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Solution of 0ptimal Control Problems Using Shifted Chebyshev Polynomial

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Abstract

This paper suggests a new and efficient method for solving linear quadratic optimal control problems. A shifted chebyshev matrix approach is implemented for solving this problem. In this method, the problem of optimal control changes into a problem of non-linear programming which can be solved easily. The corresponding nonlinear programming problem will be solved using Matlab software to find the unknown coefficients which are related to the approximate solution. Numerical examples are also given in order to compare this new method with another one.

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INTRODUCTION

Optimal control has been widely used in a variety of sciences such as aviation, electronic, robotics, etc. In the problems of optimal control, the aim is to find a control function which is applicable in dynamical system and boundary conditions and minimizes the objective functional. There are many computational methods represented for solving the problems of optimal control (Mehne & Mirjalili, 2018). These methods include two types of direct and indirect methods.

In (Razzaghi et al., 2012), a direct method for variational problems by using hybrid of block-pulse and Bernoulli polynomials was used. B-spline functions for solving constrained quadratic optimal control problems were applied in (Edrisi-Tabriz & Lakestani, 2015). A new approach using linear combination property of intervals and discretization is proposed to solve a class of nonlinear optimal control problems, containing a nonlinear system and linear functional (Skandari et al., 2011; Tohidi et al., 2011). An accurate method is proposed to solve problems such as identification, analysis and optimal control using the Bernstein orthonormal polynomials operational matrix of integration (Singh et al., 2009). Naturally, with the development of computers, many researches have focused on numerical methods for solving optimal control problems. In this paper, the aim is to introduce a new method for solving the problem of optimal control. The control problem would change into a problem of non-linear programing.

The paper is organized as follows, in Section 2, we describe the basic formulation of shifted chebyshev polynomial which is devoted to the formulation of optimal control problem. Section 3 is allocated to approximation schemes. Some numerical simulations are done in Section 4.

PROBLEM STATEMENT

Consider the following class of nonlinear systems with equality constraints,

$$x(t)=A(t)x(t)+B(t)u(t) \qquad (1)$$
$$x(a)=x^0,\ x(b)=x^1$$

where A(t) and B(t) are matrices functions, x(t) is the state vector taking values in R^n and u(t) is the control input taking values in some control set $U\subset R^m$. The problem is finding the optimal control u(t) and state trajectory x(t), satisfying (1) and minimizing the following performance index

$$Z=1/2\, x^T(b)Gx(b)+1/2\int_a^b (x^T(t)Q(t)x(t)+u^T(t)R(t)u(t))dt \qquad (2)$$

where G, $Q(t)\in R^{n\times n}$ are symmetric positive semi-definite matrices and $R(t)\in R^{m\times m}$ is a symmetric positive definite matrix.

The shifted chebyshev polynomials can be obtained with the aid of the following recurrence formula (Hesameddini & Riahi, 2018):

$$T_n^*(x)=T_n\left(\frac{2}{b-a}x-\frac{b+a}{b-a}\right),$$
$$a\le x\le b,\qquad n\ge 0$$

where

$$T_0(x)=1,\qquad T_1(x)=x,$$
$$T_n(x)=2xT_{n-1}(x)-T_{n-2}(x),$$
$$n\ge 2,\qquad -1\le x\le 1$$

In the next part, we will introduce an effective method based on using shifted chebyshev polynomial, for solving problem (2) according to condition (1).

THE PROPOSED METHOD

Suppose

$$x(t)=\sum_{j=1}^{N} a_j\, T_j^*(t) \qquad (3)$$

$$u(t)=\sum_{j=1}^{N} a_j'\, T_j^*(t) \qquad (4)$$

in which T^* is shifted chebyshev polynomial and the coefficients of (a_j) and (a_j') are unknown.

The relations (3) and (4) can be written as the following matrices:

$$[x(t)]=T^*(t)A_j$$
$$[u(t)]=T^*(t)A_j'$$

where

$$T^*(t)=[T_0^*(t),T_1^*(t),\dots,T_N^*(t)]$$
$$A_j=[a_0,\ a_1,\dots,a_N]$$
$$A_j'=[a_0',\ a_1',\dots,a_N']$$

Then, we can write the matrix form of $T^*(x)$ as follows (Hesameddini & Riahi, 2018):

$$T^*(x)=X(t)\,D^T$$

where

$$X(t)=[1\ \ t \dots t^N]$$

and D is the (N+1)×(N+1) matrix coefficients defined by

$$D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{a+b}{(a-b)^1} & \frac{-2}{(a-b)} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ \sum_{k=0}^{N} \frac{a^{N-k} b^k \binom{2N}{2k}}{(a-b)^N} & -N \sum_{k=0}^{N-1} \frac{a^{N-(k+1)} b^k \binom{2N}{2k+1}}{(a-b)^N} & \cdots & (-1)^N \frac{2^{2N-1}}{(a-b)^N} \end{bmatrix}$$

The n-th order derivative of T^* (x) is:

$$T^{*n}(t)=X^n(t)\,D^T$$

where

$$X^n(t)=X(t)(B^T)^n$$

$$B^T = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & N-1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & N \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Therefore, we rewrite problem (2) as the following:

$$Z = \frac{1}{2} A_j^T (DX^T(b) G X(b) D^T) A_j + \frac{1}{2} A_j^T \left(\int_a^b (DX^T(t) Q(t) X(t) D^T)\, dt \right) A_j + \frac{1}{2} A_j'^T \left(\int_a^b (DX^T(t) R(t) X(t) D^T)\, dt \right) A_j'$$

$$S.to: \quad (X(t) B^T D^T - A(t) X(t) D^T) A_j - (B(t) X(t) D^T) A_j' = 0,$$

$$x(a) = X(a) D^T A_j,$$

$$x(b) = X(b) D^T A_j.$$

which is a problem of non-linear programing with unknowns A_j and A_j^' . By solving this problem and putting the solutions in relations

$$x(t)=T^*(t)\,A_j, \qquad u(t)=T^*(t)\,A\_j'$$

the optimal solution of control problem would be found. It is worth mentioning that in solving the examples, "fmincon interior point algorithm", have been used (For more information, see (Byrd et al., 2000; Waltz, 2006)).

According to theorem (1) from reference (Ozturk & Gulsu, 2015), if $x_N(t)$ is an approximate of x(t), then

$$\| x(t) - x_N(t) \| \le \alpha \frac{1}{2^{2N+1}} \| x^{(N+1)}(t) \|_\infty + \sqrt{\frac{3\pi}{8}} \| A - \overline{A} \|$$

where

$$A = [a_0, a_1, \dots, a_N], \quad \overline{A} = [\overline{a}_0, \overline{a}_1, \dots, \overline{a}_N]$$

and α is a fixed amount, when

$$\overline{x}_N(t) = \sum_{r=0}^{N} \overline{a}_r\, T_r^*(t),$$

$$x_N(t) = \sum_{r=0}^{N} a_r\, T_r^*(t)$$

$x_N(t)$ are the shifted chebyshev polynomial expansion of the exact solution and $\overline{x}_N(t)$ are the approximate solution obtained by the proposed method.

NUMERICAL EXAMPLES

In this part, numerical examples are given to analyze the efficiency of the presented method. In the first example, all of the stages would be explained step by step, for a better comprehension. It is worth mentioning that these examples are taken from (Rogalsky, 2013; Yari et al., 2017) as well as from other studies cited by them in order for the readers to be able to compare methods.

Example 1: This example is adapted from (Rogalsky, 2013):

$$\min \quad Z = \int_0^4 \left(u^2(t) + x(t)\right) dt$$

$$S.\ to: \quad \dot{x}(t) = u(t),$$

$$x(0) = 0, \qquad x(4) = 1$$

Here, we solve the problem using shifted chebyshev polynomial by choosing N=5. Let

$$D=\begin{bmatrix}1&0&0&0&0&0\\-1&\frac{1}{2}&0&0&0&0\\1&-2&\frac{1}{2}&0&0&0\\-1&\frac{9}{2}&-3&\frac{1}{2}&0&0\\1&-8&10&-4&\frac{1}{2}&0\\-1&\frac{25}{2}&-25&\frac{35}{2}&-5&\frac{1}{2}\end{bmatrix}$$

$$B^T=\begin{bmatrix}0&1&0&0&0&0\\0&0&2&0&0&0\\0&0&0&3&0&0\\0&0&0&0&4&0\\0&0&0&0&0&5\\0&0&0&0&0&0\end{bmatrix}$$

$$\min\quad Z=\left(\int_0^4 (X(t)D^T)\,dt\right)A_j + A_j'^T\left(\int_0^4 (DX^T(t)X(t)D^T)\,dt\right)A_j'$$

$$S.to:\quad (X(t)B^TD^T)A_j-(X(t)D^T)A_j'=0,$$
$$X(0)D^TA_j=0,$$
$$X(4)D^TA_j=1.$$

Here, since the matrices A(t) and B(t) are independent from variable (t) and are constant matrices, we can assume the condition

$$(X(t)B^T D^T)A_j-(X(t)D^T)A_j'=0$$

as, $(B^T D^T)A_j-D^T A_j'=0$. In this case,

the unknowns of A_j and A_j' would be constant numbers. If the matrices A(t) and B(t) are some functions of (t), then the unknowns will also be the same. By solving the above problem, the solutions

$$D^T A_j=[0,-0.75,0.25,0,0,0]^T,$$
$$D^T A_j'=[-0.75,0.5,0,0,0,0]^T,$$

are in hand, such that by adding them in the relation

$x(t)=X(t)D^T A_j$, $u(t)=X(t)D^T A_j'$

we have, $x(t)=(t^2-3t)/4$ and $u(t)=(2t-3)/4$. This solution is equal to the analytic answer of the problem.

Example 2: This example is adapted from (Yari et al., 2017):

$$\min\quad Z=\int_0^1 (u^2(t)+x^2(t))dt$$
$$S.\ to:\quad \dot{x}(t)=u(t),$$
$$x(0)=0,\quad \dot{x}(1)=0,$$
$$x(1)\quad is\ indefinite$$

Let N=5, using the presented method, we obtain

$D^TA_j=[1,-0.7616,0.4996,-0.1250,0.0378,-0.0027]^T$
$D^TA_j'=[-0.7616,0.9992,-0.3749,0.1511,-0.0137,0]^T$
$Z^*=0.7615941560$

The analytical solution is (Yari et al, 2017),

$$u^*(t)=-\frac{sinh(1-t)}{cosh(1)},\quad x^*(t)=\frac{cosh(1-t)}{cosh(1)}$$

and $Z^*=0.7615941559.$

In Fig. 1 and 2, the analytic solution of the problem with the continuous line and the approximate solution obtained by the new method with * are plotted. As can be seen, the solutions are equal in interval [0,1].

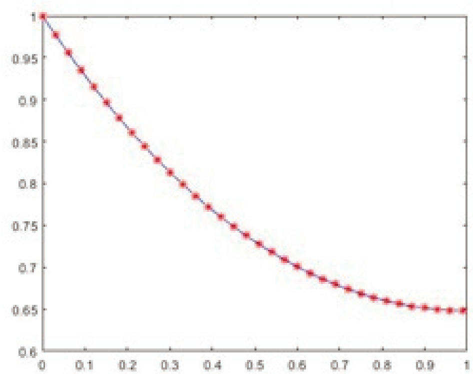

Fig. 1. Plots of the exact and approximated state function for Example (2).

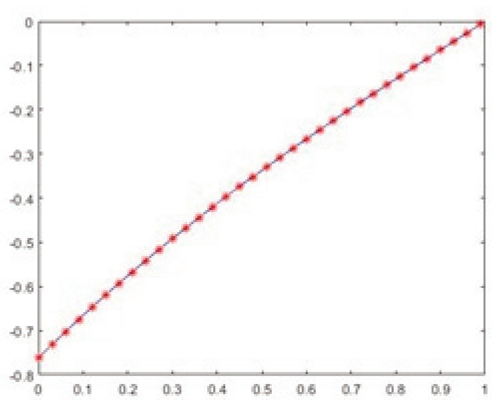

Fig.2. Plots of the exact and approximated control functions for Example (2).

The unknown state and control variables have been approximated by 6 points and in each case, L^2 error has been calculated using formula (Alimorad & Fakharzadeh, 2017):

$$L^2 error = \|x_{opt} - x_{analytical}\|_2 = \sqrt{\sum_{i=1}^{n} (x_{i_{opt}} - x_{i_{analytical}})^2},$$

and L^∞ error has been calculated using formula

$$L^\infty error = \|x_{opt} - x_{analytical}\|_\infty = max \mid x_{i_{opt}} - x_{i_{analytical}} \mid.$$

Tables 1 and 2 show the values of x_{opt}, $x_{analytical}$ and u_{opt}, $u_{analytical}$ at the same t, respectively. they also display the mentioned errors for this case.

Example 3: Find the minimum of the functional (Yari et al., 2017):

$$\min \quad Z = \int_0^T (u^2(t) + x^2(t))dt$$

$$S.\ to: \quad \dot{x}(t) + x(t) = u(t),$$

$$x(0) = x_0, \quad x(T) = 0,$$

$$T \ is\ indefinite$$

The exact solution is:

$$x^*(t) = \frac{x_0 \sinh\left(\sqrt{2}(T-t)\right)}{\sinh(\sqrt{2}T)}$$

$$u^*(t) = \frac{x_0 \sinh\left(\sqrt{2}(T-t)\right) - \sqrt{2}\cosh\left(\sqrt{2}(T-t)\right)}{\sinh(\sqrt{2}T)}$$

$$Z^* = x_0^2(0.5918916999)T$$

for N=3, we obtain

$$D^T A_j = \left[x_0, \frac{x_0}{T}(-1.\overline{10}), \frac{x_0}{T^2}(0.89\overline{39}), \frac{x_0}{T^3}(-0.31\overline{81})\right]$$

$$D^T A_j' = [x_0(0.\overline{57}), \frac{x_0}{T}(0.\overline{21}), \frac{x_0}{T^2}(-0.\overline{06}), \frac{x_0}{T^3}(-0.31\overline{81})]$$

$$Z^* = (0.59\overline{19})T x_0^2$$

We compare our computational solution obtained by the present method with the exact solutions in Table 3. Table 4 shows the exact and approximate values of Z for N=3 as well as the mentioned errors for this case. In the above examples, the results are completely acceptable and satisfactory compared to the mentioned references.

CONCLUSION

This paper proposed a novel and practical approach for obtaining the solution to optimal control problems. The shifted chebyshev polynomials were employed. Several exampls were used to show the applicability and efficiency of the presented method. Compared with other methods, this approach is more practical since the results are obtained by solving one nonlinear programming problem.

Table 1: The values of x_{opt} and $x_{analytical}$ at the same t, for Example (2).

t	x_{opt}	$x_{analytical}$	L^2-*error*	L^∞-*error*
0	1.0000000000	0.9999874504	9.9999999999e-05	5.3859114768e-05
0.2	0.8667236160	0.8667195556		
0.4	0.7682360320	0.7682361598		
0.6	0.7005849280	0.7005847785		
0.8	0.6610621440	0.6610503244		
1	0.6481000000	0.6480461408		

Table 2: The values of u_{opt} and $u_{analytical}$ at the same t, for Example (2).

t	u_{opt}	$u_{analytical}$	L^2-*error*	L^∞-*error*
0	-0.7616000000	-0.7615845983	1.1873406488e-04	9.9999999999e-05
0.2	-0.5755691200	-0.5755336544		
0.4	-0.4125843200	-0.4125808969		
0.6	-0.2661819200	-0.2661864595		
0.8	-0.1304243200	-0.1304750194		
1	0.0001000000	0		

Table 3: The exact and approximate values of Z for N=3, for Example (3).

T	**Exact**	**Presented Method**
1	0.5918916999	0.5919191919
5	2.9594584995	2.9595959595
15	8.8783748328	8.8787878787

Table 4: The values of z_{opt} and $z_{analytical}$ at the same T and x_0=1.

T	**Exact**	**Presented Method**	L^2-*error*	L^∞-*error*
1	0.5918916999	0.5919191919	0.0074347175	2.7492019191e-04
2	1.1837833998	1.1838383838		
3	1.7756750997	1.7757575757		
4	2.3675667996	2.3676767676		
5	2.9594584995	2.9595959595		
6	3.5513501994	3.5515151515		
7	4.1432418993	4.1434343434		
8	4.7351335992	4.7353535353		
9	5.3270252991	5.3272727272		
10	5.9189169990	5.9191919191		

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