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Merging DMUs Based on of the Idea Inverse DEA

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Abstract

In this paper, we propose a novel method using multiple-objective programming problems to answer the following question: if among a group of decision making units (DMUs), a subset of DMUs are required to merge and form a new DMU with specific input/output levels and a predefined efficiency target, how much should be the outputs/inputs of the merged DMU? This question answered according to the concept of inverse DEA. Sufficient conditions are established for input/output-estimation of the merged DMU using multiple-objective programming problems. A numerical example with real data is presented to illustrate the goals of this paper.

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INTRODUCTION

One of the efficient mathematical tools for evaluate the performance of the DMUs is the DEA technique which proposed by Charnes et al. (1978) and developed by many scholars, see e.g. (Banker et al., 1984; Cook & Seiford, 2009; Emrouznejad & Tavana, 2014) for some reviews. DEA proposes a technique to estimate the relative efficiency of a group of DMUs with multiple inputs and outputs such as bank branches.

Zhang and Cui (1999) presented the idea of inverse DEA. In fact, inverse DEA proposed to answer this question: among a set of DMUs, if decision maker change inputs/outputs a particular DMU under preserving the efficiency level, how much should the outputs/inputs of the DMU change? This question answered using multiple-objective linear Programming (MOLP) problems by Hadi-Vencheh et al. (2006), though its question in a special case (increase inputs/outputs) was studied by Wei et al. (2000). This question studied under improving efficiency index by Jahanshahloo et al. (2004a; 2004b). In the inverse DEA filed, the problem of simultaneous estimation of input- output levels are proposed by Jahanshahloo et al. (2014) and Ghobadi (2017). Inverse DEA has been used and developed by many scholars, see e.g. (Gattoufi et al., 2014; Ghobadi et al., 2014; Jahanshahloo et al., 2015).

Inverse DEA is important from both theoretical and practically points of view, because this technique can be used in the different framework, including preserve (improve) efficiency values (Lertworasirikul et al., 2011), resource allocation (Hadi-Vencheh et al., 2006), and firms restructuring (Amin et al., In Press). In this direction, the idea of the inverse DEA used to merging the banks by Gattoufi et al. (2014). In fact, the inverse DEA applied to answer the following question:

Question: If among a set of DMUs, a subset of DMUs are required to merge and form a new DMU with specific input/output levels and a predetermined efficiency target, how much should be the outputs/inputs of the merged DMU?

A technique was proposed to answer Question using mathematical programming by Gattoufi et al. (2014).

In this paper, we proposed a novel method to answer the above question using multiple-objective programming problems. Sufficient conditions are proposed for input/output-estimation of the merged DMU using multiple-objective programming problems. This method, unlike other proposed method (Gattoufi et al., 2014), decreases the number of the variables of the model strongly, and this decreases the computational complexity. A numerical example with real data is provided to illustrate the goals of this paper.

The structure of the paper is as follows: Section 2 gives some preliminaries from DEA. The problem of the merging DMUs investigated in section 3. This section proposes a new method to solving the problem of the merging DMUs. Sufficient conditions are proposed for input/output-estimation of the merged DMU. An example with real data provided in section 4. Section 5 gives a brief conclusion and directions for future research.

PRELIMINARIES FROM DEA

Suppose that there exist a set of n DMUs, $\{DMU_j : j=1\dots, n\}$ which DMU_j $(j=1,\dots,n)$ uses multiple positive input $x_j=(x_{1j}, x_{2j},\dots,x_{mj})$ to produce multiple positive output $y_j=(y_{1j}, y_{2j}, \dots, y_{sj})$. The following model is considered to estimate the relative efficiency of the unit under assessment DMU_o, $o=\{1, 2, \dots, n\}$, as follows:

$$\begin{aligned} \theta_o^* = \min \quad & \theta \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io}, \quad i=1,\dots,m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \quad r=1,\dots,s, \\ & \lambda \in \Omega, \end{aligned} \tag{1}$$

where:

$$\Omega = \left\{ \lambda \,\middle|\, \lambda = (\lambda_1,\dots,\lambda_n), \sigma_1\left(\sum_{j=1}^{n}\lambda_j + \sigma_2(-1)^{\sigma_3}\nu\right) = \sigma_1, \nu \ge 0, \lambda_j \ge 0, j=1,\dots,n \right\}.$$

Here σ_1, σ_2, and σ_3 are parameters with 0-1 values. It is obvious that:

(i) If $\sigma_1=0$, then model (1) is under a constant returns to scale (CRS) assumption of the production technology.

(ii) If $\sigma_1=1$ and σ_2=1"," then model (1) is under a variable returns to scale (VRS) assumption of the production technology.

(iii) If $\sigma_1 = \sigma_2 = 1$ and $\sigma_3 = 0$, then model (1) is under a non-increasing returns to scale (NIRS) assumption of the production technology.
(iv) If $\sigma_1 = \sigma_2 = \sigma_3 = 1$, then model (1) is under a non-decreasing returns to scale (NDRS) assumption of the production technology.

θ_o^* in Model (1) is called the input-oriented efficiency score of DMU_o. It is obvious that $\theta_o^* \leq 1$. If $\theta_o^* = 1$, then DMU_o is called input-oriented weakly efficient.

Model (1) is called an input-oriented DEA model. The output-oriented version of this model is as follows:

$$
\begin{aligned}
\varphi_o^* = \max \quad & \varphi \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io}, \qquad i = 1,2,\ldots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \varphi y_{ro}, \qquad r = 1,2,\ldots,s, \\
& \lambda \in \Omega.
\end{aligned} \tag{2}
$$

φ_o^* in Model (2) is called the output-oriented efficiency score of DMU_o. It is obvious that $\varphi_o^* \leq 1$. In addition, DMU_o is called output-oriented weakly efficient if $\varphi_o^* = 1$.

Remark 1.2 If $\theta_o^* < 1$ or $\phi_o^* > 1$, which is the DMU_o is inefficient, then DMU_M can be presented by the efficient DMUs. Therefore, the corresponding λ_o will be zero in optimality ($\lambda_o^* = 0$).

MERGING DMUs

In this section a new method suggested for merger DMUs using the Inverse DEA approach and MOP problems. This method allows determining the levels of inputs and outputs for a single merged DMU, following a merger between at least two DMUs.

Let us to assume that there is a set of n DMUs in which DMU_j, $j \in J = \{1, 2, \ldots, n\}$, uses m inputs x_{ij} in order to produce s outputs y_{rj}, for all $i = 1, 2, \ldots,$ m and $r = 1, 2, \ldots,$ s. Assume that the all inputs and outputs are positive. Suppose that the set DMUs, $J = \{1, 2, \ldots, n\}$, is divided into two subsets Π and Γ, where Π, Γ⊂J, $\Pi \cup \Gamma = J$, and $\Pi \cap \Gamma = \phi$. Assume that DMUs, $j \in \Pi$ are merged and are looking to create a new DMU, namely- DMU_M. In addition, suppose that $\bar{\theta}_m$ is a predetermined target for efficiency of the DMU_M.

Initially, in order to present suitable patterns to the decision maker to estimate input/output vector DMU_M, the following question is considered:

Question 1. If DMU_j, $j \in \Pi$ are required to merge and form a new unit (DMU_M), in which output vector and predetermined efficiency target of DMU_M are $Y_M = \sum_{j \in \Pi} Y_j$ and $\bar{\theta}_m$, respectively, how much should be input vector of this new DMU?

Note that DMU_M keeps the amount of outputs of all DMU_j; $j \in \Pi$, and looking to find the minimum amount of inputs of these DMUs in order to reach the pre-defined target level. The aim of the Question 1 is estimating the input vector X_M provided that the efficiency index of DMU_M *is* $\bar{\theta}_m$. In other words, the optimal value of the following model is equal $\bar{\theta}_m$.

$$
\begin{aligned}
\theta^* = \min \quad & \theta \\
s.t. \quad & \sum_{j \in \Gamma} \lambda_j x_{ij} + x_{iM} \lambda_M \leq \theta x_{iM}, \qquad i = 1,\ldots,m, \\
& \sum_{j \in \Gamma} \lambda_j y_{rj} + y_{rM} \lambda_M \geq y_{rM}, \qquad r = 1,\ldots,s, \\
& \lambda \in \bar{\Omega},
\end{aligned} \tag{3}
$$

where $y_{rM} = \sum_{(j \in)} y_{rj}$, $r = 1, 2, \ldots, s$ and:

$$
\bar{\Omega} = \left\{ \lambda \,\middle|\, \lambda = (\lambda_j : j \in \Gamma, \lambda_M) \in \mathbb{R}_{\geq 0}^{n-|\Pi|+1}, \right.
\left. \sigma_1 \left(\sum_{j \in \Gamma} \lambda_j + \lambda_M + \sigma_2 (-1)^{\sigma_3} \nu \right) = \sigma_1, \nu \geq 0 \right\}.
$$

To solve Question 1, the following multiple objective non-linear programming (MONLP) problem is considered:

$$
\begin{aligned}
\min \quad & \left(\sum_{j \in \Pi} \alpha_{ij}; \quad i = 1,2,\ldots,m \right) \\
s.t. \quad & \sum_{j \in \Gamma} \lambda_j x_{ij} + \lambda_M \left(\sum_{j \in \Pi} \alpha_{ij} \right) \leq \bar{\theta}_M \left(\sum_{j \in \Pi} \alpha_{ij} \right), \qquad i = 1,2,\ldots,m, \\
& \sum_{j \in \Gamma} \lambda_j y_{rj} + \lambda_M \left(\sum_{j \in \Pi} y_{rj} \right) \geq \sum_{j \in \Pi} y_{rj}, \qquad r = 1,2,\ldots,s, \\
& 0 \leq \sum_{j \in \Pi} \alpha_{ij} \leq \sum_{j \in \Pi} x_{ij}, \qquad i = 1,2,\ldots,m, \\
& \lambda \in \bar{\Omega}.
\end{aligned} \tag{4}
$$

Where $\bar{\theta}_m$ is a predetermined target for efficiency of the merged DMU_M. In the above model, ($\lambda_j : j \in \Gamma$, λ_M, $\alpha_{ij} : i = 1, 2, \ldots, m$, $\forall j \in \Pi$) is the variables vector. Let us to assume that ω_i, $i = 1, 2, \ldots, m$, are important degree of for each of inputs of the merged DMU_M. Therefore, model (4) converted to the following only objective model:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j\in\Pi}\omega_i\alpha_{ij} \\ s.t. \quad & \sum_{j\in\Gamma}\lambda_j x_{ij}+\lambda_M\left(\sum_{j\in\Pi}\alpha_{ij}\right)\le\bar{\theta}_M\left(\sum_{j\in\Pi}\alpha_{ij}\right), \quad i=1,2,\ldots,m, \\ & \sum_{j\in\Gamma}\lambda_j y_{rj}+\lambda_M\left(\sum_{j\in\Pi}y_{rj}\right)\ge\sum_{j\in\Pi}y_{rj}, \quad r=1,2,\ldots,s, \\ & 0\le\sum_{j\in\Pi}\alpha_{ij}\le\sum_{j\in\Pi}x_{ij}, \quad i=1,2,\ldots,m, \\ & \lambda\in\bar{\Omega}. \end{aligned} \tag{5}$$

In the real world, the most common consolidations happen between DMUs to improve their respective performances. Therefore, we can assume that the merging DMUs are inefficient. It is obvious that if $\bar{\theta}_m<1$ or even $\bar{\theta}_m=1$, then DMU_M can be presented by the other efficient DMUs, and so the corresponding λ_M will be zero in optimality ($\lambda_M^*=0$). According to the above discuss, non-linear model (5) could be converted to the following linear model:

$$\begin{aligned} \text{nin} \quad & \sum_{i=1}^{m}\sum_{j\in\Pi}\omega_i\alpha_{ij} \\ s.t. \quad & \sum_{j\in\Gamma}\lambda_j x_{ij}\le\bar{\theta}_M\left(\sum_{j\in\Pi}\alpha_{ij}\right), \quad i=1,2,\ldots,m, \\ & \sum_{j\in\Gamma}\lambda_j y_{rj}\ge\sum_{j\in\Pi}y_{rj}, \quad r=1,2,\ldots,s, \\ & 0\le\sum_{j\in\Pi}\alpha_{ij}\le\sum_{j\in\Pi}x_{ij}, \quad i=1,2,\ldots,m, \\ & \lambda\in\hat{\Omega}, \end{aligned} \tag{6}$$

where

$$\hat{\boldsymbol{\Omega}}=\left\{\lambda\,\middle|\,\lambda=(\lambda_j: j\in\boldsymbol{\Pi})\in\mathbb{R}_{\ge0}^{n-|\boldsymbol{\Pi}|},\sigma_1\left(\sum_{j\in\boldsymbol{\Gamma}}\lambda_j+\sigma_2(-1)^{\sigma_3}\nu\right)=\sigma_1,\nu\ge0\right\}.$$

Remark 1.3 In Model (6), $\sum_{j\in\Pi}\alpha_{ij}$ is unknown. Considering $\alpha i\ M=\sum_{j\in\Pi}\alpha_{ij}$; ($i=1, 2, \ldots, m$) as a new variable in Model (6), the number of variables is strongly reduced, and so computational complexity is reduced.

The following theorem shows how Model (6) can be used for input estimation of DMU_M.

Theorem 1.3 Let DMU_j , $\forall j\in\Pi$ be inefficient. If $\Lambda=(\lambda_j^*: j\in\Gamma, \alpha_{ij}^*: i=1,2, \ldots, m, \forall j\in\Pi)$ is an optimal solution to model (6), then efficiency score DMU_M with the input vector $x_M=\sum_{j\in\Pi}\alpha_j^*$ and output vector $y_M=\sum_{j\in\Pi}y_j$ is equal to $\bar{\theta}_M$.

Proof. It is obvious that $\Delta=((\lambda_j^*: j\in\Gamma, \lambda_M^*=0), \theta=\bar{\theta}_M)$ is a feasible solution to model (3). Therefore, $\theta^*\le\bar{\theta}_M$. By contradiction assume that $\tilde{\Lambda}=((\tilde{\lambda}_j: j\in\Gamma, \tilde{\lambda}_M, \theta^*)$ is an optimal solution to model (4) such that, $\theta^*<\bar{\theta}_M$. Feasibility of $\nabla=(\lambda_j^*, \sum_{j\in\Pi}\alpha_{ij}^*)$ for LP (6), implies

$$\sum_{j\in\Gamma}\lambda_j^* x_{ij}\le\bar{\theta}_M\left(\sum_{j\in\Pi}\alpha_{ij}^*\right)\le\left(\sum_{j\in\Pi}\alpha_{ij}^*\right), i=1,2,\ldots,m, \tag{7}$$

$$\sum_{j\in\Gamma}\lambda_j^* y_{rj}\ge\left(\sum_{j\in\Pi}y_{rj}\right), \quad r=1,2,\ldots,s, \tag{8}$$

$$0\le\sum_{j\in\Pi}\alpha_{ij}^*\le\sum_{j\in\Pi}x_{ij}, i=1,2,\ldots,m, \tag{9}$$

$$\lambda^*\in\hat{\Omega}. \tag{10}$$

By Eqs 7-10 and feasibility of $\tilde{\Lambda}=(\tilde{\lambda}_j: j\in\Gamma, \tilde{\lambda}_M, \theta^*)$ for problem (3), we have:

$$\theta^*_{\square}x_{iM}\ge x_{iM}\tilde{\lambda}_M+\sum_{j\in\Gamma}\tilde{\lambda}_j x_{ij}\ge\tilde{\lambda}_M\sum_{j\in\Gamma}\lambda_j^* x_{ij}+\sum_{j\in\Gamma}\tilde{\lambda}_j x_{ij}=\sum_{j\in\Gamma}\tilde{\lambda}_M\lambda_j^* x_{ij}+\sum_{j\in\Gamma}\tilde{\lambda}_j x_{ij}. \tag{11}$$

$$y_{rM}\le y_{rM}\tilde{\lambda}_M+\sum_{j\in\Gamma}\tilde{\lambda}_j y_{rj}\le\tilde{\lambda}_M\sum_{j\in\Gamma}\lambda_j^* y_{rj}+\sum_{j\in\Gamma}\tilde{\lambda}_j y_{rj}=\sum_{j\in\Gamma}\tilde{\lambda}_M\lambda_j^* y_{rj}+\sum_{j\in\Gamma}\tilde{\lambda}_j y_{rj}. \tag{12}$$

Set λ¯_j:=(λ_j ̃) +λ_j^* λ ̃_M "," for each ∀j∈Γ"," then

$$\sum_{j\in\Gamma}\bar{\lambda}_j x_{ij}\le\theta^* x_{iM} \qquad i=1,2,\ldots,m, \tag{13}$$

$$\sum_{j\in\Gamma}\bar{\lambda}_j y_{rj}\ge y_{rM} \qquad i=1,2,\ldots,m. \tag{14}$$

It is easily seen that:

$$\bar{\lambda}=(\bar{\lambda}_j; j\in\Gamma)\in\hat{\Omega}. \tag{15}$$

By Eq 13 and θ^*<θ¯_M, we have:

$$\sum_{j\in\Gamma}\bar{\lambda}_j x_{ij}\le\theta^* x_{iM}<\bar{\theta}_M x_{iM}=\bar{\theta}_M\sum_{j\in\Pi}\alpha_{ij}^*. \tag{16}$$

Without loos of generality, we assume that $\alpha_1 k^*>0$, because $x_M\neq0$. By Eq 16, we get

$$\sum_{j\in\Gamma}\bar{\lambda}_j x_{1j}<\bar{\theta}_M x_{iM}=\bar{\theta}_M\sum_{j\in\Pi}\alpha_{1j}^*.$$

Therefore, there exists a positive scalar $\mu>0$, such that

$$\sum_{j\in\Gamma}\bar{\lambda}_j x_{1j}\le\bar{\theta}_M\Big(\sum_{j\in\Pi-\{k\}}\alpha_{1j}^*+(\alpha_{1k}^*-\mu)\Big), \tag{17}$$

and $\alpha_1 k^*-\mu\ge0$.

Now, define

$$\bar{\alpha}_{ij}=\begin{cases}\alpha_{ij}^*-\mu & i=1, j=k\\ \alpha_{ij}^* & otherwise.\end{cases}$$

According to Eqs 13, 14, 15, and 17, it is obvious that is a feasible solution to model (6). The value of the objective function of LP (6) at this feasible point is equal:

$$\sum_{i=1}^{\dots}\sum_{j\in\Pi}\omega_i\bar{\alpha}_{ij}=\sum_{i=2}^{\dots}\sum_{j\in\Pi-\{k\}}(\omega_i\bar{\alpha}_{ij})+\omega_1((\sum_{j\in\Pi-\{k\}}\bar{\alpha}_{1j})+\bar{\alpha}_{1k})$$
$$=\sum_{i=2}^{m}\sum_{j\in\Pi-\{k\}}(\omega_i\alpha^*_{ij})+\omega_1((\sum_{j\in\Pi-\{k\}}\alpha^*_{1j})+(\alpha^*_{1k}-\mu)) \quad (18)$$
$$<\sum_{i=2}^{m}\sum_{j\in\Pi-\{k\}}\omega_i\alpha^*_{ij}+\omega_1\sum_{j\in\Pi}\alpha^*_{jk}=\sum_{i=1}^{m}\sum_{j\in\Pi}\omega_i\alpha^*_{ij}.$$

This contradicts the assumption and completes the proof.

Now consider in the following question:

Question 2. . If DMU_j, $j\in\Pi$ are required to merge and form a new unit (DMU_M), in which input vector and predetermined efficiency target of DMU_M are $x_M=\sum_{j\in\Pi}x_j$ and $\bar{\varphi}_M$, respectively, how much should be input vector of this new DMU?

Here, suppose DMU_M keeps the amount of inputs of all DMU_j; $j\in\Pi$, and looking to find the maximum amount of outputs of these DMUs in order to reach the pre-defined target level. The aim of the Question 2 is estimating the output vector y_M provided that the efficiency index of DMU_M *is* $\bar{\varphi}_M$. In other words, the optimal value of the following model is equal $\bar{\varphi}_M$.

$$\begin{aligned}\varphi^*=\max\quad &\varphi\\ s.t.\quad &\sum_{j\in\Gamma}\lambda_j x_{ij}+x_{iM}\lambda_M\le x_{iM}, \quad i=1,\dots,m,\\ &\sum_{j\in\Gamma}\lambda_j y_{rj}+y_{rM}\lambda_M\ge\varphi y_{rM}, \quad r=1,\dots,s,\\ &\lambda\in\bar{\Omega},\end{aligned} \quad (19)$$

where $x_{iM}=\sum_{j\in\Pi}x_{ij}$, $i=1, 2\dots, m$ and

$$\bar{\Omega}=\left\{\lambda\left|\lambda=(\lambda_j:j\in\Gamma,\lambda_M)\in\mathbb{R}^{n-|\Pi|+1}_{\ge0},\right.\right.$$
$$\left.\sigma_1\left(\sum_{j\in\Gamma}\lambda_j+\lambda_M+\sigma_2(-1)^{\sigma_3}v\right)=\sigma_1, v\ge0\right\}.$$

To estimate output vector of DMU_M the following multiple objective non-linear programming (MONLP) problem is considered:

$$\begin{aligned}\max\quad &\left(\sum_{j\in\Pi}\beta_{rj};\quad r=1,2,\dots,s\right)\\ s.t.\quad &\sum_{j\in\Gamma}\lambda_j x_{ij}+\lambda_M\left(\sum_{j\in\Pi}x_{ij}\right)\le\sum_{j\in\Pi}x_{ij}, \quad i=1,2,\dots,m,\\ &\sum_{j\in\Gamma}\lambda_j y_{rj}+\lambda_M\left(\sum_{j\in\Pi}\beta_{rj}\right)\ge\bar{\varphi}_M\sum_{j\in\Pi}\beta_{rj}, \quad r=1,2,\dots,s,\\ &\sum_{j\in\Pi}\beta_{rj}\ge\sum_{j\in\Pi}y_{rj}, \quad r=1,2,\dots,s,\\ &\lambda\in\bar{\Omega}.\end{aligned} \quad (20)$$

Where $\bar{\varphi}_M$ is a predetermined target for efficiency of the merged DMU_M. In the above model, (λ_j :$j\in\Gamma$, λ_M , β_{rj} :$r=1, 2, \dots, s$, $\forall j\in\Pi$) is the variables vector. Let us to assume that ω_r , $r=1, 2,\dots, s$, are important degree of for each of outputs of the merged DMU_M.Therefore, model (20) converted to the following only objective model:

$$\begin{aligned}\max\quad &\left(\sum_{r=1}^{s}\sum_{j\in\Pi}\omega_r\beta_{rj}\quad r=1,2,\dots,s\right)\\ s.t.\quad &\sum_{j\in\Gamma}\lambda_j x_{ij}+\lambda_M\left(\sum_{j\in\Pi}x_{ij}\right)\le\sum_{j\in\Pi}x_{ij}, \quad i=1,2,\dots,m,\\ &\sum_{j\in\Gamma}\lambda_j y_{rj}+\lambda_M\left(\sum_{j\in\Pi}\beta_{rj}\right)\ge\bar{\varphi}_M\sum_{j\in\Pi}\beta_{rj}, \quad r=1,2,\dots,s,\\ &\sum_{j\in\Pi}\beta_{rj}\ge\sum_{j\in\Pi}y_{rj}, \quad r=1,2,\dots,s,\\ &\lambda\in\bar{\Omega}.\end{aligned} \quad (21)$$

Similar to the discussion raised in the conversion of model (5) to model (6), the nonlinear model (21) can be transformed into the following linear model:

$$\begin{aligned}\max\quad &\sum_{r=1}^{s}\sum_{j\in\Pi}\omega_r\beta_{rj}\\ s.t.\quad &\sum_{j\in\Gamma}\lambda_j x_{ij}\le\sum_{j\in\Pi}x_{ij}, \quad i=1,2,\dots,m,\\ &\sum_{j\in\Gamma}\lambda_j y_{rj}\ge\bar{\varphi}_M\sum_{j\in\Pi}\beta_{rj}, \quad r=1,2,\dots,s,\\ &\sum_{j\in\Pi}\beta_{rj}\ge\sum_{j\in\Pi}y_{rj}, \quad r=1,2,\dots,s,\\ &\lambda\in\hat{\Omega},\end{aligned} \quad (22)$$

$$\hat{\Omega}=\left\{\lambda\left|\lambda=(\lambda_j:j\in\Gamma)\in\mathbb{R}^{n-|\Pi|}_{\ge0},\sigma_1\left(\sum_{j\in\Gamma}\lambda_j+\sigma_2(-1)^{\sigma_3}v\right)=\sigma_1, v\ge0\right.\right\}.$$

where

Remark 2.3 In Model (22), $\sum_{j\in\Pi}\beta_{rj}$ is unknown. Considering $\beta_{rM}=\sum_{j\in\Pi}\beta_{rj}$:(r=1, 2, …, s), as a new variable in Model (22), the number of variables is strongly reduced, and so computational complexity is reduced.

The proof of the following theorem is omitted because it is similar to the proof of Theorem 2 in (Gattoufi et al., 2014).

Theorem 2.3 Let (φ^* , λ_j^* ; $\forall$ $j\in$"Γ," $\lambda_{n+1}^*\ge0$) be an optimal solution the following model:

$$\varphi^* = \max \ \varphi$$
$$s.t. \quad \sum_{j\in\Gamma} \lambda_j x_{ij} + \lambda_{n+1} x_{in+1} \le x_{in+1}, \qquad i=1,2,\ldots,m, \tag{23}$$
$$\sum_{j\in\Gamma} \lambda_j y_{rj} + \lambda_{n+1} y_{rn+1} \ge \varphi y_{rn+1}, \qquad r=1,2,\ldots,s,$$
$$\lambda \in \bar{\Omega} = \left\{ \lambda \middle| \lambda = (\lambda_j : j \in \Gamma, \lambda_{n+1}) \in \mathbb{R}_{\ge 0}^{n-|\Pi|+1}, \right.$$
$$\left. \sigma_1\left(\sum_{j\in\Gamma} \lambda_j + \lambda_{n+1} + \sigma_2(-1)^{\sigma_3} v\right) = \sigma_1, v \ge 0 \right\},$$

where, $x_{in+1}=\sum_{j\in\Pi} x_{ij};(i=1, 2, \ldots, m)$ and $y_{rn+1}=\sum_{j\in\Pi} y_{rj};(r=1, 2, \ldots, s)$.

Then, Model (22) is feasible if and only if $\varphi^* \ge \bar{\varphi}_M$.

With the minor changes in the proof of Theorem 1.3, the following theorem can be proved. Therefore, the proof of the theorem 3.3 is omitted. Theorem 3.3 shows how Model (22) can be used to estimate of outputs of DMU_M

Theorem 3.3 Let $DMU_j, \forall j \in \Pi$ be inefficient. If $\Lambda=(\lambda_j^* : j\in\Gamma, \beta_{rj}^* : r=1, 2, \ldots, s, \ \forall j \in J)$ is an optimal solution with optimal value of $\sum_{r=1}^{s}\sum_{j\in\Pi} \omega_i \beta_{rj}^*$ to model (22), then efficiency score of DMU_M with input vector $x_M=\sum_{j\in\Pi} x_{ij}$ and output vector $y_M=\sum_{j\in\Pi} \beta_{rj}$ is $\bar{\varphi}_M$.

AN EXAMPLE WITH REAL DATA

Consider a static technology comprising of 14 the educational departments in Islamic Azad University of Khomeinishahr-Iran as DMU, in which each DMU to produce two different continuous-valued outputs, Satisfaction of the students (y_1) and Satisfaction of the professors and staff (y_2), uses two different continuous-valued inputs, Facilities (x_1) and Amount of the attention paid to the department by the university (x_2). The data is obtained from the work of Ghobadi & Jahangiri (2015). The data of inputs, outputs and efficiency score (considering input-oriented BCC model) are shown in Table 1:

Table 1: The data and efficiency score under BCC.

Departments	x_1	x_2	y_1	y_2	Efficiency Score
DMU1	0.385854	0.782695	11.76842	13.97176	0.9772
DMU2	0.53634	0.786386	12.24444	10.01111	0.9520
DMU3	0.972344	0.852564	12.43333	15.98	1.0000
DMU4	0.554214	0.712929	11.27391	14.41182	1.0000
DMU5	0.358756	0.912581	12.50481	13.07813	0.8889
DMU6	0.417995	0.672647	9.646154	14.51444	1.0000
DMU7	0.511568	0.784326	12.31864	14.53929	0.9976
DMU8	0.388259	0.837351	13.24667	10.3875	1.0000
DMU9	0.558262	0.829015	12.28824	12.72222	0.9065
DMU10	0.272026	0.81424	12.34615	14.11111	1.0000
DMU11	0.198246	0.883972	11.55625	12.77	1.0000
DMU12	0.546817	0.748349	12.48148	14.41182	1.0000
DMU13	0.558458	0.952591	13.03182	14.41182	1.0000
DMU14	1	1	12.15287	14.41182	0.7387

As can be seen, DMU_5 and DMU_{14} are inefficient DMUs.

Suppose that the decision maker wants to establish a new DMU (DMU_M) by merging these DMUs, in which output vector and predetermined efficiency target of DMU_M are $(y_{1M}, y_{2M})=(24.65768, 27.48995)$ and $\bar{\theta}_M=0.9215$, respectively. To determine input vector DMU_M, Model (7) corresponding to DMU_M is written as follows:

$$\min \quad \omega_1\alpha_M^1 + \omega_2\alpha_M^2$$
$$s.t. \quad \sum_{j\in\Gamma} \lambda_j x_j^1 \le \bar{\theta}_M \alpha_M^1,$$
$$\sum_{j\in\Gamma} \lambda_j x_j^2 \le \bar{\theta}_M \alpha_M^2,$$
$$\sum_{j\in\Gamma} \lambda_j y_j^1 \ge y_5^1 + y_{14}^1 = 24.65768, \tag{24}$$
$$\sum_{j\in\Gamma} \lambda_j y_j^2 \ge y_5^2 + y_{14}^2 = 27.48995,$$

$$0 \le \alpha_M^1 \le 1.358756,$$
$$0 \le \alpha_M^2 \le 1.912581,$$
$$\alpha_M^1 \ge 0, \alpha_M^2 \ge 0,$$
$$\lambda_j \ge 0, \lambda_M \ge 0, \quad j \in \Gamma = J - \{5,14\}.$$

Where $J=\{1, 2, \ldots, 14\}$.

Considering different important degree ω_1 and ω_2, in which $\omega_1+\omega_2=1$ for each of inputs of DMU_M, the following solutions are generated:
$(x_M^1 , x_M^2)=(\alpha_M^{1*}), \alpha_M^{2*})=(1.17 ,1.91)$,
$(x_M^1$ "," $x_M^2)=(\alpha_M^{1*}), \alpha_M^{2*})=(1.26 ,1.91)$.

Therefore, if the educational departments of DMU_5 and DMU_{14} are required to merge and form a new DMU with predetermined efficiency target of and output vector then DMU_M must receive inputs such as one of the above solutions.

CONCLUSION

In the present paper, a novel method proposed to estimate inputs/outputs in the problem of merging DMUs in order to reach a predetermined efficiency target. Sufficient conditions are introduced to find the minimum/maximum amount of inputs/outputs of merging DMUs in order to reach the pre-defined target level. Our method, unlike other proposed method (Gattoufi et al., 2014), decreases the number of the variables of the model strongly, and this decreases the computational complexity. Also, a numerical example with real data is presented to confirm the credibility and applicability of our method.

Here, following research topics are recommended:

- Obtaining necessary conditions to estimate inputs/outputs.
- Similar models can be investigated for merging efficient DMUs.
- Similar models can be developed in presence of fuzzy data.
- Similar models can be developed in presence of stochastic data.

REFERENCES

Amin, G. R., Emrouznejad, A., & Gattoufi, S. (2017). Modelling generalized firms' restructuring using inverse DEA. *Journal of Productivity Analysis*, *48*(1), 51-61.

Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management science*, *30*(9), 1078-1092.

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European journal of operational research*, *2*(6), 429-444.

Cook, W. D., & Seiford, L. M. (2009). Data envelopment analysis (DEA)–Thirty years on. *European journal of operational research*, *192*(1), 1-17.

Emrouznejad, A., & Tavana, M. (2014). *Performance measurement with fuzzy data envelopment analysis*. Springer.

Gattoufi, S., Amin, G. R., & Emrouznejad, A. (2014). A new inverse DEA method for merging banks. *IMA Journal of Management Mathematics*, *25*(1), 73-87.

Ghobadi, S. (2017). Inputs and Outputs Estimation in Inverse DEA. *Iranian Journal of Optimization*, *9*(2), 119-129.

Ghobadi, S., & Jahangiri, S. (2015). Inverse DEA: review, extension and application. *International Journal of Information Technology & Decision Making*, *14*(04), 805-824.

Ghobadi, S., Jahanshahloo, G. R., Lotfi, F. H., & Rostamy-Malkhalifeh, M. (2014). Dynamic inverse DEA in the presence of fuzzy data. *Advances in Environmental Biology*, 139-152.

Hadi-Vencheh, A., & Foroughi, A. A. (2006). A generalized DEA model for inputs/outputs estimation. *Mathematical and Computer Modelling*, *43*(5-6), 447-457.

Jahanshahloo, G. R., Vencheh, A. H., Foroughi, A. A., & Matin, R. K. (2004a). Inputs/outputs estimation in DEA when some factors are undesirable. *Applied Mathematics and Computation*, *156*(1), 19-32.

Jahanshahloo, G. R., Lotfi, F. H., Shoja, N., Tohidi, G., & Razavyan, S. (2004b). The outputs estimation of a DMU according to improvement of its efficiency. *Applied Mathematics and Computation*, *147*(2), 409-413.

Jahanshahloo, G. R., Hosseinzadeh Lotfi, F., Rostamy-Malkhalifeh, M., & Ghobadi, S. (2014). Using enhanced Russell model to solve inverse data envelopment analysis problems. *The Scientific World Journal*, *2014*.

Jahanshahloo, G. R., Soleimani-Damaneh, M., & Ghobadi, S. (2015). Inverse DEA under intertemporal dependence using multiple-objective programming. *European Journal of Operational Research*, *240*(2), 447-456.

Lertworasirikul, S., Charnsethikul, P., & Fang, S. C. (2011). Inverse data envelopment analysis model to preserve relative efficiency values: The case of variable returns to scale. *Computers & Industrial Engineering*, *61*(4), 1017-1023.

Wei, Q., Zhang, J., & Zhang, X. (2000). An inverse DEA model for inputs/outputs estimate. *European Journal of Operational Research*, *121*(1), 151-163.

Zhang, X. S., & Cui, J. C. (1999). A project evaluation system in the state economic information system of china an operations research practice in public sectors. *International Transactions in Operational Research*, *6*(5), 441-452.