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Inputs and Outputs Estimation in Inverse DEA

Saeid Ghobadi

Department of Mathematics, Khomeinishahr Branch, Islamic Azad University, Khomeinishahr, Isfahan, Iran

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Abstract

The present study addresses the following question: if among a group of decision making units, the decision maker is required to increase inputs and outputs to a particular unit in which the DMU, with respect to other DMUs, maintains or improves its current efficiency level, how much should the inputs and outputs of the DMU increase? This question is considered as a problem of inverse data envelopment analysis, and a method is introduced to answer this question. Using (weak) pareto solutions of multiple-objective linear programming, necessary and sufficient conditions for inputs and outputs estimation are established. An application of inverse DEA using real data (for choosing a suitable strategy for spreading educational departments in a university) is presented. In addition, two new optimal notions are introduced for multiple-objective programming problems: semi-pareto and semi-weak pareto optimal notions. The aforementioned solutions are used to answer the above question.

***Correspondence E-mail**: ghobadi@iaukhsh.ac.ir

INTRODUCTION

Data envelopment analysis (DEA) was introduced by Charnes, Cooper, and Rhodes (CCR model) (1978) and extended by Banker et al. (BCC model) (1984). DEA is a well-known nonparametric technique in operation research and management science which is based on linear programming to estimate relative efficiencies of a decision making unit (DMU). In this technique, it is assumed that the assessed units are homogenous and consume the same multiple inputs for producing the same multiple outputs. DEA has been used and developed by many researchers, see e.g. Cook and Seiford (2009), Cooper et al. (1999), and Hatami-Marbini et al. (2011) for some reviews. Moreover, relationships between DEA and multi-objective linear programming (MOLP) have been studied from several viewpoints by many researchers, see, e.g., Golany (1988), Hosseinzadeh Lotfi et al. (2010a, 2010b), Joro et al. (2003), Lins et al. (2004), Quariguasi Frota Neto and Angulo-Meza (2007), Thanassoulis and Allen (1998), Wong et al. (2009), and Yang et al. (2009) among others.

DEA and MOLP can be applied as mathematical tools in management control and planning. Whilst these two types of models are similar in structure, DEA is directed to assess past performances as part of the management control, MOLP is to planning future performance targets (Yang et al., 2009).

The idea of the inverse DEA first appeared in Zhang and Cui (1999), though inverse DEA was formally studied at first in a worthwhile paper by Wei et al. (2000). In Zhang and Cui (1999) the input increases of a DMU are estimated for its given output increases under the CCR efficiency-fixed constraints. Wei et al. (2000) have studied the following important question:

Question 1. If among a group of DMUs, the decision maker increases certain inputs to a particular unit and assumes that the DMU, with respect to other DMUs, maintains its current efficiency level, how much should the outputs of the DMU increase? They used multiple-objective linear programming (MOLP) problem to estimate the desired outputs when the DMU is inefficient, though it was answered by solving only an LP when the DMU is (at least) weakly efficient. After the initial work in inverse DEA by Wei et al. (2000), it has been remarkably considered by some scholars in the DEA field, see, e.g., Gattoufi et al. (2012), Ghobadi and Jahangiri (2015), Hadi-Vencheh et al. (2006, 2008), Hatami-Marbini et al. (2011), Jahanshahloo et al. (2004a, 2004b, 2005, 2014, 2015), Lertworasirikul et al. (2011), Lin (2010), and Yan et al. (2002) for some reviews. The following question, along the lines of (Wei et al., 2000), was investigated in inverse DEA filed by Hadi- Vencheh et al. (2008):

Question 2. If among a group of DMUs, the decision maker increases certain outputs to a particular unit and assumes that the DMU, with respect to other DMUs, maintains its current efficiency level, how much should the inputs of the DMU increase? Hadi-Vencheh et al. (2008) used (weak) pareto solutions of MOLP problems to estimate the desired inputs. Both Questions, input-estimation and output-estimation, are investigated under inter-temporal dependence assumption by Jahanshahloo et al. (2015). They are introduced a new optimality notion for multiple-objective programming problems, periodic weak Pareto optimality. These points were used to answer the above questions under inter-temporal dependence assumption. Recently, Jahanshahloo et al. (2014) studied the following question in the inverse DEA field:

Question 3. If among a group of DMUs, the decision maker is required to increase inputs and outputs to a particular unit in which the DMU, with respect to other DMUs, maintains its current efficiency level, how much should the inputs and outputs of the DMU increase? The aim of Question 3 is estimating the minimum increase of inputs and the maximum increase of outputs provided that the DMU maintains its current efficiency level. Jahanshahloo et al. (2014) utilized multiple-objective linear programming tools for input and output estimation under preserving the efficiency score. It is worth to mention that Question 3 was answered only for the efficient DMUs.

The aim of the present study is to investigate and develop the question proposed by Jahanshahloo et al. (2014). In other words, the present paper addresses the following question: if among a group of DMUs, the decision maker is required to increase inputs and outputs to a particular unit in which the DMU, with respect to other DMUs, maintains or improves its current efficiencylevel, how much should the inputs and outputs of the DMU increase? Necessary and sufficient conditions to estimate input and output levels simultaneously are introduced using pareto solutions of multiple-objective linear programming problems. In addition, two new optimal notions are introduced for MOLP problems: semi-pareto and semi-weak pareto optimality. These points are utilized in inverse DEA, and it is shown that all these can be found by a simple alteration in weighted sum scalarization technique.

Solving the above question is taken into considerations both theoretically and practically, because it provides new connections between DEA and MOLP. Moreover, it can help the decision maker to make better decisions in order to extend DMUs. That is to say that the decision makers can take necessary actions by choosing a suitable strategy for spreading the DMU. In other words, these can be used for sensitivity analysis (Jahanshahloo et al., 2004, 2005), preserve (improve) efficiency values (Jahanshahloo et al., 2004, 2005; Lertworasirikul et al., 2011; Wei et al., 2000; Yan et al., 2002) resource allocation (Hadi-Vencheh et al., 2008), merging the banks (Gattoufi et al., 2012), and setting revenue target (Lin, 2010).

The rest of the paper unfolds as follows: In section 2, some preliminaries in clouding multiple-objective optimization and some of the basic models in DEA are reviewed. In section 3, the input and output estimation problem is (simultaneously) dealt with. This section is devoted to the main results of the paper.In section 4, an application of inverse DEA using real data (for choosing a suitable strategy for spreadingeducational departments in a university) is presented. In section 5, two new optimalitynotions for MOLP problems are introduced. It is proven that this solutions can be characterized by a simple manipulating in the weighted sum method (Ehrgott, 2005). Concluding remarks are provided in the section 6.

PRELIMINARIES

Multiple-objective programming

A multiple-objective programming (MOP) problem is written

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & S=\left\{x\in \mathbb{R}^n : g_j(x)\le 0, \quad j=1,2,\dots,k\right\}, \end{aligned} \quad (1)$$

Where $f:\mathbb{R}^n\to\mathbb{R}^m$ and $g:\mathbb{R}^n\to\mathbb{R}^k$ are two given vector-valued functions, i.e.,

$f(x)=(f_1(x), f_2(x),\dots, f_m(x))$, $g(x)=(g_1(x), g_2(x),\dots, g_k(x))$

f_i s are the objective functions of this MOP. The set $s\subseteq\mathbb{R}^n$ is called the set of feasible solutions of MOP (1). "min" indicates that the purpose is to minimizes all objectives simultaneously. There is usually no solution $x\in s$ that simultaneously minimizes all objective functions. Therefore, (weak) Pareto/efficient solutions are defined instead ofoptimal solutions.

Definition 2.1 (Ehrgott, 2005). A feasible solution $x^*\in s$ is called a Pareto solution to MOP (1) if there does not exist $x^o\in s$ such that

$fi(x^\circ)\le fi(x^*)$ for each $i=1, 2, \dots, m$
$fi(x^\circ)\le fi(x^*)$ for some $i=1, 2, \dots, m$

Definition 2.2 (Ehrgott, 2005) A feasible solution $x^*\in s$ is called a weak Pareto solution to MOP (1) if there does not exist $x^o\in s$ such that $f_i(x^o)\le f_i(x^*)$ for each $i= 1,2,\dots,m$.

Some of the Basic Models in DEA

Let us to consider a set of n $DMUs$, $\{DMU_j\}::j=1,\dots,n\}$, in which DMU_j produce multiple positive Outputs y_{rj} $(r=1,\dots,s)$, by utilizing multiple positive inputs x_{ij} $(i=1,\dots,m)$. Let input and output for DMU_j be denoted by $x_i=(x_{1i},x_{2i},\dots,x_{mi})$ and $Y_j=(y_{1j},x_{2j},\dots,x_{sj})$, respectively. To measure the relative efficiency of the unit under assessment of DMU_o, o={1,2,…,n}, the following input-oriented generalized DEA model (Yu and Wei, 1996; Wei and Yu, 1997) is considered:

$$\begin{aligned} \theta_o^* = \min \quad & \theta \\ s.t. \quad & \sum_{j=1}^n \lambda_j x_{ij} \le \theta x_{io}, \quad i=1,\dots,m, \\ & \sum_{j=1}^n \lambda_j y_{rj} \ge y_{ro}, \quad r=1,\dots,s, \\ & \lambda\in\Omega, \end{aligned} \quad (2)$$

where

$$\Omega=\left\{\lambda \,\middle|\, \lambda=(\lambda_1,\dots,\lambda_n), \sigma_1\left(\sum_{j=1}^n \lambda_j + \sigma_2(-1)^{\sigma_3}\nu\right)\right.$$

$$\left. =\sigma_1, \nu\ge 0, \lambda_j\ge 0, j=1,\dots,n\right\}.$$

In the above model σ_1, σ_2 and σ_3 are parameters with 0-1 values. It is easy to see that:

If $\sigma_1=0$ then (2) is under a constant returns to scale (CRS) assumption of the production technology. This model is the first basic DEA model which has been provided by Charnes, Cooper, and Rhodes (CCR model) (1978). If $\sigma_1=1$ and $\sigma_2=1$ then (2) is called BCC model which has been introduced by Banker et al. (1984).This model is under a variable returns to scale (VRS) assumption of the production technology. If $\sigma_1=\sigma_2=1$ and $\sigma_3=0$, then (2) known as FG model which has been proposed by Fare and Grosskopf (1985). This model is under a non-increasing returns to scale (NIRS) assumption of the production technology. If $\sigma_1=\sigma_2=\sigma_3=1$, then model (2) is under a non-decreasing returns to scale (NDRS) assumption of the production technology. This model suggested by Seiford and Thrall (1990) is known as ST model.

The optimal value θ_0^* of the model (2) is called the input-oriented efficiency score of DMU_o. If $\theta_0^*=1$, then DMU_o is called input-oriented (at least) weakly efficient.It is easy to see that $\theta_0^* \le 1$.

The following model is output-oriented version of the model (2):

$$\begin{aligned} \varphi_o^* = \max \quad & \varphi \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io}, \quad i=1,2,\dots,m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge \varphi y_{ro}, \quad r=1,2,\dots,s, \\ & \lambda \in \Omega. \end{aligned} \tag{3}$$

In model (3), φ_0^* is called the output-oriented efficiency score of DMUo. It is easy to see that $\varphi_0^* \ge 1$ DMU_o is called output-oriented (at least) weakly efficient if $\varphi_0^*=1$.

INVERSE DEA

This section is devoted to studying and extending Question 3, provided by Jahanshahloo et al. (2014). In other words, the following question is addressed: if among a group of DMUs, the decision makeris required to increase inputs and outputs to a particularunit in which the DMU, with respect to other DMUs, maintains its current efficiencylevel or improves it to the amount η-percent, how much should the inputs andoutputs of the DMU increase?

The aim of the study is estimating the minimum increase ofinput vector and the maximum increase of output vector provided that the DMU_o, with respect to other units, maintains its current efficiency level, that is θ_0^* , or improves it to the amount η-percent. In fact,

$$\alpha_o^* = \left(\alpha_{1o}^*, \alpha_{20}^*, \dots, \alpha_{mo}^*\right)^t = X_o + \Delta X_o, \quad \Delta X_o \ge 0,$$
$$\beta_o^* = \left(\beta_{1o}^*, \beta_{20}^*, \dots, \beta_{so}^*\right)^t = Y_o + \Delta Y_o, \quad \Delta Y_o \ge 0.$$

Assume that DMUnew represents DMUo after changing the input and output vectors. The following model is considered toestimate the efficiency score of DMU_{new}:

$$\begin{aligned} \theta_o^{new} = \min \quad & \theta \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + \alpha_{io}^* \lambda_{new} \le \theta \alpha_{io}^*, \quad i=1,\dots,m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} + \beta_{ro}^* \lambda_{new} \ge \beta_{ro}^*, \quad r=1,\dots,s, \\ & \lambda \in \Omega_{new}. \end{aligned} \tag{4}$$

where

$$\Omega_{new} = \left\{ \lambda \middle| \lambda = \left(\lambda_1, \dots, \lambda_2, \lambda_{new}\right), \sigma_1\left(\sum_{j=1}^{n} \lambda_j + \lambda_{new} + \sigma_2 (-1)^{\sigma_3} \nu\right) = \sigma_1, \nu \ge 0, \lambda_{new} \ge 0, \lambda_j \ge 0, j=1,\dots,n \right\}.$$

Definition 3.1 Suppose that θ_0^* and θ_0^{new} are the optimal values of problems (2) and (4), respectively. Then

i) If $\theta_0^*=\theta_0^{new}$ then it is said that the efficiency score of DMU_o remains unchanged, i.e., $eff(\alpha_0^*,\beta_0^*)=eff(X_0,Y_0)$.

ii) If $\theta_0^{new} = [1+\eta/100]\,\theta_0^*$ then it is said that the amount of improvement of the efficiency of DMU_o is η-percent of the θ_0^*, i.e.,

$$eff\left(\alpha_o^*, \beta_o^*\right) = \left(1 + \frac{\eta}{100}\right) eff\left(X_o, Y_o\right).$$

To answer the above question, i.e, to estimate the minimum increase of input vector and the maximum increase of output vector, in which the amount of improvement of the efficiency of DMU_o is η-percent of θ_0^*, the following MOLP problem is considered:

$$\begin{aligned} \min \quad & (\alpha_{1o},\dots,\alpha_{mo}) \\ \max \quad & (\beta_{1o},\dots,\beta_{ro}) \\ s.t. \quad & \sum_{j=1}^{n}\lambda_j x_{ij} \le \left(1+\frac{\eta}{100}\right)\theta_o^*\alpha_{io}, \quad i=1,\dots,m, \\ & \sum_{j=1}^{n}\lambda_j y_{rj} \ge \beta_{ro}, \quad r=1,\dots,s, \\ & \alpha_{io} \ge x_{io}, \quad i=1,\dots,m, \\ & \beta_{ro} \ge y_{ro}, \quad r=1,\dots,s, \\ & \beta_o \in \Gamma, \alpha_o \in \Lambda, \\ & \lambda \in \Omega, \end{aligned} \tag{5}$$

Where θ_0^* and are the optimal value of problem (2) and the certain amount of improvement of current efficiency level of DMU_o, respectively. Γ and Λ are bounded sets and represent the increasing variation rate of inputs and outputs of the DMU_o which are considered by the decision maker.

Remark 3.2 If the efficiency score of DMU_o is θ_0^*, then η must be $0 \le \eta \le 1- \theta_0^*/ \theta_0^* * 100$ If $\eta=0$ then DMU_o, with respect to other units, maintains its current efficiency score. If $\eta=1- \theta_0^*/ \theta_0^* * 100$ then DMU_o will be efficient.

The following theorem shows how the above MOLP can be used forinputs and outputs estimation.

Theorem 3.3 Suppose that ($\lambda^*,\theta_0^*=\theta^*$) is an optimal solutionto problem (2).

Let ($\lambda^{\hat{}*}$, $\alpha\hat{}_0^*$, $\beta\hat{}_0^*$) be a pareto solution to problem(5). Suppose that the inputs and outputs of DMU_o are increased to $\alpha\hat{}_0^*$ and $\beta\hat{}_0^*$, respectively. Then,

If DMU_o be inefficient and $\alpha\hat{}_0^* \ge\ne x_0$ then

$$eff\left(\alpha_o^*,\beta_o^*\right)=\left(1+\frac{\eta}{100}\right)eff\left(X_o,Y_o\right).$$

If DMU_o maintains its current efficiency level, i.e., $\eta=0$, and $\alpha\hat{}_0^* \ge x_0$ then $eff(\alpha\hat{}_0^*, \beta\hat{}_0^*)=(X_0, Y_0)$

Remark 3.4 If $\alpha\hat{}_0^* = x_0$ and $\beta\hat{}_0^* \ge\ne Y_0$, then $\beta\hat{}_0^*- yr_0$ indicates the lack-output amount in rth output component of the DMU_o. In other words, the decision maker can preserve the efficiency score of the DMU_o while the outputs increase from Y_0 to $\beta\hat{}_0^*$ without the inputs increase from X_0. In this case, projection point of DMU_o is on the weak efficiency frontier of the production possible set.

Proof. To prove the theorem, $\theta_0^{new}=[1+\eta/100]\theta_0^*$ should be shown. Because and ($\lambda^{\hat{}*}$, $\alpha\hat{}_0^*$, $\beta\hat{}_0^*$) is a feasible solution for MOLP, the following relations are held:

$$\sum_{j=1}^{n}\hat{\lambda}_j^* x_{ij} \le \left(1+\frac{\eta}{100}\right)\theta_o^*\hat{\alpha}_{io}^* \le \hat{\alpha}_{io}^*, \quad i=1,\dots,m \tag{6}$$

$$\sum_{j=1}^{n}\hat{\lambda}_j^* y_{rj} \ge \hat{\beta}_{ro}^*, \quad r=1,\dots,s \tag{7}$$

$$\hat{\alpha}_{io}^* \ge x_{io}, \quad i=1,\dots,m \tag{8}$$

$$\hat{\beta}_o^* \in \Gamma, \hat{\alpha}_o^* \in \Lambda, \tag{9}$$

$$\hat{\lambda}^* \in \Omega \tag{10}$$

(11)

Let $\lambda^{-}=(\lambda^{-}_1,\dots,\lambda^{-}_n,\lambda^{-}_{new})$, in which $\lambda^{-}_j=\lambda\hat{}_j^*$ for each j=1,…,n and $\lambda^{-}_{new}=0$ It is clear that $\lambda^{-}\in\Omega_{new}$ Since $\lambda^{-}\in\Omega_{new}$, because of (6) and (7), (λ^{-} $(1+\eta/100)\,\theta_0^*)$ is a feasible solution to problem (4). Therefore, $\theta_0^{new} \le (1+\eta/100)\,\theta_0^*$.

Let $(\lambda^{+*}=\lambda_1^{+*},\dots,\lambda_n^{+*},\lambda_{new}^{+*},\theta^{+*}=\theta_0^{new})$ be an optimal solution to problem (4). The inequalities (6) and (7) will be used in problem (4), the following results are obtained:

$$\begin{aligned} &\theta_o^{new}\hat{\alpha}_{io}^* \ge \sum_{j=1}^{n}\lambda_j^{+*}x_{ij}+\lambda_{new}^{+*}\hat{\alpha}_{io}^* = \sum_{j=1}^{n}\lambda_j^{+*}x_{ij}+\lambda_{new}^{+*}\left(\sum_{j=1}^{n}\hat{\lambda}_j^* x_{ij}\right), \\ &\theta_o^{new}\hat{\alpha}_{io}^* \ge \sum_{j=1}^{n}(\lambda_j^{+*}+\lambda_{new}^{+*}\hat{\lambda}_j^*)x_{ij}, \quad i=1,\dots,m, \end{aligned} \tag{12}$$

$$\begin{aligned} &\hat{\beta}_{ro}^* \le \sum_{j=1}^{n}\lambda_j^{+*}y_{rj}+\lambda_{new}^{+*}\hat{\beta}_{ro}^* \le \sum_{j=1}^{n}\lambda_j^{+*}y_{rj}+\lambda_{new}^{+*}\left(\sum_{j=1}^{n}\hat{\lambda}_j^* y_{rj}\right), \\ &\hat{\beta}_{ro}^* \le \sum_{j=1}^{n}(\lambda_j^{+*}+\lambda_{new}^{+*}\hat{\lambda}_j^*)y_{rj}, \quad r=1,\dots,s. \end{aligned} \tag{13}$$

Set $\tilde{\lambda}_j:=\lambda_j^{+*}+\lambda_{new}^{+*}\hat{\lambda}_j^*$ for each j=1,2,…,n. It is easily seen that $\tilde{\lambda}=(\tilde{\lambda}_j,\dots,\tilde{\lambda}_n)\in\Omega$.

By contradiction assume that $\theta_0^{new}< (1+ \eta/100)\,\theta_0^*$.

Taking Eq. 12 and $\theta_0^{new}< (1+ \eta/100)\,\theta_0^*$, the following inequality is obtained:

$$\sum_{j=1}^{n}\lambda_j x_{ij} \le \theta_o^{new}\hat{\alpha}_{io}^* < \left(1+\frac{\eta}{100}\right)\theta_o^*\hat{\alpha}_{io}^*, \quad i=1,\dots,m. \tag{14}$$

If assumption (i) holds, then $\alpha^{\hat{}0*}\ne X_0$ Therefore, there exists some i=∈{1,…,m} in which $\alpha\hat{}_{i0}^* > x_{i0}$ Let $I=\{\alpha\hat{}_{i0}^* > x_{i0}\}$ If

$$\mu = \min\left\{ \min_{i\in I}\left\{ \frac{(1+\frac{\eta}{100})\theta_o^*\hat{\alpha}_{io}^* - \sum_{j=1}^{n}\lambda_j x_{ij}}{(1+\frac{\eta}{100})\theta_o^*} \right\}, \min_{i\in I}\left\{\hat{\alpha}_{io}^* - x_{io}\right\}, \right\} \quad (15)$$

Then $\mu > 0$. Now, define $\beta_0 = \hat{\beta}_0^*$ and

$$\alpha_{io} = \begin{cases} \hat{\alpha}_{io}^* - \mu & if\ i \in I, \\ \hat{\alpha}_{io}^* & if\ i \notin I. \end{cases}$$

Considering (15), thefollowing inequalities are obtained:

$$\mu \le \hat{\alpha}_{io}^* - x_{io} \Rightarrow x_{io} \le \hat{\alpha}_{io}^* - \mu = \alpha_{io}, \quad i \in I,$$

and

$$\mu \le \frac{(1+\frac{\eta}{100})\theta_o^*\hat{\alpha}_{io}^* - \sum_{j=1}^{n}\lambda_j x_{ij}}{(1+\frac{\eta}{100})\theta_o^*} \Rightarrow$$
$$\sum_{j=1}^{n}\lambda_j x_{ij} \le (1+\frac{\eta}{100})\theta_o^*(\hat{\alpha}_{io}^* - \mu) = (1+\frac{\eta}{100})\theta_o^*\alpha_{io}, \quad i \in I, \quad (16)$$

which implies that $\tilde{\alpha}_0 \ge X_0$, because $\tilde{\alpha}_{i0} = \hat{\alpha}_{i0}^*$ ™$_{xi0}$ for each $i \notin I$ In addition, $\tilde{\alpha}_0 \in$, Λ because $X_0 \le \tilde{\alpha}_0$ and $\hat{\alpha}_0^* \in \Lambda$ by (10).

According to Eqs. 13, 14 and 16 thefollowing inequalities are obtained:

$$\sum_{j=1}^{n}\lambda_j x_{ij} < (1+\frac{\eta}{100})\theta_o^*\hat{\alpha}_{io}^* = (1+\frac{\eta}{100})\theta_o^*\alpha_{io}, \quad i \notin I, \quad (17)$$

$$\sum_{j=1}^{n}\lambda_j x_{ij} \le (1+\frac{\eta}{100})\theta_o^*\alpha_{io}, \quad i \in I, \quad (18)$$

$$\sum_{j=1}^{n}\lambda_j y_{rj} \ge \hat{\beta}_{ro}^* = \beta_{ro}, \qquad r = 1,\dots,s. \quad (19)$$

Since $\tilde{\lambda} \in \Omega$, because of (17)-(19), $X_0 \le \tilde{\alpha}_0 \in \Lambda$, and $Y_0 \le \hat{\beta}_{i0}^* = \tilde{\beta}_0 \in \Gamma$, $(\tilde{\lambda}, \tilde{\alpha}_0, \tilde{\beta}_0)$ is a feasible solution to problem (5), where $\tilde{\alpha}_{i0} \le \hat{\alpha}_{i0}^*$ and $\tilde{\beta}_{r0} \ge \hat{\beta}_{r0}^*$ for all i,r, and $\tilde{\alpha}_{i0} \le \hat{\alpha}_{i0}^*$ for some i=1,…,m.

This contradicts the assumption that $(\lambda^{\sim *}, \hat{\alpha}_0^*, \hat{\beta}_0^*)$ is a pareto solution to problem (5), and the proof of case (i) is completed.

The proof under assumption (ii) is similar to the proof of case (i) when replacing the notation η by constant value zero. The only difference is in case (ii) when $\hat{\alpha}_0^* = X_0$. Note that if $\hat{\alpha}_0^* = X_0$, then by (12) and (13), thefollowing inequalities are obtained:

$$\sum_{j=1}^{n}\lambda_j x_{ij} \le \theta_o^{new}\hat{\alpha}_{io}^* = \theta_o^{new} x_{io}, \quad i = 1,\dots,m, \quad (20)$$

$$\sum_{j=1}^{n}\lambda_j y_{rj} \ge \hat{\beta}_{ro}^* \ge y_{ro}, \quad r = 1,\dots,s. \quad (21)$$

Because $\tilde{\lambda} \in \Omega$, (20), and (21) imply that $(\tilde{\lambda}, \theta_0^{new})$ is a feasible solution to problem (2), such that $\theta_0^{new} < \theta_0^*$. But it is impossible because θ_0^* is the optimal value of problem (2).

Remark 3.5 Theorem 3.3 will remain valid if one replacesthe objective function of MOLP 5 with

"min $(\alpha_{1o},\dots,\alpha_{mo})$".

Theorem 3.6 is converse version of Theorem 3.3.

Theorem 3.6 Suppose that $(\lambda^*, \theta_0^* = \theta^*)$ is an optimal solution to problem (2). Let $(\bar{\lambda}, \bar{\alpha}_0, \bar{\beta}_0)$ be a feasible solution to the problem (5). If $eff(\bar{\alpha}_o, \bar{\beta}_o) = (1+\frac{\eta}{100})eff(X_o, Y_o)$ in which $0 \le \eta \le 1 - \theta_0^{*/}\theta_0^{*\ *}100$, then $(\bar{\lambda}, \bar{\alpha}_0, \bar{\beta}_0)$ must be a (semi-) weak pareto solution to problem (5).

Proof. If $(\bar{\lambda}, \bar{\alpha}_0, \bar{\beta}_0)$ is not a weak pareto solution to problem (5), then there existsanother feasible solution of problem (5), $(\tilde{\lambda}, \tilde{\alpha}_0, \tilde{\beta}_0)$, such that $\tilde{\alpha}_{i0} < \bar{\alpha}_{i0}$ and $\tilde{\beta}_{r0} < \bar{\beta}_{r0}$ for all i, r. Therefore

$$\sum_{j=1}^{n}\lambda_j x_{ij} \le (1+\frac{\eta}{100})\theta_o^*\alpha_{io} < (1+\frac{\eta}{100})\theta_o^*\bar{\alpha}_{io}, \quad i = 1,\dots,m, \quad (22)$$

$$\sum_{j=1}^{n}\lambda_j y_{rj} \ge \beta_{ro} > \bar{\beta}_{ro}, \quad r = 1,\dots,s, \quad (23)$$

$$\lambda \in \Omega \quad (24)$$

If

$$\mu = \max\left\{ \frac{\sum_{j=1}^{n}\lambda_j x_{ij}}{(1+\frac{\eta}{100})\theta_o^*\bar{\alpha}_{io}} \ |i = 1,\dots,m \right\}, \quad (25)$$

then $0 < \mu < 1$. Taking relation (25), thefollowing inequality is obtained:

$$\sum_{j=1}^{n} \lambda_j x_{ij} \le \mu(1+\frac{\eta}{100})\theta_o^* \bar{\alpha}_{io}, \quad i=1,\dots,m. \quad (26)$$

According to Eqs. (23), (24), and (26), *($\lambda=\lambda$, $\lambda_{new}=0$, $\theta=\mu(1+\eta/100)\theta_0^*$)* is a feasible solution to problem (4) (considering $\alpha_0^*=\bar{\alpha}_0$ and $\beta_0^*=\bar{\beta}_0$ in problem (4)). The valueof the objective function of LP (4) at this feasiblepoint is equal to $\mu(1+\eta/100)\theta_0^*$ Therefore,

$$eff\,(\bar{\alpha}_o, \bar{\beta}_o) = \theta_o^{new} \le \mu(1+\frac{\eta}{100})\theta_o^*$$

$$< (1+\frac{\eta}{100})\theta_o^* = (1+\frac{\eta}{100}) eff\,(X_o, Y_o).$$

This contradicts the assumption and completes the proof.

AN APPLICATION OF INVERSE DEA

Consider a static technology comprising of 14 the educationaldepartments in Islamic Azad university of Khomeinishahr-Iran as DMU, in which each DMU uses two different continuous-valued inputs to produce two different continuous-valued outputs.The data is obtained from the work of Ghobadi and Jahangiri (2015). The data of inputs, outputs and efficiency score (considering input-oriented BCC model) areshown in Table 1:

As can be seen, D_5 is an inefficient DMU. Assume that the decision maker is required toincrease the input and output in which D_5, with respect to other DMUs, improves current efficiency level to mount 5-percentof its current efficiency level ($\theta_5^*=0.8889$).Suppose that the decision maker identifiedthe variations rate of increase inputs and outputs for this DMU as:

$$0.358756 \le x_5^1 \le 0.398756, \quad 0.912581 \le x_5^2 \le 0.942581,$$
$$12.50481 \le y_5^1 \le 13.10481, \quad 13.07813 \le y_5^2 \le 13.72121.$$

MOLP (5) corresponding to is D_5 written as follows:

$$\begin{aligned}
\min \quad & (\alpha_5^1, \alpha_5^2) \\
\max \quad & (\beta_5^1, \beta_5^2) \\
s.t. \quad & \sum_{j=1}^{14} \lambda_j x_j^1 \le \left(1+\frac{5}{100}\right)\theta_5^* \alpha_5^1, \\
& \sum_{j=1}^{14} \lambda_j x_j^2 \le \left(1+\frac{5}{100}\right)\theta_5^* \alpha_5^2, \\
& \sum_{j=1}^{n} \lambda_j y_j^1 \ge \beta_5^1, \\
& \sum_{j=1}^{n} \lambda_j y_j^2 \ge \beta_5^2, \\
& 0.358756 \le x_5^1 \le 0.398756, \\
& 0.912581 \le x_5^2 \le 0.942581, \\
& 12.50481 \le y_5^1 \le 13.10481, \\
& 13.07813 \le y_5^2 \le 13.72121. \\
& \sum_{j=1}^{14} \lambda_j = 1, \\
& \lambda_j \ge 0, \quad j = 1,2,\dots,14. \quad (27)
\end{aligned}$$

Using the weight-sum method (Ehrgott, 2005) for MOLP model (27), the following pareto solutions are generated:

Table 1: The data and efficiency score under VRS.

Departments	x_1	x_2	y_1	y_2	Efficiency Score
D_1	0.385854	0.782695	11.76842	13.97176	0.9772
D_2	0.53634	0.786386	12.24444	10.01111	0.9520
D_3	0.972344	0.852564	12.43333	15.98	1.0000
D_4	0.554214	0.712929	11.27391	14.41182	1.0000
D_5	0.358756	0.912581	12.50481	13.07813	0.8889
D_6	0.417995	0.672647	9.646154	14.51444	1.0000
D_7	0.511568	0.784326	12.31864	14.53929	0.9976
D_8	0.388259	0.837351	13.24667	10.3875	1.0000
D_9	0.558262	0.829015	12.28824	12.72222	0.9065
D_{10}	0.272026	0.81424	12.34615	14.11111	1.0000
D_{11}	0.198246	0.883972	11.55625	12.77	1.0000
D_{12}	0.546817	0.748349	12.48148	14.41182	1.0000
D_{13}	0.558458	0.952591	13.03182	14.41182	1.0000
D_{14}	1	1	12.15287	14.41182	0.7387

$$\left(\alpha_5^{1*},\alpha_5^{2*},\beta_5^{1*},\beta_5^{2*}\right)=\left(0.39876,0.91966,12.66600,13.72121\right),$$
$$\left(\alpha_5^{1*},\alpha_5^{2*},\beta_5^{1*},\beta_5^{2*}\right)=\left(0.39876,0.91374,12.77006,13.07813\right),$$
$$\left(\alpha_5^{1*},\alpha_5^{2*},\beta_5^{1*},\beta_5^{2*}\right)=\left(0.39876,0.91258,12.64869,13.72121\right).$$

Therefore, the decision makers are able to make better decisions and choosing a suitable strategy to expand D_5. That is to say that the decisionmaker can take necessary actions by choosing a suitable strategy for spreading D_5 ,in which with respect to other DMUs, improves current efficiency level to amount 5-percent, i.e. the efficiency score of new DMU is 0.9333.

As can be seen, D_{14} is an inefficient unit. Suppose that the decision maker is required toincrease the input and output in which D_{14}, with respect to other DMUs, improves current efficiency level to amount 20-percentof its current efficiency level $(\theta_5^*=0.7387)$. The decision maker identifiedthe variations rate of increase inputs and outputs for D_{14} as follows:

$$1.00000 \le x_{14}^1 \le 1.22205, \qquad 1.00000 \le x_{14}^2 \le 1.19221,$$
$$12.15287 \le y_{14}^1 \le 12.90481, \qquad 14.41182 \le y_{14}^2 \le 15.12121.$$

MOLP (5) corresponding to D_{14} has been written and the following results were obtained:

$$(\alpha_{14}^{1*},\alpha_{14}^{2*},\beta_{14}^{1*},\beta_{14}^{2*})=(1.00000,1.02358,12.76108,1512121),$$
$$(\alpha_5^{1*},\alpha_5^{2*},\beta_5^{1*},\beta_5^{2*})=(1.00000,1.00000,12.70476,1512121),$$
$$(\alpha_5^{1*},\alpha_5^{2*},\beta_5^{1*},\beta_5^{2*})=(1.00000,1.02145,12.90481,1441182).$$

OPTIMALITY NOTION FOR MOLP

In this section, based on the special structure of MOLP (5), two new optimal notions are introduced for MOLP: semi-pareto and semi-weak pareto optimal notions. It is proven that semi-pareto and semi-weak pareto solutions can be characterized by a simple manipulating in the weighted sum method (Ehrgott, 2005).

The semi-pareto and semi-weak pareto concepts are defined as follows:

Definition 5.1 Let $(\lambda^*,\alpha_0^*,\beta_0^*)$ be a feasible solution to problem (5). If there is no feasible solution $(\lambda,\alpha_0,\beta_0)$ of (5) such that $(\alpha_0-\beta_0) \le (\alpha_0^*-\beta_0^*)$ and $\alpha_{i0}<\alpha_{i0}^*$ for some i∈{1,2,…,m}, then $(\lambda^*,\alpha_0^*,\beta_0^*)$ is called a semi-pareto (semi-strongly efficient) solution to problem (5). Let semi-pareto solutions set of MOLP (5) be denoted by X_{sp}.

Definition 5.2 Let $(\lambda^*,\alpha_0^*,\beta_0^*)$ be a feasible solution to problem (5). If there is no feasible solution $(\lambda,\alpha_0,\beta_0)$ of (5) such that $\alpha_0<\alpha_0^*$ and $\beta_0\ge\beta_0^*$, then is called a $(\lambda^*,\alpha_0^*,\beta_0^*)$ semi-weak pareto (semi-efficient) solution to problem (5). Let semi-weak pareto solutions set of MOLP (5) be denoted by X_{sw}.

Let pareto and weak pareto solutions set of MOLP (5) be denoted by X_p and X_w respectively. It is obvious that $X_p \subseteq X_{sp} \subseteq X_{sw} \subseteq X_w$ Therefore, semi-weak pareto optimality and semi-pareto optimality are two notions between the pareto optimality and weak pareto optimality.

Remark 5.3 It is easy to see that the Theorem 3.3 is valid if one replaces the “pareto" assumption with “semi-pareto " assumption.

This section continues with a discussion about semi-pareto and semi-weak pareto solutions. In order to the following example is considered:

Example 5.4 The following MOLP is considered:

$$\begin{aligned} &\min \quad (\alpha_1,\alpha_2) \\ &\max \quad (\beta_1,\beta_2) \\ &s.t.\ X=\{A,B,C,D,E,F\}, \end{aligned}$$

where $A=(1,1,5,6)^t$,$B=(1,1,5,5)^t$, $D=(2,1,4,5)^t$, $E=(2,2,5,5)^t$, and $F=(2,2,4,4)^t$

It can be seen that $X_p=\{A\}$, $X_{sp}=\{A,B\}$, $X_{sw}=\{A,B,C,D\}$ and $X_w=\{A,B,C,D,E\}$

The above example addresses a situation in which the inclusions $X_p \subseteq X_{sp} \subseteq X_{sw} \subseteq X_w$ are strict. This section is ended with theorems 5.5 and 5.6 that show the weight-sum method (Ehrgott, 2005; Isermann, 1977; Steuer 1986) can be used to characterize semi-pareto and semi-weak pareto solutions, respectively.

Theorem5.5 Let $\prod=(\bar{\lambda},\bar{\alpha}_0,\bar{\beta}_0)$ be a feasible solution to MOLP (5). ∏ is asemi-pareto solution of MOLP (5) if and only if there exist positive weight vector $v=(v_1,\dots,v_m)\in \square^m$ and non-positive weight vector $U=(u_1,\dots,u_2)\in \square^s$ such that ∏ is an optimal solution tothe following LP:

$$\begin{aligned} &\min \sum_{i=1}^{m} v_i\alpha_{io}+\sum_{r=1}^{s}u_r\beta_{ro} \\ &s.t.\ \text{The constraints of MOLP(5).} \end{aligned} \tag{29}$$

Proof. If ∏ is not a semi-Pareto solution to MOLP (5), then there exists another feasible solution of MOLP (5)(and hence feasible to LP (29), $(\tilde{\lambda},\tilde{\alpha}_0, \beta_0)$ such that $\tilde{\alpha}_0 \leq \bar{\alpha}_{i0}$ and $\tilde{\beta}_{r0} \geq \bar{\beta}_{r0}$ for all i, r, and $\tilde{\alpha}_0 < \bar{\alpha}_{i0}$ for some i. Because the V and U are positive and non-positive weight vectors, then $\sum_{i=1}^{m} v_i \alpha_{io} < \sum_{i=1}^{m} v_i \bar{\alpha}_{io}$ and $\sum_{r=1}^{s} u_r \beta_{ro} \leq \sum_{r=1}^{s} u_r \bar{\beta}_{ro}$. Therefore

$$\sum_{i=1}^{m} v_i \alpha_{io} + \sum_{r=1}^{s} u_r \beta_{ro} < \sum_{i=1}^{m} v_i \bar{\alpha}_{io} + \sum_{r=1}^{s} u_r \bar{\beta}_{ro},$$

which implies that ∏ is not an optimal solution for LP (29).

Conversely, let ∏ be a semi-pareto solution to MOLP (5). The following auxiliary model is considered:

$$\begin{aligned} \max\ & \sum_{i=1}^{m} \alpha_i \\ s.t.\ & \text{The constraints of MOLP(5).} \\ & \alpha_{io} + \alpha_i = \bar{\alpha}_{io}, \quad i = 1,\dots,m, \\ & -\beta_{ro} + \sum_{i=1}^{m} \alpha_i = -\bar{\beta}_{ro}, \quad r = 1,\dots,s, \\ & \alpha_i \geq 0, \quad i = 1,\dots,m. \end{aligned}$$

In this model, α_i is a variable corresponding to the ith input. Since ∏ is a semi-pareto solution to MOLP (5), thereforethe optimal value of LP (30) is zero. Note that LP (30) is always feasible. Considering the dual of LP (30) and in similar manner to the proof of Theorem 6.11 in (Ehrgott, 2005), non negative weight vector $v=(v_1,\dots,v_m) \in \square^m$ and non-positive weight vector $U=(u_1,\dots,u_s) \in {}^s$ are obtained such that $v_i + \sum_{r=1}^{s} u_r \geq 1$ for each $i=1,\dots,m$, and ∏ is an optimal solution to LP (29). These complete the proof.

The most important point in Theorem 5.5 is that, all of the weights corresponding to inputs are positive, butall of the weights corresponding to outputs are non-positive.

Theorem 5.6 Let ∏= $(\bar{\lambda}, \bar{\alpha}_0, \bar{\beta}_0)$ be a feasible solution to MOLP (5). ∏ is a semi-weak pareto solution of MOLP (5) if and only if there exist **nonzero nonnegative** weight vector $v=(v_1,\dots,v_m) \in \square^m$ and non-positive weight vector $U=(u_1,\dots,u_2) \in \square^s$ such that ∏ is an optimal solution tothe following LP:

$$\begin{aligned} \min\ & \sum_{i=1}^{m} v_i \alpha_{io} + \sum_{r=1}^{s} u_r \beta_{ro} \\ s.t.\ & \text{The constraints of MOLP(5).} \end{aligned} \tag{31}$$

Proof. The proof is similar to the proof of Theorem 5.5. Note that, the following auxiliary model is considered:

$$\begin{aligned} \max\ & \alpha \\ s.t.\ & \text{The constraints of MOLP(5).} \\ & \alpha_{io} + \alpha_i = \bar{\alpha}_{io}, \quad i = 1,\dots,m, \\ & -\beta_{ro} + \alpha = -\bar{\beta}_{ro}, \quad r = 1,\dots,s, \\ & \alpha \geq 0, \end{aligned} \tag{32}$$

In this model, α is scaler variable. Since ∏ is a semi-weak pareto solution to MOLP (5), therefore the optimal value of LP (32)is zero. Considering the dual of LP (32) and in similar manner to the proof of Theorem 6.11 in (Ehrgott, 2005), nonnegative weight vector $v=(v_1,\dots,v_m) \in \square^m$ and non-positive weight vector $U=(u_1,\dots,u_2) \in \square^s$ are obtained such that $\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r \geq 1$ and ∏ is an optimal solution to LP (31). These complete the proof.

The most important point in Theorem 5.6 is that, all of the weights corresponding to inputs are nonzero nonnegative, but all of the weights corresponding to outputs are non-positive.

CONCLUSION

This paper studied the inverse DEA problem to estimate the minimum of inputs increase and the maximum of outputs increase of the DMU under preserving or improving the current efficiency level. Necessary and sufficient conditions were established using MOLP tools for inputs and outputs estimation provided that the DMU maintains or improves its current efficiencylevel. In addition, two new optimal notions (semi-Pareto and semi-weak Pareto optimality) for MOLP problems were introduced and investigated.

The results can be used to present patterns to decision makers to increase inputs and outputs (extending decision making units) of the DMU either efficient or inefficient while the efficiency levelremains unchanged or, to the certain amount, improved.

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