Numerical Optimal Control of The Wave Equation

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Abstract
   In this paper, we present a spectral method for approximating the boundary optimal control problems of a well-known wave equation by the linear optimal control problems. The method is based upon constructing the Mth degree interpolation polynomials, using Chebyshevs nodes, to approximate the wave equation. Necessary conditions for optimal control functions are obtained by using the Pontryagin's maximum principle. Moreover, the control parameterization enhancing technique (CPET) is used to obtain the piecewise constant sub-optimal control functions. Finally, the efficiency of the proposed method is confirmed by a numerical example.

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INTRODUCTION

Consider a one-dimensional wave equation

$$u_{tt}(z,t) = \alpha^2 u_{zz}(z,t), \quad (z,t) \in (0,\ell) \times (0,\tau), \tag{1}$$

with the initial conditions,

$$u(z,0) = u_1(z), \quad u_t(z,0) = u_2(z), \quad z \in (0,\ell), \tag{2}$$

and the boundary conditions,

$$u(0,t) = \upsilon_2(t), \quad u(\ell,t) = \upsilon_1(t), \quad t \in (0,\tau), \tag{3}$$

and the end conditions,

$$u(z,\tau) = s_1(z), \quad u_t(z,\tau) = s_2(z), \quad z \in (0,\ell), \tag{4}$$

where υ_1 and υ_2 are measurable control functions which are assumed to be constrained as

$$\kappa_j \leq \upsilon_j(t) \leq \sigma_j, \quad j = 1,2, \quad t \in (0,\tau) \tag{5}$$

Consider the problem of minimizing the functional

$$J(\tau, \upsilon_1, \upsilon_2) = \int_0^\tau |\upsilon_1(t)|^p + |\upsilon_2(t)|^p \, dt, \tag{6}$$

subject to the constraints (1)-(5).

Optimal control problems of linear distributed parameter systems have been studied by many authors (Kamyad et al., 1991; Sakawa et al., 1999). A full discretization method based on the appropriate finite differences is used to solve a special case of this problem by Gerdts et al. (2008), where the functions s_1 and s_2 in the end conditions (4) are taken to be zero, the final time τ is fixed and there are no constraints (5) on the control functions. Therefore, the problem considered in this paper is more general than the problem considered by Gerdts et al. (2008). Zarei and Bahrmand (2014) obtained an explicit solution for equations (1)-(3) and proposed a numerical method to solve a multi-objective optimal control of the wave equation. Optimal boundary control of the wave equation is studied by Farahi et al.

(1996), using a measure theoretical approach. Moreover, the optimal control problems for the wave equation are studied by Manita (2008).

The spectral methods as an effective tool, have been used to solve the optimal control problems for lumped (Elnagar and Razzaghi, 1997) and, in recent years, distributed parameter systems (Chen et al., 2011(a); 2011(b); Zarei, 2015). In this paper, we use a spectral method to minimize the functional (6) subject to the constraints (1)-(5). The method is outlined in the next section. In section III, the distributed parameter system (1)-(4) is approximated by a lumped parameter system and the necessary conditions for optimal controls are derived, when $p = 1, 2$. The control parameterization enhancing technique (CPET) to obtain the sub-optimal control functions in piecewise constant form is outlined in section IV. Section V includes the numerical results. The last section is devoted to the conclusion.

THE PROPOSED METHOD

Let $T_M(x)$, $x \in [-1,1]$ denotes the Chebyshev polynomial of degree M, then the collocation points

$$x_j = \cos\left(\frac{\pi j}{M}\right), \quad j = 0, 1, 2, \ldots, M, \tag{7}$$

are the zeros of $(1-x^2)\dot{T}_M(x)$, $x \in [-1,1]$ The Mth degree interpolation polynomials to $u(x,t)$ is given by

$$u^M(x,t) \simeq \sum_{j=0}^{M} a_j(t)\, \varphi_j(x), \tag{8}$$

where φ_j s are the Lagrange polynomials that

$$\varphi_j(x_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

Let $D = (d_{k\ell})$ denotes the Chebyshev derivative matrix defined by

$$d_{k\ell} = \begin{cases} \dfrac{c_k(-1)^{k+\ell}}{c_\ell(t_k - t_\ell)} & \text{if } k \neq \ell \\[2mm] \dfrac{2M^2+1}{6} & \text{if } k = \ell = 0 \\[2mm] -\dfrac{2M^2+1}{6} & \text{if } k = \ell = M \\[2mm] -\dfrac{t_\ell}{2(1-t_\ell^2)} & \text{if } 1 \leq k = \ell \leq M-1, \end{cases}$$

where, $c_0 = c_M = 2$, $c_k = 1$, $1 \le k \le M - 1$.

Then $D^2 = \left(d_{kj}^{(2)} \right)$ is the second order Chebyshev derivative matrix which is used to compute the values of the function $u_{xx}^M(x,t)$ at the Chebyshev nodes x_k, $k = 0,1,2,...,M$ as

$$u_{xx}^M(x_k,t) = \sum_{j=0}^{M} d_{kj}^{(2)} a_j(t) \qquad (9)$$

In the next section we approximate the equations (1)-(4) by a linear control system, applying the proposed method.

OPTIMAL CONTROL FORMULATION

In order to use the Chebyshev nodes we introduce the transformation $z = \dfrac{\ell}{2}(1+x)$. In this way the equations (1)-(4) convert to

$$u_{tt}(x,t) = \beta^2 u_{xx}(x,t), \ (x,t) \in (-1,1) \times (0,\tau), \qquad (10)$$

with the initial conditions,

$$u(x,0) = u_1(x), \quad u_t(x,0) = u_2(x), \quad x \in (-1,1), \qquad (11)$$

and the end conditions,

$$u(x,\tau) = s_1(x), \quad u_t(x,\tau) = s_2(x), \quad x \in (-1,1), \qquad (12)$$

and the boundary conditions,

$$u(-1,t) = v_2(t), \quad u(1,t) = v_1(t), \quad t \in (0,\tau), \qquad (13)$$

where $\beta = \dfrac{2}{\ell}\alpha$. We approximate the equations (10)- (12) by

$$u_{tt}(x_k,t) = \beta^2 u_{xx}(x_k,t), \ t \in (0,\tau), \qquad (14)$$

$$u(x_k,0) = u_1(x_k), \quad u_t(x_k,0) = u_2(x_k), \qquad (15)$$

$$u(x_k,\tau) = s_1(x_k), \quad u_t(x_k,\tau) = s_2(x_k),$$
$$k = 1,2,...,M-1 \qquad (16)$$

Substituting the equations (8) and (9) into the equations (14)-(16), we get a system of linear second order differential equations as

$$\ddot{a}_k(t) = \sum_{j=0}^{M} \beta^2 d_{kj}^{(2)} a_j(t), \ t \in (0,\tau), \qquad (17)$$

$$a_k(0) = u_1(x_k), \ \dot{a}_k(0) = u_2(x_k), \qquad (18)$$

$$a_k(\tau) = s_1(x_k), \ \dot{a}_k(\tau) = s_2(x_k),$$
$$k = 1,2,...,M-1. \qquad (19)$$

Moreover, from the equation (13) $a_0(t) = v_1(t)$, $a_M(t) = v_2(t)$, $t \in (0,\tau)$

we have ; therefore, the equations (17)-(19) can be written in the matrix form as

$$\ddot{a}(t) = Ca(t) + Bv(t), \qquad (20)$$

$$a(0) = a_i, \ \dot{a}(0) = a_i', \qquad (21)$$

$$a(\tau) = a_f, \ \dot{a}(\tau) = a_f', \qquad (22)$$

where,

$$C = \beta^2 \begin{bmatrix} d_{11}^{(2)} & \cdots & d_{1M-1}^{(2)} \\ \vdots & \ddots & \vdots \\ d_{M-11}^{(2)} & \cdots & d_{M-1M-1}^{(2)} \end{bmatrix}, \ B = \beta^2 \begin{bmatrix} d_{10}^{(2)} & d_{1M}^{(2)} \\ \vdots & \vdots \\ d_{M-10}^{(2)} & d_{M-1M}^{(2)} \end{bmatrix},$$

$a(t) = (a_1(t),...,a_{M-1}(t))^T$, $v(t) = (v_1(t),v_2(t))^T$, $a_i = (u_1(x_1),..., u_1(x_{M-1}))^T$, $= (u_2(x_1),..., u_2(x_{M-1}))^T$, $a_f = (s_1(x_1),..., s_1(x_{M-1}))^T$ and $d_f = (s_2(x_1),..., s_2(x_{M-1}))^T$.

Setting the control system (20)-(22) can be written as the following first order linear control system

$$\dot{Y}(t) = EY(t) + Fv(t), \qquad (23)$$

$$Y(0) = \begin{bmatrix} a_i' \\ a_i \end{bmatrix}, \ Y(\tau) = \begin{bmatrix} a_f' \\ a_f \end{bmatrix} \qquad (24)$$

where,

$$Y(t) = \begin{bmatrix} b(t) \\ a(t) \end{bmatrix}, \ E = \begin{bmatrix} 0 & C \\ I & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

Minimum energy Problem

As a minimum energy problem (MEP), we first consider the objective functional (6) with $p=2$. In

order to obtain the necessary conditions for the optimal controls $v^*(t) = (v_1^*(t), v_2^*(t))^T$ and the optimal time τ^* we define the Hamiltonian as

$$H(a,b,u_1,v_2,\lambda_1,\lambda_2) = v_1^2 + v_2^2 + \lambda_1(Ca + Bv) + \lambda_2 b. \tag{25}$$

According to the Pontryagin's maximum principle (Kirk, 1970), we should have

$$H(a^*,b^*,u_1^*,v_2^*,\lambda_1^*,\lambda_2^*) \le H(a^*,b^*,u_1,v_2,\lambda_1^*,\lambda_2^*), \tag{26}$$

for all control functions v_1 and v_2 satisfying the constraint (5), where $b^*(t)$ and $a^*(t)$ are the solutions of (23)-(24) corresponding to $v^*(t)$ and λ_1^* and λ_2^* are the costate variables which satisfy

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial b} = -\lambda_2, \tag{27}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial a} = -\lambda_1 C \tag{28}$$

The partial derivatives of the Hamiltonian with respect to $v_j, j=1,2$ are zero. Since $\frac{\partial H}{\partial v_j} = 2v_j + \lambda_1 B_j = 0$, we have $v_j(t) = -\frac{1}{2}\lambda_1^*(t)B_j$ where $B_j, j=1,2$ the jth column of B. is to determine an explicit expression for the optimal controls, we consider the following three cases. (i) If

$\kappa_j \le -\frac{1}{2}\lambda_1^*(t)B_j \le \sigma_j$, then $v_j^*(t) = -\frac{1}{2}\lambda_1^*(t)B_j$.
(ii) If $-\frac{1}{2}\lambda_1^*(t)B_j > \sigma_j$, then $v_j^*(t) = \sigma_j$.
(iii) If $-\frac{1}{2}\lambda_1^*(t)B_j < \kappa_j$, then $v_j^*(t) = \kappa_j$.
Combining these three cases gives

$$v_j^*(t) = \max\{\kappa_j, \min\{\sigma_j, -\frac{1}{2}\lambda_1^*(t)B_j\}\}, j=1,2. \tag{29}$$

Minimum fuel Problem
Setting $p=1$ in functional (6), leads to the minimum fuel problem (MFP). Assume that the bounds on the control functions in (5) are such that $\kappa_j < 0$ and $\sigma_j > 0$. The Hamiltonian function for this problem is $H(a,b,v_1,v_2,\lambda_1,\lambda_2) =$

$$|v_1| + |v_2| + \lambda_1(Ca + Bv) + \lambda_2 b. \tag{30}$$

According to the Pontryagin's maximum principle, the optimal controls $v^*(t) = (v_1^*(t), v_2^*(t))^T$ and the corresponding states $b^*(t)$ and $a^*(t)$ satisfy

$$H(a^*,b^*,v_1^*,v_2^*,\lambda_1^*,\lambda_2^*) \le H(a^*,b^*,v_1,v_2,\lambda_1^*,\lambda_2^*), \tag{31}$$

for all control functions v_1 and v_2 satisfying the constraint (5), where λ_1^* and λ_2^* are the costate variables which satisfy the equations (27)-(28). From (31), it is concluded that

$$v_j^*(t) = \begin{cases} \sigma_j & if\ \lambda_1^*(t)B_j \le -1 \\ 0 & if\ -1 < \lambda_1^*(t)B_j < 1 \\ \kappa_j & if\ \lambda_1^*(t)B_j \ge 1. \end{cases} \tag{32}$$

Moreover, since H is explicitly independent of t and the final time τ is free, we also know that (See Kirk (1970) for more details)

$$|v_1^*(t)| + |v_2^*(t)| + \lambda_1^*(t)(Ca^*(t) + Bv^*(t)) + \lambda_2^*(t)b^*(t) = 0, \ t \in [0,\tau] \tag{33}$$

According to the equation (32), a singular interval to exist it is necessary that $\lambda_1^*(t)B_j$ is to be either +1 or -1 during a time interval $[t_1,t_2]$. This implies that $\lambda_1^*(t)B_j=0, t \in [t_1,t_2]$. Hence, from the equation (27) we have $\lambda_2^*(t)B_j=0, t\in[t_1,t_2]$. Clearly, this condition occurs if $\lambda_2^*(t)=0$ for $t\in[t_1,t_2]$. But, this cannot happen, because from the equations (27)-(28) we obtain $\lambda_1^*(t)=\lambda_2^*(t)=0$, $t\in[0,\tau]$; hence, from the equation (33) we obtain $v_1^*(t)=v_2^*(t)=0$, for $t\in[0,\tau]$, which means that the optimal control functions do not affect the system at all. We have a similar situation when $B_j=0$. So, we consider the case that $\lambda_2^*(t)$ is nonzero for any $t\in[t_1,t_2]$. Form the equations (27)-(28) we have $\ddot{\lambda}_2^*(t) = -\lambda_2^*(t)C$, and hence twice differentiating the equation $\lambda_2^*(t)B_j=0$ gives $\lambda_2^*(t)CB_j=0$. Similarly, twice differentiating the equation $\lambda_2^*(t)CB_j=0$ gives $\lambda_2^*(t)C^2B_j=0$. Continuing this pattern gives

$$\lambda_2^*(t)[B_j | CB_j | ... | C^{M-1}B_j] = 0, \ t \in [t_1,t_2] \tag{34}$$

Obviously, if the equation (34) is to be satisfied, the matrix

$$[B_j \mid CB_j \mid ... \mid C^{M-1}B_j] \qquad (35)$$

must be singular. Therefore, if the matrices $[B_j \mid CB_j \mid \cdots \mid C^{M-1}B_j]$, j=1, 2 are nonsingular, then there are no singular intervals and the optimal controls are completely determined by (32).

We note that the optimality systems of the MEP and the MFP contain the state system (23) with the boundary conditions (24), the costate system (27)-(28), together with the expressions (29) and (32) for the control functions and a complicated nonlinear equation for the optimal time τ^* as

$$H(a^*(\tau^*), b^*(\tau^*), \upsilon_1^*(\tau^*), \upsilon_2^*(\tau^*), \lambda_1^*(\tau^*), \lambda_2^*(\tau^*)) = 0,$$

$$(36)$$

where, H is the corresponding Hamiltonian function. Due to difficulties in solving the optimality systems of the MEP and the MFP, in the next section we directly use the control parameterization enhancing technique (CPET) to optimize the functional (6) subject to the constraints (5) and (23)-(24). The CPET introduced by Lee et al. (1997) maps all the switching points of the original problem onto the set of integers, so that the time of the switching points can be accurately determined.

CONTROL PARAMETRIZASION ENHANCING TECHNIQUE

Using the CPET, an optimal control problem can be approximated by an optimal parameter selection problem which can be solved efficiently by the software package MISER3 (Jennings et al., 1997).

CPET1 for MEP

In order solve the MEP we set $\mho = [\kappa_1, \sigma_1] \times [\kappa_2, \sigma_2]$ and we assume that the control function υ has a piecewise constant form as:

$$\upsilon^N(t) = \sum_{k=1}^{N} \alpha_k \chi_{[t_{k-1}, t_k)}(t) \qquad (37)$$

where N is the number of control subintervals and $\chi_{[t_{k-1}, t_k)}$ is the characteristic function for the interval $[t_{k-1}, t_k)$. We note that $\alpha_k \in \mho$ and $t_k, k = 1, 2, ..., N$ the decision variables characterizing υ

where, $0 = t_0 \leq t_1 \leq \cdots \leq t_N = \tau$. Substituting the function (37) into (6) and (23)-(24) gives the following optimal control problem:

Minimize $\hat{J}(\upsilon^N) = \sum_{k=1}^{N} \alpha_k^2 \int_{t_{k-1}}^{t_k} \chi_{[t_{k-1}, t_k)}(t)dt \qquad (38)$

Subject to

$$Y^.(t) = E Y(t) + F \alpha_k, \alpha_k \in \mho, t \in [t_{k-1}, t_k), \qquad (39)$$

$$Y(0) = \begin{bmatrix} a_i' \\ a_i \end{bmatrix}, Y(\tau) = \begin{bmatrix} a_f' \\ a_f \end{bmatrix}, k = 1, ..., N. \qquad (40)$$

It has been proved that $\lim_{N \to \infty} \hat{J}(\upsilon^{N*}) = J(\tau^*, \upsilon^*)$ where υ^{N*} is the optimal solution of (38)-(40) and (τ^*, υ^*) is the optimal solution of MEP satisfying the optimality conditions (29) and (36) (See chapter 6 of Teo et al. (1999)). Now, we relate the new time variable $t \in [0, \tau)$ to the original time variable through the ordinary differential equation

$$\frac{dt}{ds} = \vartheta(s), t(0) = 0, \qquad (41)$$

where $\vartheta : [0, M) \to R$ is a new nonnegative piecewise constant function defined by

$$\vartheta(s) = \sum_{k=1}^{N} \beta_k \chi_{[k-1, k)}(s) \qquad (42)$$

with $\beta_k = t_k - t_{k-1}$; hence, $t_k = \sum_{j=1}^{k} \beta_j$, k=1,...,M. Let $\hat{Y}(s) = Y(t(s))$. Under the CPET transformation, the problem (38)-(40) becomes:

Minimize $\hat{J}(\upsilon^N) = \sum_{k=1}^{N} \beta_k \alpha_k^2 \qquad (43)$

Subject to

$$\frac{d\hat{Y}(s)}{ds} = \beta_k (E \hat{Y}(s) + F \alpha_k), s \in [k-1, k), \qquad (44)$$

$$\hat{Y}(0) = \begin{bmatrix} a_i' \\ a_i \end{bmatrix}, \hat{Y}(N) = \begin{bmatrix} a_f' \\ a_f \end{bmatrix}, \alpha_k \in \mho, \beta_k \geq 0, k = 1, .. \qquad (45)$$

CPET2 for MFP

For the MFP, we use the CPET for optimal discrete-valued control problems introduced by Lee et al. (1999). To this end, we assume that the con-

trol variable can take values in $\mho = \{\upsilon^1, \upsilon^2, ..., \upsilon^m\}$ According to the necessary conditions for the nonsingular optimal controls given by (32, we have $\mho = \{(\upsilon_1, \upsilon_2)^T \mid \upsilon_j \in \{\kappa_j, 0, \sigma_j\}, j = 1, 2\}$ Hence, $m=9$. In this technique, we construct a set $\{\alpha_i\}_{i=1}^N$, $N = nm$ where $\alpha_1 = \upsilon^1, ..., \alpha_m = \upsilon^m$, $\alpha_{m+1} = \upsilon^1, ..., \alpha_{2m} = \upsilon^m$, ..., $\alpha_{(n-1)m+1} = \upsilon^1$, ..., $\alpha_{nm} = \upsilon^m$ and we seek a control function in the form of (37). Therefore, in this case is now determined only by t_j, $j= 1,2,...N$. Moreover, we use the time scale control (42) to relate the new time variable $s \in [0, M)$ to the original time variable through the ordinary differential equation (41) where $\beta_k = t_k - t_{k-1}$. Following a procedure similar to CPET1, we obtain an optimal parameter selection problem as

Minimize $\qquad \hat{J}(\upsilon^N) = \sum_{k=1}^N \beta_k \alpha_k^2 \qquad (46)$
Subject to

$$\frac{d\hat{Y}(s)}{ds} = \beta_k (E\,\hat{Y}(s) + F\,\alpha_k), \; s \in [k-1, k), \qquad (47)$$

$$\hat{Y}(0) = \begin{bmatrix} a'_i \\ a_i \end{bmatrix}, \; \hat{Y}(N) = \begin{bmatrix} a'_f \\ a_f \end{bmatrix}, \beta_k \geq 0, \, k = 1, ..., N \qquad (48)$$

Note that the unknowns in problem (46)-(48)

are only β_k, $k=1,...,N$, while in problem (38)-(40) α_k, $k=1,...,N$ are also unknowns. Obviously, the CPETs introduced in this paper lead to the optimal parameter selection problems, which can be solved by the optimal control software MISER 3.

NUMERICAL RESULTS

As a numerical example, we consider a problem with $\ell=100$, $\alpha=1$, $u_1(z) = \frac{4}{\ell^2}\left(z - \frac{\ell}{2}\right)^2$,

$u_2(Z)=0$, $s_1(z) = \sin\left(\frac{2\pi z}{\ell}\right)$, $s_2(Z)=0$, $k_j=-1$, $\sigma_j=1$, $j=1,2$.

In our implementation we set M=10. Fig. 1 and Fig. 2, respectively, show the optimal boundaries υ_j^*, $j= 1,2$ and the corresponding optimal state $u^M(x,t)$ for the MEP obtained by CPET1 with $N=10$. The optimal value for objective functional (6) obtained by CPET1 is $J(\tau^*, \upsilon_1^*, \upsilon_2^*)=23.3801$ which is corresponding to $\tau^*=147.1297$. The optimal controls obtained by the CPET2 with $n=5$ and the corresponding optimal state $u^M(x,t)$ are depicted in Fig. 3 and Fig. 4, respectively. Moreover, the value of objective functional (6) resulting by CPET2 is $J(\tau^*, \upsilon_1^*, \upsilon_2^*)=58.1754$ which is corresponding to $\tau^*=50.8477$.
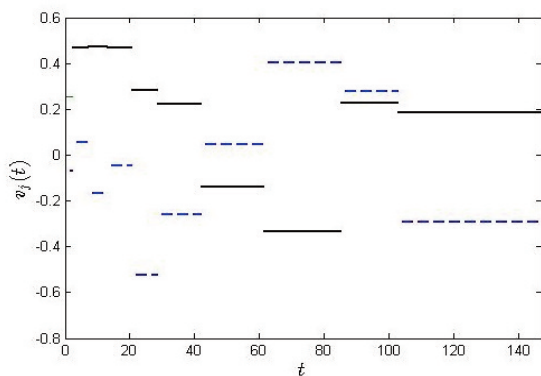

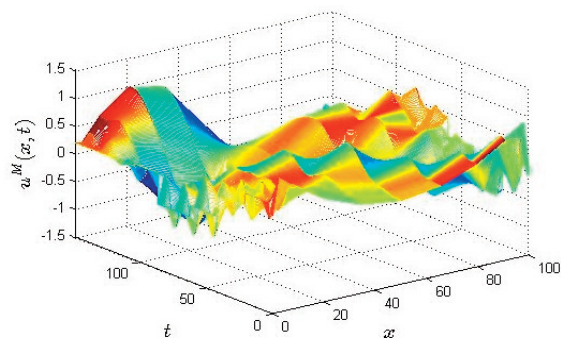Fig. 1. Control functions v_1(- - -) and v_2(−) for MEP obtained by CPET1.

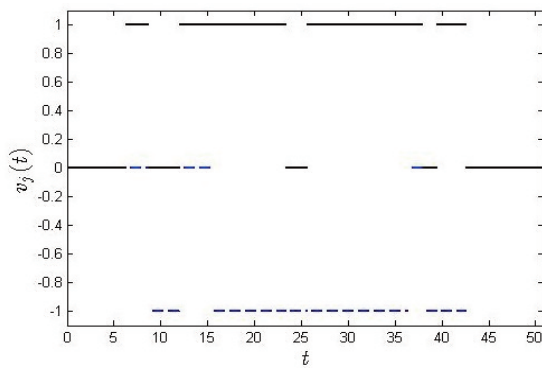Fig. 2. $u^M(x,t)$ for MEP obtained by CPET1.

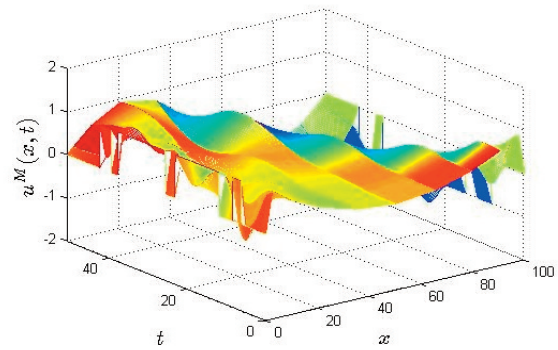Fig. 1. Control functions v_1(- - -) and v_2(–) for MEP obtained by CPET2.

Fig. 2. $u^M(x,t)$ for MEP obtained by CPET2.

CONCLUSION

In this paper, we approximated the boundary optimal control problem of the wave equation by a linear optimal control problem using a spectral method. When the objective functions indicate the minimum energy problem and the minimum fuel problem, the necessary optimality conditions are obtained. Moreover, the control parameterization enhancing technique is used to obtain the sub-optimal control functions in piecewise constant form. The numerical results confirmed the capability of the proposed method for solving the boundary optimal control problem of the one-dimensional wave equation.

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