_i(\overline{x},v(\overline{x}))}{g_i(\overline{x},v(\overline{x}))} \geq 0$, $i = 1, ..., p$. Assume that there exist non-negative real numbers a and b such that $0 < a \leq g_i(x, v(x)) \leq b$, for all $i \in \{1, ..., p\}$ and $x \in \mathcal{A}$. Then $\overline{x}$ is a properly efficient solution of $(\mathcal{BMFP})$ if and only if there exists $(\lambda_1, ..., \lambda_p) \in \mathrm{int}(\mathbb{R}^p_+)$ such that $\overline{x}$ is an optimal solution of the following scalar convex optimization problem*

$$(\mathcal{PP}_\eta) \;\; \min\Big\{\sum_{i=1}^{p} \lambda_i \Big(f_i(x, v(x)) - \eta_i g_i(x, v(x))\Big) : x \in \mathcal{A}\Big\}.$$

Proof It suffices to show that $\overline{x}$ is a properly efficient solution of $(\mathcal{BMFP})$ if and only if $\overline{x}$ is a properly efficient solution of $(\mathcal{P}_\eta)$ and apply Proposition 1. So, Assume that $\overline{x}$ is a properly efficient solution of $(\mathcal{BMFP})$, then by Definition 1 it follows that $\overline{x}$ is efficient and there exists $\alpha > 0$ such that for all $i \in \{1, ..., p\}$ and all $x \in \mathcal{A}$ satisfying $f_i(x, v(x)) - \eta_i g_i(x, v(x)) < 0 = f_i(\overline{x}, v(\overline{x})) - \eta_i g_i(\overline{x}, v(\overline{x}))$, there exists one $j \in \{1, ..., p\}$ such that

$$f_j(\overline{x}, v(\overline{x})) - \eta_j g_j(\overline{x}, v(\overline{x})) = 0 < f_j(x, v(x)) - \eta_j g_j(x, v(x))$$

and

$$\begin{aligned}
&\Big(f_i(\overline{x}, v(\overline{x})) - \eta_i g_i(\overline{x}, v(\overline{x}))\Big) - \Big(f_i(x, v(x)) - \eta_i g_i(x, v(x))\Big)\\
&\leq \alpha \frac{g_i(x, v(x))}{g_j(x, v(x))} \Big[\Big(f_j(x, v(x)) - \eta_j g_j(x, v(x))\Big) - \Big(f_j(\overline{x}, v(\overline{x})) - \eta_j g_j(\overline{x}, v(\overline{x}))\Big)\Big]\\
&\leq \alpha \frac{b}{a} \Big[\Big(f_j(x, v(x)) - \eta_j g_j(x, v(x))\Big) - \Big(f_j(\overline{x}, v(\overline{x})) - \eta_j g_j(\overline{x}, v(\overline{x}))\Big)\Big].
\end{aligned}$$

Thus, $\overline{x}$ is a properly efficient solution of $(\mathcal{P}_\eta)$. Similarly we prove the reciprocal implication. □

Now, let $\varphi : \mathbb{R}^m \to \mathbb{R}^{m+1}$ be a function defined by

$$\varphi(x) := (x, v(x)), \; x \in \mathbb{R}^m.$$

It is clear that the function $\varphi : \mathbb{R}^m \to \mathbb{R}^{m+1}$ is proper, $\mathbb{R}^{m+1}_+$-convex, $\mathbb{R}^{m+1}_+$-epi closed and $\varphi(\mathrm{dom}\varphi) \subseteq \mathbb{R}^{m+1}$ since the function $v : \mathbb{R}^m \to \mathbb{R}$ is finite, convex and continuous. Furthermore, we have

$$\mathrm{epi}\varphi = \mathbb{E} := \{(x, y, r) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} : x \leqq_{\mathbb{R}^m_+} y \text{ and } v(x) \leq r\}.$$

Lemma 3 *Let $(\overline{x},\overline{y}) \in \mathbb{R}^m \times B$ such that $v(\overline{x}) = f(\overline{x},\overline{y})$. Then for any $x^* \in \mathbb{R}^m_+$ and $t \geq 0$ it holds*

$$\partial\Big((x^*,t)\circ\varphi\Big)(\overline{x}) = \mathcal{S}(t,\overline{x},\overline{y})$$

where

$$\mathcal{S}(t,\overline{x},\overline{y}) := \{x^*\} + \{y^* \in \mathbb{R}^m : (y^*,0) \in t\partial f(\overline{x},\overline{y}) + \{0\} \times N_B(\overline{y})\}.$$

Proof Let $x^* \in \mathbb{R}^m_+$ and $t \geq 0$. It is clear that

$$\partial\Big((x^*,t)\circ\varphi\Big)(\overline{x}) = \partial(x^* + tv)(\overline{x}) = \{x^*\} + \partial(tv)(\overline{x}).$$

From [12, Proposition 5.1], it results that

$$\begin{aligned}
y^* \in \partial(tv)(\overline{x}) &\iff (tv)(\overline{x}) + (tv)^*(y^*) = \langle y^*, \overline{x}\rangle \\
&\iff (y^*,0) \in \partial(tf + \delta_{\mathbb{R}^m\times B})(\overline{x},\overline{y}) \\
&\iff (y^*,0) \in \partial(tf)(\overline{x},\overline{y}) + \partial\delta_{\mathbb{R}^m\times B}(\overline{x},\overline{y}) \\
&\iff (y^*,0) \in t\partial f(\overline{x},\overline{y}) + N_{\mathbb{R}^m\times B}(\overline{x},\overline{y}) \\
&\iff (y^*,0) \in t\partial f(\overline{x},\overline{y}) + N_{\mathbb{R}^m}(\overline{x}) \times N_B(\overline{y}) \\
&\iff (y^*,0) \in t\partial f(\overline{x},\overline{y}) + \{0\} \times N_B(\overline{y}).
\end{aligned}$$

This completes the proof. □

Now, we are able to state the main results of this section.

Theorem 3 *Let $\overline{x} \in \mathcal{A}$, $\overline{y} := (\overline{x}, v(\overline{x})) \in \mathbb{R}^{m+1}$, $\overline{z} := h(\overline{y}) \in \mathbb{R}^s$, $\eta \in \mathbb{R}^p_+$ with $\eta_i := \frac{f_i(\overline{x},v(\overline{x}))}{g_i(\overline{x},v(\overline{x}))} \geq 0$, $i = 1,...,p$, and $\overline{\beta} = (f_1(\overline{y}) - \eta_1 g_1(\overline{y}),..., f_p(\overline{y}) - \eta_p g_p(\overline{y}))$. Then $\overline{x}$ is an efficient solution of $(\mathcal{BMFP})$ if and only if there exist sequences $\{(x_n,y_n,\theta_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{E}$, $\{b_n\}_{n\in\mathbb{N}} \subseteq B$, $f(x_n,b_n) = v(x_n)$, $\{(x_n^*,y_n^*,\theta_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m \times \mathbb{R}^m_+ \times \mathbb{R}_+$, $\{z_n\}_{n\in\mathbb{N}} \subseteq -K_s$, $\{z_n^*\}_{n\in\mathbb{N}} \subseteq K_s^*$, $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq -\mathbb{R}^p_+$, $\{\alpha_n^*\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^p_+$, $\{(r_n,t_n,\beta_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1} \times \mathbb{R}^s \times \mathbb{R}^p$ with $\{(r_n,t_n)\}_{n\in\mathbb{N}} \subseteq \operatorname{epi} h$ and $\{(r_n,\beta_{i,n})\}_{n\in\mathbb{N}} \subseteq \operatorname{epi}(f_i - \eta_i g_i)$, $i = 1,...,p$, $\{(r_n^*,t_n^*,\ \beta_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1} \times K_s^* \times \mathbb{R}^p_+$, $\{w_n\}_{n\in\mathbb{N}} \subseteq A$, $\{w_n^*\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m$, $\{(u_n^i,w_n^i)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$, $\{(u_n^{i*},w_n^{i*})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$, $i = 1,...,p$, satisfying*

$$\begin{cases}
w_n^* \in N_A(w_n),\ u_n^{i*} \in \partial f_i(u_n^i),\ w_n^{i*} \in \partial(-\eta_i g_i)(w_n^i)\ (i = 1,...,p), \\
x_n^* - y_n^* \in \mathcal{S}(\theta_n^*, x_n, b_n),\ \langle y_n^*, y_n - x_n\rangle + \theta_n^*(\theta_n - v(x_n)) = 0, \\
\langle z_n^*, z_n\rangle = 0,\ \langle \alpha_n^*, \alpha_n\rangle = 0,\ r_n^* \in \partial\Big(t_n^* \circ h + \displaystyle\sum_{i=1}^p \beta_{i,n}^*(f_i - \eta_i g_i)\Big)(r_n), \\
\langle t_n^*, t_n - h(r_n)\rangle + \displaystyle\sum_{i=1}^p \beta_{i,n}^*[\beta_{i,n} - (f_i(r_n) - \eta_i g_i(r_n))] = 0,
\end{cases}$$

$$\begin{cases} w_n^* + x_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} 0,\ \sum_{i=1}^p u_n^{i*} + \sum_{i=1}^p w_n^{i*} + r_n^* - (y_n^*, \theta_n^*) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} 0, \\ (z_n^*, \alpha_n^*) - (t_n^*, \beta_n^*) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s\times\mathbb{R}^p}} 0, \end{cases}$$

$$\begin{cases} w_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ x_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ (y_n, \theta_n) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}, \\ r_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y},\ u_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y},\ w_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}\ (i = 1, ..., p), \\ (t_n, \beta_n) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s\times\mathbb{R}^p}} (\overline{z}, \overline{\beta}),\ (z_n, \alpha_n) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s\times\mathbb{R}^p}} (\overline{z}, \overline{\beta}), \end{cases}$$

and

$$\begin{cases} f_i(u_n^i) - f_i(\overline{y}) - \langle u_n^{i*}, u_n^i - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \\ (-\eta_i g_i)(w_n^i) - (-\eta_i g_i)(\overline{y}) - \langle w_n^{i*}, w_n^i - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \\ \langle y_n^*, y_n - \overline{x}\rangle + \theta_n^*(\theta_n - v(\overline{x})) - \langle x_n^*, x_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0, \\ -\langle z_n^*, z_n - \overline{z}\rangle - \langle \alpha_n^*, \alpha_n - \overline{\beta}\rangle \xrightarrow[n\to+\infty]{} 0, \\ \langle t_n^*, t_n - \overline{z}\rangle + \langle \beta_n^*, \beta_n - \overline{\beta}\rangle - \langle r_n^*, r_n - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0, \\ -\langle w_n^*, w_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0. \end{cases}$$

Proof By Proposition 2, we have $\overline{x} \in \mathcal{A}$ is an efficient solution of $(\mathcal{BMFP})$ if and only if it is an optimal solution of the scalar problem $(\mathcal{EP}_\eta)$ and this is equivalent to

$$0 \in \partial\Big(\sum_{i=1}^p f_i \circ \varphi + \sum_{i=1}^p (-\eta_i g_i) \circ \varphi + \delta_{-(K_s\times\mathbb{R}^p_+)} \circ h' \circ \varphi + \delta_A\Big)(\overline{x})$$

where the function $h' : \mathbb{R}^{m+1} \to (\mathbb{R}^s \times \mathbb{R}^p) \cup \{+\infty_{(\mathbb{R}^s\times\mathbb{R}^p)}\}$ is defined by

$$h'(x) := \Big(h(x), f_1(x) - \eta_1 g_1(x), ..., f_p(x) - \eta_p g_p(x)\Big),\ x \in \mathbb{R}^{m+1}.$$

Obviously, the functions δ_A, $\delta_{-(K_s\times\mathbb{R}^p_+)}$, $f_1, ..., f_p$, $g_1, ..., g_p$, h' and φ satisfy all the assumptions of Theorem 2 (note that for the monotonicity of $\delta_{-(K_s\times\mathbb{R}^p_+)}$ one can see [22]). Hence, there exist sequences $\{(x_n, y_n, \theta_n)\}_{n\in\mathbb{N}} \subseteq \mathrm{epi}\varphi = \mathbb{E} \subseteq \mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}$, $\{b_n\}_{n\in\mathbb{N}} \subseteq B$, $f(x_n, b_n) = v(x_n)$, $\{(x_n^*, y_n^*, \theta_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m\times\mathbb{R}^m_+\times\mathbb{R}_+$, $\{(z_n, \alpha_n)\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}\delta_{-(K_s\times\mathbb{R}^p_+)} = -(K_s\times\mathbb{R}^p_+)$, $\{(z_n^*, \alpha_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^s\times\mathbb{R}^p$, $\{(r_n, t_n, \beta_n)\}_{n\in\mathbb{N}} \subseteq \mathrm{epi}h' \subseteq \mathbb{R}^{m+1}\times\mathbb{R}^s\times\mathbb{R}^p$ (i.e. $\{(r_n, t_n)\}_{n\in\mathbb{N}} \subseteq \mathrm{epi}h$ and $\{(r_n, \beta_{i,n})\}_{n\in\mathbb{N}} \subseteq \mathrm{epi}(f_i - \eta_i g_i)$, $i = 1, ..., p$), $\{(r_n^*, t_n^*, \beta_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}\times K_s^*\times\mathbb{R}^p_+$, $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}\delta_A = A$, $\{w_n^*\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m$, $\{(u_n^i, w_n^i)\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}f_i \times$

$\mathrm{dom}(-\eta_i g_i) = \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$, $\{(u_n^{i*}, w_n^{i*})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$, $i = 1, ..., p$, satisfying

$$\begin{cases} w_n^* \in N_A(w_n),\ u_n^{i*} \in \partial f_i(u_n^i),\ w_n^{i*} \in \partial(-\eta_i g_i)(w_n^i)\ (i = 1, ..., p), \\ x_n^* - y_n^* \in \mathcal{S}(\theta_n^*, x_n, b_n),\ \langle y_n^*, y_n - x_n\rangle + \theta_n^*(\theta_n - v(x_n)) = 0, \\ (z_n^*, \alpha_n^*) \in N_{-(K_s\times\mathbb{R}_+^p)}(z_n, \alpha_n) = N_{-K_s}(\alpha_n) \times N_{-\mathbb{R}_+^p}(z_n), \\ r_n^* \in \partial\Big(t_n^* \circ h + \sum_{i=1}^p \beta_{i,n}^*(f_i - \eta_i g_i)\Big)(r_n), \\ \langle t_n^*, t_n - h(r_n)\rangle + \sum_{i=1}^p \beta_{i,n}^*\Big[\beta_{i,n} - (f_i(r_n) - \eta_i g_i(r_n))\Big] = 0, \end{cases} \tag{11}$$

$$\begin{cases} w_n^* + x_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} 0,\ \sum_{i=1}^p u_n^{i*} + \sum_{i=1}^p w_n^{i*} + r_n^* - (y_n^*, \theta_n^*) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} 0, \\ (z_n^*, \alpha_n^*) - (t_n^*, \beta_n^*) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s\times\mathbb{R}^p}} 0, \end{cases}$$

$$\begin{cases} w_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ x_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ (y_n, \theta_n) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}, \\ r_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y},\ u_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y},\ w_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}\ (i = 1, ..., p), \\ (t_n, \beta_n) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s\times\mathbb{R}^p}} (\overline{z}, \overline{\beta}),\ (z_n, \alpha_n) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s\times\mathbb{R}^p}} (\overline{z}, \overline{\beta}), \end{cases}$$

and

$$\begin{cases} f_i(u_n^i) - f_i(\overline{y}) - \langle u_n^{i*}, u_n^i - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \\ (-\eta_i g_i)(w_n^i) - (-\eta_i g_i)(\overline{y}) - \langle w_n^{i*}, w_n^i - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \\ \langle y_n^*, y_n - \overline{x}\rangle + \theta_n^*(\theta_n - v(\overline{x})) - \langle x_n^*, x_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0, \\ -\langle z_n^*, z_n - \overline{z}\rangle - \langle \alpha_n^*, \alpha_n - \overline{\beta}\rangle \xrightarrow[n\to+\infty]{} 0, \\ \langle t_n^*, t_n - \overline{z}\rangle + \langle \beta_n^*, \beta_n - \overline{\beta}\rangle - \langle r_n^*, r_n - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0, \\ -\langle w_n^*, w_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0. \end{cases}$$

To end up the proof, it remains to note that (11) is equivalent to

$$(z_n^*, \alpha_n^*) \in K_s^* \times \mathbb{R}_+^p,\ \langle z_n^*, z_n\rangle = 0 \text{ and } \langle \alpha_n^*, \alpha_n\rangle = 0.$$

□

Theorem 4 *Let $\overline{x} \in \mathcal{A}$, $\overline{y} := (\overline{x}, v(\overline{x})) \in \mathbb{R}^{m+1}$, $\overline{z} := h(\overline{y}) \in \mathbb{R}^s$ and $\eta \in \mathbb{R}^p_+$ with $\eta_i := \frac{f_i(\overline{x},v(\overline{x}))}{g_i(\overline{x},v(\overline{x}))} \geq 0$, $i = 1, ..., p$. Then, $\overline{x}$ is a weakly efficient solution of $(\mathcal{BMFP})$ if and only if there exist $(\lambda_1, ..., \lambda_p) \in \mathbb{R}^p_+ \setminus \{0\}$ and sequences $\{(x_n, y_n, \theta_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{E}$, $\{b_n\}_{n\in\mathbb{N}} \subseteq B$, $f(x_n, b_n) = v(x_n)$, $\{(x^*_n, y^*_n, \theta^*_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m \times \mathbb{R}^m_+ \times \mathbb{R}_+$, $\{z_n\}_{n\in\mathbb{N}} \subseteq -K_s$, $\{z^*_n\}_{n\in\mathbb{N}} \subseteq K^*_s$, $\{(r_n, t_n)\}_{n\in\mathbb{N}} \subseteq$ epi$h$, $\{(r^*_n, t^*_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1} \times K^*_s$, $\{w_n\}_{n\in\mathbb{N}} \subseteq A$, $\{w^*_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m$, $\{(u^i_n, w^i_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$, $\{(u^{i*}_n, w^{i*}_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$, $i = 1, ..., p$, satisfying*

$$\begin{cases} w^*_n \in N_A(w_n),\ u^{i*}_n \in \partial(\lambda_i f_i)(u^i_n),\ w^{i*}_n \in \partial(-\eta_i\lambda_i g_i)(w^i_n)\ (i = 1, ..., p), \\ x^*_n - y^*_n \in \mathcal{S}(\theta^*_n, x_n, b_n),\ \langle y^*_n, y_n - x_n\rangle + \theta^*_n(\theta_n - v(x_n)) = 0, \\ \langle z^*_n, z_n\rangle = 0,\ r^*_n \in \partial(t^*_n \circ h)(r_n),\ \langle t^*_n, t_n - h(r_n)\rangle = 0, \end{cases}$$

$$\begin{cases} w^*_n + x^*_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} 0,\ \displaystyle\sum_{i=1}^{p} u^{i*}_n + \sum_{i=1}^{p} w^{i*}_n + r^*_n - (y^*_n, \theta^*_n) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} 0, \\ z^*_n - t^*_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} 0, \end{cases}$$

$$\begin{cases} w_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ x_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ (y_n, \theta_n) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}, \\ r_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y},\ u^i_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y},\ w^i_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}\ (i = 1, ..., p), \\ t_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z},\ z_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z}, \end{cases}$$

and

$$\begin{cases} (\lambda_i f_i)(u^i_n) - (\lambda_i f_i)(\overline{y}) - \langle u^{i*}_n, u^i_n - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \\ (-\eta_i\lambda_i g_i)(w^i_n) - (-\eta_i\lambda_i g_i)(\overline{y}) - \langle w^{i*}_n, w^i_n - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \\ \langle y^*_n, y_n - \overline{x}\rangle + \theta^*_n(\theta_n - v(\overline{x})) - \langle x^*_n, x_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0, \\ -\langle z^*_n, z_n - \overline{z}\rangle \xrightarrow[n\to+\infty]{} 0, \\ \langle t^*_n, t_n - \overline{z}\rangle - \langle r^*_n, r_n - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0, \\ -\langle w^*_n, w_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0. \end{cases}$$

Proof According to Proposition 3, it is clear that $\overline{x}$ is a weakly efficient solution of $(\mathcal{BMFP})$ if and only if there exist $(\lambda_1, ..., \lambda_p) \in \mathbb{R}^p_+ \setminus \{0\}$ such that

$$0 \in \partial\Big(\sum_{i=1}^{p}(\lambda_i f_i) \circ \varphi + \sum_{i=1}^{p}(-\eta_i\lambda_i g_i) \circ \varphi + \delta_{-K_s} \circ h \circ \varphi + \delta_A\Big)(\overline{x}).$$

Thus, by Theorem 2 it follows that there exist sequences $\{(x_n, y_n, \theta_n)\}_{n\in\mathbb{N}} \subseteq$ epi$\varphi = \mathbb{E} \subseteq \mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}$, $\{b_n\}_{n\in\mathbb{N}} \subseteq B$, $f(x_n, b_n) = v(x_n)$, $\{(x^*_n, y^*_n, \theta^*_n)\}_{n\in\mathbb{N}} \subseteq$

$\mathbb{R}^m \times \mathbb{R}^m_+ \times \mathbb{R}_+$, $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}\delta_{-K_s} = -K_s$, $\{z_n^*\}_{n\in\mathbb{N}} \subseteq K_s^*$, $\{(r_n,t_n)\}_{n\in\mathbb{N}} \subseteq \mathrm{epi}h \subseteq \mathbb{R}^{m+1} \times \mathbb{R}^s$, $\{(r_n^*,t_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1} \times K_s^*$, $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}\delta_A = A$, $\{w_n^*\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m$, $\{(u_n^i,w_n^i)\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}(\lambda_i f_i) \times \mathrm{dom}(-\eta_i\lambda_i g_i) = \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$, $\{(u_n^{i*},w_n^{i*})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$, $i=1,...,p$, satisfying

$$\begin{cases} w_n^* \in N_A(w_n),\ u_n^{i*} \in \partial(\lambda_i f_i)(u_n^i),\ w_n^{i*} \in \partial(-\eta_i\lambda_i g_i)(w_n^i)\ (i=1,...,p), \\ x_n^* - y_n^* \in \mathcal{S}(\theta_n^*, x_n, b_n),\ \langle y_n^*, y_n - x_n\rangle + \theta_n^*(\theta_n - v(x_n)) = 0,\ \langle z_n^*, z_n\rangle = 0, \\ r_n^* \in \partial(t_n^* \circ h)(r_n),\ \langle t_n^*, t_n - h(r_n)\rangle = 0, \end{cases}$$

$$\begin{cases} w_n^* + x_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} 0,\ \displaystyle\sum_{i=1}^p u_n^{i*} + \sum_{i=1}^p w_n^{i*} + r_n^* - (y_n^*, \theta_n^*) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} 0, \\ z_n^* - t_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} 0, \end{cases}$$

$$\begin{cases} w_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ x_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ (y_n,\theta_n) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}, \\ r_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y},\ u_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y},\ w_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}\ (i=1,...,p), \\ t_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z},\ z_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z}, \end{cases}$$

and

$$\begin{cases} (\lambda_i f_i)(u_n^i) - (\lambda_i f_i)(\overline{y}) - \langle u_n^{i*}, u_n^i - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i=1,...,p), \\ (-\eta_i\lambda_i g_i)(w_n^i) - (-\eta_i\lambda_i g_i)(\overline{y}) - \langle w_n^{i*}, w_n^i - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i=1,...,p), \\ \langle y_n^*, y_n - \overline{x}\rangle + \theta_n^*(\theta_n - v(\overline{x})) - \langle x_n^*, x_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0, \\ -\langle z_n^*, z_n - \overline{z}\rangle \xrightarrow[n\to+\infty]{} 0, \\ \langle t_n^*, t_n - \overline{z}\rangle - \langle r_n^*, r_n - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0, \\ -\langle w_n^*, w_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0. \end{cases}$$

Hence, the proof is complete. □

Now, by applying Proposition 5 and Theorem 2 we get the following result.

Theorem 5 *Let $\overline{x} \in \mathcal{A}$, $\overline{y} := (\overline{x}, v(\overline{x})) \in \mathbb{R}^{m+1}$, $\overline{z} := h(\overline{y}) \in \mathbb{R}^s$ and $\eta \in \mathbb{R}^p_+$ with $\eta_i := \frac{f_i(\overline{x},v(\overline{x}))}{g_i(\overline{x},v(\overline{x}))} \geq 0$, $i=1,...,p$. Assume that there exist non-negative real numbers a and b such that $0 < a \leq g_i(x,v(x)) \leq b$, for all $i \in \{1,...,p\}$ and all $x \in \mathcal{A}$. Then, $\overline{x}$ is a properly efficient solution of $(\mathcal{BMFP})$ if and only if there exist $(\lambda_1,...,\lambda_p) \in \mathrm{int}(\mathbb{R}^p_+)$ and sequences $\{(x_n,y_n,\theta_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{E}$, $\{b_n\}_{n\in\mathbb{N}} \subseteq B$, $f(x_n,b_n) = v(x_n)$, $\{(x_n^*,y_n^*,\theta_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m \times \mathbb{R}^m_+ \times \mathbb{R}_+$, $\{z_n\}_{n\in\mathbb{N}} \subseteq -K_s$, $\{z_n^*\}_{n\in\mathbb{N}} \subseteq K_s^*$, $\{(r_n,t_n)\}_{n\in\mathbb{N}} \subseteq \mathrm{epi}h$, $\{(r_n^*,t_n^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1} \times K_s^*$, $\{w_n\}_{n\in\mathbb{N}} \subseteq$*

A, $\{w_n^*\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m$, $\{(u_n^i, w_n^i)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}\times\mathbb{R}^{m+1}$, $\{(u_n^{i*}, w_n^{i*})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}\times\mathbb{R}^{m+1}$, $i = 1, ..., p$, satisfying

$$\begin{cases} w_n^* \in N_A(w_n),\ u_n^{i*} \in \partial(\lambda_i f_i)(u_n^i),\ w_n^{i*} \in \partial(-\eta_i\lambda_i g_i)(w_n^i)\ (i = 1, ..., p), \\ x_n^* - y_n^* \in \mathcal{S}(\theta_n^*, x_n, b_n),\ \langle y_n^*, y_n - x_n\rangle + \theta_n^*(\theta_n - v(x_n)) = 0, \\ \langle z_n^*, z_n\rangle = 0,\ r_n^* \in \partial(t_n^* \circ h)(r_n),\ \langle t_n^*, t_n - h(r_n)\rangle = 0, \end{cases}$$

$$\begin{cases} w_n^* + x_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} 0,\ \sum_{i=1}^{p} u_n^{i*} + \sum_{i=1}^{p} w_n^{i*} + r_n^* - (y_n^*, \theta_n^*) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} 0, \\ z_n^* - t_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} 0, \end{cases}$$

$$\begin{cases} w_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ x_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x},\ (y_n, \theta_n) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}, \\ r_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y},\ u_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y},\ w_n^i \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}\ (i = 1, ..., p), \\ t_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z},\ z_n \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z}, \end{cases}$$

and

$$\begin{cases} (\lambda_i f_i)(u_n^i) - (\lambda_i f_i)(\overline{y}) - \langle u_n^{i*}, u_n^i - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \\ (-\eta_i\lambda_i g_i)(w_n^i) - (-\eta_i\lambda_i g_i)(\overline{y}) - \langle w_n^{i*}, w_n^i - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0\ (i = 1, ..., p), \\ \langle y_n^*, y_n - \overline{x}\rangle + \theta_n^*(\theta_n - v(\overline{x})) - \langle x_n^*, x_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0, \\ -\langle z_n^*, z_n - \overline{z}\rangle \xrightarrow[n\to+\infty]{} 0, \\ \langle t_n^*, t_n - \overline{z}\rangle - \langle r_n^*, r_n - \overline{y}\rangle \xrightarrow[n\to+\infty]{} 0, \\ -\langle w_n^*, w_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0. \end{cases}$$

Proof We apply Proposition 5 and Theorem 2 and we follow the same reasonings as in the proof of Theorem 4. □

Next, we close this section by providing an example illustrating sequential optimality conditions given in Theorem 3, Theorem 4 and Theorem 5 where for instance the Slater's constraint qualification fails. Let us recall that the set $\mathcal{A}$ is said to satisfy Slater's constraint qualification if there exists $\widehat{x} \in A$ such that $h(\widehat{x}, v(\widehat{x})) \in -\mathrm{int}(K_s)$ (see [23]).

Example 1 Let

$$d := 1,\ m := 1,\ p := 2,\ s := 1,\ A = B := \left[0, \frac{1}{2}\right],\ K_s := \mathbb{R}_+,$$

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(x,y) := x + y,$$

$$f_1 : \mathbb{R}^2 \to \mathbb{R},\ f_1(x,y) := \begin{cases} x^2 + 1, \ \text{if } x \geq 0, \\ 1, \ \text{if } x < 0, \end{cases}$$

$$f_2 : \mathbb{R}^2 \to \mathbb{R}\ f_2(x,y) := \begin{cases} y^2, \ \text{if } y \geq 0, \\ 0, \ \text{if } y < 0, \end{cases}$$

$$g_1 : \mathbb{R}^2 \to \mathbb{R},\ g_1(x,y) := 1,$$

$$g_2 : \mathbb{R}^2 \to \mathbb{R},\ g_2(x,y) := -x - y + 2,$$

$$h : \mathbb{R}^2 \to \mathbb{R},\ h(x,y) = y.$$

Then, our bilevel multiobjective fractional programming problem that we consider can be formulated as follows

$$(\mathcal{BMFP})\ \text{v-}\min \left\{ \left(x^2 + 1, \frac{x^2}{2 - 2x} \right) : x \in \mathcal{A} \right\}$$

where $\mathcal{A} := \{x \in \left[0, \frac{1}{2}\right] : x \leq 0\} \neq \emptyset$ and $\min(\mathcal{FP}_x) = v(x) = x$, for all $x \in \mathbb{R}$. Clearly, f is convex, $f_i, -g_i$ are convex and $\mathbb{R}^2_+$-nondecreasing, $i = 1,2$ and h is proper, convex, lower semicontinuous and $\mathbb{R}^2_+$-nondecreasing. Moreover, one can see that $0 < a = 1 \leq g_i(x, v(x)) \leq b = 2$, for all $i = 1,2$ and all $x \in \mathcal{A}$. It is a simple matter to check that $\overline{x} = 0$ is a properly and weakly efficient solution of $(\mathcal{BMFP})$ where $\mathcal{A}$ does not satisfy the Slater's constraint qualification. Nevertheless, the sequential optimality conditions given in Theorem 4 and also Theorem 5 are satisfied. Take $\overline{y} := (0,0)$, $\overline{z} := 0$, $\eta_1 := 1$, $\eta_2 := 0$, $\lambda_1 = \lambda_2 := 1$, $x_n = y_n = \theta_n := \frac{1}{n+1}$, $b_n := 0$, $x_n^* = y_n^* := \frac{1}{n+1}$, $\theta_n^* := 0$, $z_n := 0$, $z_n^* := \frac{1}{n+1}$, $r_n := (\frac{1}{n+1}, \frac{1}{n+1})$, $t_n := \frac{1}{n+1}$, $r_n^* := (0,0)$, $t_n^* := 0$, $w_n = w_n^* := 0$, $u_n^1 = u_n^2 := (\frac{1}{n+1}, \frac{1}{n+1})$, $u_n^{1*} := (\frac{2}{n+1}, 0)$, $u_n^{2*} := (0, \frac{2}{n+1})$, $w_n^1 = w_n^2 := (\frac{1}{n+1}, \frac{1}{n+1})$, $w_n^{1*} = w_n^{2*} := (0,0)$, for all $n \in \mathbb{N}$. Thus, we can see easily that

$$\begin{cases} 0 \in N_A(0),\ (\frac{2}{n+1}, 0) \in \partial(\lambda_1 f_1)(\frac{1}{n+1}, \frac{1}{n+1}),\ (0, \frac{2}{n+1}) \in \partial(\lambda_2 f_2)(\frac{1}{n+1}, \frac{1}{n+1}), \\ (0,0) \in \partial(-\eta_1 \lambda_1 g_1)(\frac{1}{n+1}, \frac{1}{n+1}),\ (0,0) \in \partial(-\eta_2 \lambda_2 g_2)(\frac{1}{n+1}, \frac{1}{n+1}), \\ x_n^* - y_n^* = 0 \in \mathcal{S}(0, \frac{1}{n+1}, 0),\ y_n^*(y_n - x_n) + \theta_n^*(\theta_n - v(x_n)) = 0, \\ z_n^* z_n = 0,\ (0,0) \in \partial(t_n^* h)(\frac{1}{n+1}, \frac{1}{n+1}),\ t_n^*(t_n - h(r_n)) = 0, \end{cases}$$

$$\begin{cases} w_n^* + x_n^* = \frac{1}{n+1} \xrightarrow[n \to +\infty]{} 0, \\ u_n^{1*} + u_n^{2*} + w_n^{1*} + w_n^{2*} + r_n^* - (y_n^*, \theta_n^*) = (\frac{1}{n+1}, \frac{2}{n+1}) \xrightarrow[n \to +\infty]{\|\cdot\|_{\mathbb{R}^2}} (0,0), \\ z_n^* - t_n^* = \frac{1}{n+1} \xrightarrow[n \to +\infty]{} 0, \end{cases}$$

$$\begin{cases} w_n = z_n = 0 \xrightarrow[n\to+\infty]{} 0,\ x_n = \frac{1}{n+1} \xrightarrow[n\to+\infty]{} 0, \\ (y_n, \theta_n) = r_n = (\frac{1}{n+1}, \frac{1}{n+1}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^2}} (0,0), \\ u_n^1 = u_n^2 = w_n^1 = w_n^2 = (\frac{1}{n+1}, \frac{1}{n+1}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^2}} (0,0), \\ t_n = \frac{1}{n+1} \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

and

$$\begin{cases} (\lambda_i f_i)(u_n^i) - (\lambda_i f_i)(\overline{y}) - \langle u_n^{i*}, u_n^i - \overline{y}\rangle = -\frac{1}{(n+1)^2} \xrightarrow[n\to+\infty]{} 0 \ (i=1,2), \\ (-\eta_i\lambda_i g_i)(w_n^i) - (-\eta_i\lambda_i g_i)(\overline{y}) - \langle w_n^{i*}, w_n^i - \overline{y}\rangle = 0 \xrightarrow[n\to+\infty]{} 0 \ (i=1,2), \\ y_n^*(y_n - \overline{x}) + \theta_n^*(\theta_n - v(\overline{x})) - x_n^*(x_n - \overline{x}) = 0 \xrightarrow[n\to+\infty]{} 0, \\ -z_n^*(z_n - \overline{z}) = 0 \xrightarrow[n\to+\infty]{} 0,\ t_n^*(t_n - \overline{z}) - \langle r_n^*, r_n - \overline{y}\rangle = 0 \xrightarrow[n\to+\infty]{} 0, \\ -w_n^*(w_n - \overline{x}) = 0 \xrightarrow[n\to+\infty]{} 0. \end{cases}$$

For the sequential optimality conditions given in Theorem 3, it suffices to set $\overline{y} = \overline{\beta} := (0,0)$, $\overline{z} := 0$, $\eta_1 := 1$, $\eta_2 := 0$, $x_n = y_n = \theta_n := \frac{1}{n+1}$, $b_n := 0$, $x_n^* = y_n^* := \frac{1}{n+1}$, $\theta_n^* := 0$, $z_n := 0$, $\alpha_n := (0,0)$, $z_n^* := \frac{1}{n+1}$, $\alpha_n^* := (0,0)$, $r_n := (\frac{1}{n+1}, \frac{1}{n+1})$, $t_n := \frac{1}{n+1}$, $\beta_n := (\frac{1}{(n+1)^2}, \frac{1}{(n+1)^2})$, $r_n^* := (0,0)$, $t_n^* := 0$, $\beta_n^* := (0,0)$, $w_n = w_n^* := 0$, $u_n^1 = u_n^2 := (\frac{1}{n+1}, \frac{1}{n+1})$, $u_n^{1*} := (\frac{2}{n+1}, 0)$, $u_n^{2*} := (0, \frac{2}{n+1})$, $w_n^1 = w_n^2 := (\frac{1}{n+1}, \frac{1}{n+1})$, $w_n^{1*} = w_n^{2*} := (0,0)$, for all $n \in \mathbb{N}$.

5 Conclusions

In this work, without assuming any qualification condition, we have obtained sequential calculus rules for the subdifferential of finite sums involving composed and multi-composed functions under convexity and lower semicontinuity hypotheses, in terms of limits of subgradients at nearby points to the nominal point.

Next, we have deduced sequential optimality conditions characterizing properly or weakly efficient solutions of a bilevel multiobjective fractional programming problem with an extremal value function.

As final conclusion, we think that several results of this work will be useful in order to improve the actual resolution techniques and develop new methods to solve multiobjective fractional mathematical programs.

References

1. Migdalas, A.: Bilevel programming in traffic planning: Models, methods and challenge. J. Global Optim. **7**(4), 381–405 (1995)

2. Yin, Y.: Multiobjective bilevel optimization for transportation planning and management problems. J. Adv. Transp. **36**(1), 93–105 (2002)
3. Eichfelder, G.: Multiobjective bilevel optimization. Math. Program. **123**(2), 419–449 (2010)
4. Ahmad, I., Zhang, F., Liu, J., Anjum, M.N., Zaman, M., Tayyab, M., Waseem, M., Farid, H.U.: A linear bi-level multi-objective program for optimal allocation of water resources. Plos one **13**(2), 1–25 (2018)
5. Bard, J.F.: Practical Bilevel Optimization: Algorithms and Applications, vol. 30. Springer Science & Business Media (2013)
6. Dempe, S.: Foundations of Bilevel Programming. Springer Science & Business Media (2002)
7. Dempe, S., Kalashnikov, V., Pérez-Valdés, G.A., Kalashnykova, N.: Bilevel Programming Problems. Springer, Berlin, Heidelberg (2015)
8. Shimizu, K., Ishizuka, Y., Bard, J.F.: Nondifferentiable and Two-level Mathematical Programming. Springer Science & Business Media (2012)
9. Floudas, C.A., Pardalos, P.M.: Encyclopedia of Optimization. Springer Science & Business Media (2009)
10. Colson, B., Marcotte, P., Savard, G.: An overview of bilevel optimization. Ann. Oper. Res. **153**(1), 235–256 (2007)
11. Shimizu, K., Ishizuka, Y.: Optimality conditions and algorithms for parameter design problems with two-level structure. IEEE Trans. Autom. Control **30**(10), 986–993 (1985)
12. Aboussoror, A., Adly, S.: A Fenchel-Lagrange duality approach for a bilevel programming problem with extremal-value function. J. Optim. Theory Appl. **149**(2), 254–268 (2011)
13. Wang, H., Zhang, R.: Duality for multiobjective bilevel programming problems with extremal-value function. J. Math. Res. Appl. **35**(3), 311–320 (2015)
14. Boţ, R.I., Vargyas, E., Wanka, G.: Conjugate duality for multiobjective composed optimization problems. Acta Math. Hungarica **116**(3), 177–196 (2007)
15. Stancu-Minasian, I.M.: Fractional Programming: Theory, Methods and Applications, vol. 409. Springer Science & Business Media (2012)
16. Stancu-Minasian, I.M.: A ninth bibliography of fractional programming. Optimization **68**(11), 2125–2169 (2019)
17. Thibault, L.: A generalized sequential formula for subdifferentials of sums of convex functions defined on Banach spaces. Lecture Notes Econom. Math. Syst. (1995)
18. Thibault, L.: Sequential convex subdifferential calculus and sequential Lagrange multipliers. SIAM J. Control Optim. (1997)
19. Boţ, R.I., Grad, S.M., Wanka, G.: Duality in Vector Optimization. Springer Science & Business Media (2009)
20. Laghdir, M., Dali, I., Moustaid, M.B.: A generalized sequential formula for subdifferential of multi-composed functions defined on Banach spaces and applications. Pure Appl. Funct. Anal. **5**(4), 999–1023 (2020)

21. Dinkelbach, W.: On nonlinear fractional programming. Manag. Sci. **13**(7), 492–498 (1967)
22. Combari, C., Laghdir, M., Thibault, L.: Sous-différentiels de fonctions convexes composées. Ann. Sci. Math. Québec **18**(2), 119–148 (1994)
23. Mangasarian, O.L.: Nonlinear Programming. SIAM (1994)