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A Numerical Method in the Hilbert Space for Solving the Time-Fractional Reaction-Diffusion Equation with Time Delay

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Abstract. This paper aims to obtain an approximate solution through a simplified reproducing kernel method (SRKM) for the time-fractional delay reaction-diffusion equation. By a simple transformation, we first homogenize the considered reaction-diffusion equation with delay. Then, we recall some reproducing kernel Hilbert spaces and their properties and build up the reproducing kernel Hilbert space that we need throughout the solution scheme. This new reproducing kernel space satisfies the delay condition, the property that reduces the computational complexity. Next, the nth-term approximation $\mathcal{U}_n$ of the exact solution $\mathcal{U}$ is obtained without the Gram–Schmidt orthogonalization process. The properties of completeness and orthogonal projection of the considered basis are stated and proved. Eventually, given to the various examples represented, the efficiency and accuracy of the method are scrutinized. It is shown that the proposed method works well for various values of fractional order derivatives and even for large mode N.

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1. Introduction

Fractional partial differential equations have pervasive applications in varied scientific disciplines and play an important role in modeling most of natural phenomena [3, 10, 15]. Over the past decades, due to the crucial roles of fractional partial differential equations, solving them has been the center of attention of many researchers [11, 17, 19, 30].

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This paper deals with the numerical solution of the following time-fractional delay reaction-diffusion equation:

$$\begin{cases} \frac{\partial^\alpha \varpi(\mathrm{x},\tau)}{\partial \tau^\alpha} = \frac{\partial^2 \varpi(\mathrm{x},\tau)}{\partial \mathrm{x}^2} + \mathrm{a}(\mathrm{x},\tau)\varpi(\mathrm{x},\tau) + \mathrm{b}(\mathrm{x},\tau)\varpi(\mathrm{x},\tau-\theta) + \mathrm{h}(\mathrm{x},\tau), & 0 \leqslant \mathrm{x} \leqslant 1, \\ & 0 \leqslant \tau \leqslant \mathbb{T}, ; \\ \varpi(\mathrm{x},0) = \omega_0(\mathrm{x}), \quad \varpi(0,\tau) = \vartheta_1(\tau), \quad \varpi(1,\tau) = \vartheta_2(\tau), & 0 \leqslant \tau \leqslant \mathbb{T}; \\ \varpi(\mathrm{x},\tau) = \phi(\mathrm{x},\tau) & -\theta \leqslant \tau \leqslant 0, \end{cases} \tag{1}$$

where $\mathrm{a}(\mathrm{x},\tau)$, $\mathrm{b}(\mathrm{x},\tau)$, $\nu_0(\mathrm{x})$, $\vartheta_1(\tau)$, $\vartheta_2(\tau)$ and $\mathrm{h}(\mathrm{x},\tau)$ are given functions with $0 \leqslant \alpha \leqslant 1$ and $\theta > 0$ is delay term. Here, $\frac{\partial^\alpha \varpi(\mathrm{x},\tau)}{\partial \tau^\alpha}$ is the Caputo derivative that is defined by

$$\frac{\partial^\alpha \varpi(\mathrm{x},\tau)}{\partial \tau^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^\tau \frac{1}{(\tau-\zeta)^\alpha} \frac{\partial \varpi(\mathrm{x},\zeta)}{\partial \zeta} \mathrm{d}\zeta$$

for $0 < \alpha < 1$, $\Gamma(\mathrm{z})$ being the Gamma function. Moreover, $\mathrm{x} \in [0,1]$ is a variable in space and $\tau \in [0,\mathbb{T}] \subset \mathbb{R}$ is a variable in time.

Delay differential equations are widely considered in the modeling of a variety of phenomena in the natural sciences and mathematical models, e.g., transportation scheduling, engineering control [1], nuclear engineering [4]. Research on delay differential equations has been the focus of much research and there are valuable resources available for finding the numerical solution of ODEs and PDEswith delay [2, 7, 12, 13, 21, 22, 25, 28, 31, 32, 35]. A fractional reaction-diffusion equation with delay can model complex biological systems like neural networks or epidemics where the effects of past events (delay) and non-local diffusion (fractional derivative) are significant. The interplay between these factors can lead to intricate patterns and behaviors that are not captured by standard models. Rihan [20] studied the time-fractional parabolic PDEs based on the ν-methods. The homotopy perturbation method was used to numerically solve time-fractional PDEs with proportional delays in [23]. The semilinear convection-reaction-diffusion equation with fractional derivative and delay term was solved by a linearized compact finite difference scheme and spectral collocation methods in [14, 34]. Hosseinpour et al. [9] proposed a collocation scheme for solving time-fractional delay reaction-diffusion equations, and Sun [24] solved this kind of equations with a linearized compact difference scheme.

The reproducing kernel method (RKM) is a powerful numerical method to investigate various scientific models. This method has been improved by many researchers to arrive at an efficient and fast algorithm for solving different types of problems such as perturbed problems [8], integro-differential equations [5], Telegraph equation [26] and space-time-fractional equations [27].

To overcome the problem of time-consumption of the Schmidt orthogonalization process, Xu and Lin [33] proposed the simplified reproducing kernel method (SRKM) for solving delay fractional differential equations. Then, this method was utilized to solve impulsive delay differential equations [16]. Recently, Niu et al. [18] used SRKM for the numerical solution of heat conduction equations with delay.

The main aim of the present work is to develop an numerical method based on the SRKM for the time-fractional delay reaction-diffusion equation given in Eq. (2). To set the initial and boundary conditions in (1) into $\mathcal{W}_{(3,2)}(\Omega)$, which is constructed in the following section, these conditions are needed to homogenized. Put $\nu(\mathrm{x},\tau) = \varpi(\mathrm{x},\tau) - \omega_0(\mathrm{x}) - \mathrm{H}(\mathrm{x},\tau) + H_0$ where $H(\mathrm{x},\tau) = \vartheta_1(\tau)(1-\mathrm{x}) + \vartheta_2(\tau)\mathrm{x}$

and $H_0(\mathrm{x}) = H(\mathrm{x}, 0)$. Hence Eq (1) transformed to the following equation

$$\begin{cases} \frac{\partial^\alpha \nu(\mathrm{x},\tau)}{\partial \tau^\alpha} = \frac{\partial^2 \nu(\mathrm{x},\tau)}{\partial \mathrm{x}^2} + \mathrm{a}(\mathrm{x},\tau)\nu(\mathrm{x},\tau) + \mathrm{b}(\mathrm{x},\tau)\nu(\mathrm{x},\tau-\theta) + \mathrm{F}(\mathrm{x},\tau), & 0 \leqslant \tau \leqslant \mathbb{T}; \\ \nu(\mathrm{x},0) = 0, \quad \nu(0,\tau) = 0, \quad \nu(1,\tau) = 0, & 0 \leqslant \tau \leqslant \mathbb{T}; \\ \nu(\mathrm{x},\tau) = \Phi(\mathrm{x},\tau). & -\theta \leqslant \tau \leqslant 0, \end{cases} \tag{2}$$

where

$$\mathrm{F}(\mathrm{x},\tau) = \mathrm{h}(\mathrm{x},\tau) + \frac{\partial^2}{\partial \mathrm{x}^2}\Big(H(\mathrm{x},\tau) + \omega_0(\mathrm{x}) - H_0(\mathrm{x})\Big) + \frac{\partial^\alpha H(\mathrm{x},\tau)}{\partial \tau^\alpha} + \mathrm{a}(\mathrm{x},\tau)H(\mathrm{x},\tau) + \mathrm{b}(\mathrm{x},\tau)H(\mathrm{x},\tau-\theta),$$
$$\Phi(\mathrm{x},\tau) = \phi(\mathrm{x},\tau) + H(\mathrm{x},\tau) + \omega_0(\mathrm{x}) - H_0(\mathrm{x}).$$

For convenience, we should homogenize the delay condition $\nu(\mathrm{x},\tau) = \Phi(\mathrm{x},\tau)$ in Eq. (2). To this end, set $\mathcal{U}(\mathrm{x},\tau) = \nu(\mathrm{x},\tau) - \rho(\mathrm{x},\tau)$, where

$$\rho(\mathrm{x},\tau) = \begin{cases} 0, & 0 \leqslant \tau \leqslant \mathbb{T}; \\ \Phi(\mathrm{x},\tau), & -\theta \leqslant \tau \leqslant 0. \end{cases}$$

Hence, the problem (2) can be transformed to the following homogeneous problem:

$$\begin{cases} \frac{\partial^\alpha \mathcal{U}(\mathrm{x},\tau)}{\partial \tau^\alpha} = \frac{\partial^2 \mathcal{U}(\mathrm{x},\tau)}{\partial \mathrm{x}^2} + \mathrm{a}(\mathrm{x},\tau)\mathcal{U}(\mathrm{x},\tau) + \mathrm{b}(\mathrm{x},\tau)\mathcal{U}(\mathrm{x},\tau-\theta) + \mathcal{G}(\mathrm{x},\tau), & 0 \leqslant \tau \leqslant \mathbb{T}; \\ \mathcal{U}(\mathrm{x},0) = 0, \quad \mathcal{U}(0,\tau) = 0, \quad \mathcal{U}(1,\tau) = 0, & 0 \leqslant \tau \leqslant \mathbb{T}; \\ \mathcal{U}(\mathrm{x},\tau) = 0 & -\theta \leqslant \tau \leqslant 0. \end{cases} \tag{3}$$

where

$$\mathcal{G}(\mathrm{x},\tau) = \begin{cases} \mathrm{F}(\mathrm{x},\tau) + \Phi(\mathrm{x},\tau-\theta), & 0 \leqslant \tau \leqslant \theta; \\ \mathrm{F}(\mathrm{x},\tau), & \theta \leqslant \tau \leqslant \mathbb{T}. \end{cases}$$

The novelty of our work is as follows. First, a novel reproducing kernel space that matches with the structure of (2) is derived. Second, the idea of transforming the original problem (1) to the homogeneous problem (3) is new. Third, the logic behind the derivation of the proposed method is proved via two theorems.

In Section 2, we recall some required concepts and properties of some reproducing kernel Hilbert spaces. In Section 3, we give a brief description of the SRKM approach and bring the detailed theorems and formulations of the SRKM for the problem (3). Several examples are solved using the SRKM in Section 4. Finally, conclusions are given in 5.

2. Reproducing kernel spaces

To solve the problem (3) using the SRKM, we derive a new reproducing kernel Hilbert space and a novel reproducing kernel function. In what follows, we recall some reproducing kernel Hilbert spaces and their properties and build up the reproducing kernel Hilbert space that we need throughout the solution scheme.

Reproducing Kernel Hilbert Spaces (RKHSs) are Hilbert spaces where the evaluation functional at any point is continuous. This means there's a special kernel function that allows you to "reproduce" the value of a function at a given point by taking the inner product of the function with the kernel. Different RKHSs are distinguished by the input space, the chosen kernel function, and the structure of the Hilbert space itself. The input space determines the type of functions the RKHS can represent. For instance, the space of continuous functions on a compact set might be a different RKHS than the space of square-integrable functions. The choice of input space impacts the type of problems the RKHS can be used to solve. The kernel function is a function on the input space that determines the inner product of two functions in the RKHS. Different kernels capture different types of smoothness and complexity in the functions represented by the RKHS. Examples of kernels include the Gaussian kernel, the linear kernel, and the polynomial kernel. The Hilbert space itself, including its inner product and norm, defines the properties of the functions within the RKHS. Different Hilbert spaces can be created using different choices of input spaces and inner products.

2.1 *The space* $\mathcal{W}_2^1[\mathrm{a},\mathrm{b}]$

The reproducing kernel Hilbert space $\mathcal{W}_2^1[\mathrm{a},\mathrm{b}]$, is the set of all absolutely continuous functions such that the first derivative of these functions belongs to $L^2[\mathrm{a},\mathrm{b}]$. This space is complete in the concept of reproducing kernel spaces [6], and we have

- For all $\nu(\mathrm{x}),\varpi(\mathrm{x})\in\mathcal{W}_2^1[\mathrm{a},\mathrm{b}]$, the inner product and norm for this space are defined as follow:

$$< \nu(\mathrm{x}),\varpi(\mathrm{x}) >_{\mathcal{W}_2^1}= \nu(\mathrm{a})\varpi(\mathrm{a}) + \int_{\mathrm{a}}^{\mathrm{b}} \nu'(\mathrm{x})\varpi'(\mathrm{x})\mathrm{dx},$$
$$\|\nu\|_{\mathcal{W}_2^1} = \sqrt{< \nu,\nu >_{\mathcal{W}_2^1}},$$

- The reproducing kernel function of this space is given by

$$\mathcal{K}_1(\eta,\mathrm{x}) = \begin{cases} 1+\eta, & \eta \leqslant \mathrm{x}; \\ 1+\mathrm{x}, & \eta > \mathrm{x}. \end{cases} \tag{4}$$

2.2 *The space* $\mathcal{W}_{2,0}^2[0,\mathbb{T}]$

The reproducing kernel space $\mathcal{W}_{2,0}^2[0,\mathbb{T}]$ is defined as the set of all real-valued functions ν so that ν and ν' are absolutely continuous in $[0,\mathbb{T}]$ and $\nu(0)=0$ and $\nu''\in L^2[0,\mathbb{T}]$. By [6], we can show that $\mathcal{W}_{2,0}^2[0,\mathbb{T}]$ is a complete reproducing kernel space and we have

- For all $\nu(\tau),\varpi(\tau)\in\mathcal{W}_{2,0}^2[0,\mathbb{T}]$ the inner product and norm for this space are defined by

$$< \nu(\mathrm{x}),\varpi(\mathrm{x}) >_{\mathcal{W}_{2,0}^2[0,\mathbb{T}]}= \nu'(0)\varpi'(0) + \int_0^{\mathbb{T}} \nu''(\tau)\varpi''(\tau)\mathrm{d}\tau,$$
$$\|\nu\|_{\mathcal{W}_{2,0}^2} = \sqrt{< \nu,\nu >_{\mathcal{W}_{2,0}^2}}.$$

- The reproducing kernel function of this space is

$$\mathcal{K}_2(\xi,\tau)=\begin{cases}-\frac{1}{6}\tau(\tau^2-6\xi-3\tau\xi), & \xi\leqslant\tau;\\ -\frac{1}{6}\xi(-6\tau-3\tau\xi+\xi^2), & \xi>\tau.\end{cases} \tag{5}$$

2.3 *The space* $\mathcal{W}^2_{2,\theta}[-\theta,\mathbb{T}]$

The linear space $\mathcal{W}^2_{2,\theta}[-\theta,\mathbb{T}]$ includes all real-valued functions such that for all $\nu(\tau)\in\mathcal{W}^2_{2,\theta}[-\theta,\mathbb{T}]$ the following property holds true

$$\begin{cases}\nu=0, & \text{If } \tau\in[-\theta,0];\\ \nu\in\mathcal{W}^2_{2,0}[0,\mathbb{T}], & \text{If } \tau\in[0,\mathbb{T}].\end{cases}$$

For the reproducing kernel space $\mathcal{W}^2_{2,\theta}[-\theta,\mathbb{T}]$ we have (see [6], [33]):

- For all $\nu(\tau),\varpi(\tau)\in\mathcal{W}^2_{2,\theta}[-\theta,\mathbb{T}]$, the inner product and norm for this space are defined as

$$<\nu(\mathrm{x}),\varpi(\mathrm{x})>_{\mathcal{W}^2_{2,\theta}[-\theta,\mathbb{T}]}=\nu'_+(0)\varpi'_+(0)+\int_0^{\mathbb{T}}\nu''(\tau)\varpi''(\tau)\mathrm{d}\tau,$$
$$\|\nu\|_{\mathcal{W}^2_{2,\theta}}=\sqrt{<\nu(\mathrm{x}),\nu(\mathrm{x})>_{\mathcal{W}^2_{2,\theta}}}.$$

- The reproducing kernel function is

$$R_2(\xi,\tau)=\begin{cases}\mathcal{K}_2(\xi,\tau), & 0\leqslant\xi<\mathbb{T}\text{ and }0\leqslant\tau<\mathbb{T};\\ 0, & -\theta\leqslant\tau<0.\end{cases} \tag{6}$$

where $\mathcal{K}_2(\xi,t)$ is given in (5).

2.4 *The space* $\mathcal{W}^3_{2,0}[0,1]$

The reproducing kernel space $\mathcal{W}^3_{2,0}[0,1]$ represents the space of all functions, which for each function belongs to this space , such as ν, all functions ν,ν' and ν'' are real-valued and absolutely continuous on the interval $[0,1]$. Furthermore, $\nu'''\in L^2[0,1]$ and $\nu(0)=\nu(1)=0$.

- For all $\nu(\tau),\varpi(\tau)\in\mathcal{W}^3_{2,0}[0,1]$, the inner product and norm for this space are defined as

$$<\nu(\mathrm{x}),\varpi(\mathrm{x})>_{\mathcal{W}^3_{2,0}[0,1]}=\nu'(0)\varpi'(0)+\int_0^1\nu''(\mathrm{x})\varpi''(\mathrm{x})\mathrm{dx},$$
$$\|\nu\|_{\mathcal{W}^3_{2,0}}=\sqrt{<\nu,\nu>_{\mathcal{W}^3_{2,0}}}$$

- The reproducing kernel function is

$$\mathcal{K}_3(\eta,\mathrm{x})=\begin{cases}\mathcal{K}(\eta,\mathrm{x}), & \eta\leqslant\mathrm{x};\\ \mathcal{K}(\mathrm{x},\eta), & \eta>\mathrm{x}.\end{cases} \tag{7}$$

where $\mathcal{K}(\eta,\mathrm{x}) = \frac{-1}{18720}(-1+\mathrm{x})\eta\Big(156\eta^4 + 6\mathrm{x}^2(120+30\eta+10\eta^2-5\eta^3+\eta^4) - 4\mathrm{x}^3(120+30\eta+10\eta^2-5\eta^3+\eta^4) + \mathrm{x}^4(120+30\eta+10\eta^2-5\eta^3+\eta^4) + 12\mathrm{x}(360-300\eta-100\eta^2-15\eta^3+3\eta^4)\Big)$.

2.5 *The spaces* $\mathcal{W}^{(3,2)}_{2,\theta}(\Omega)$ *and* $\mathcal{W}^{(1,1)}_{2}(\tilde{\Omega})$

Let $\Omega = [0,1]\times[-\theta,\mathbb{T}]$. The reproducing kernel space $\mathcal{W}^{(3,2)}_{2,\theta}(\Omega) = \mathcal{W}^3_{2,0}[0,1]\otimes\mathcal{W}^2_{2,\theta}[-\theta,\mathbb{T}]$ and its reproducing kernel is defined by (see [18])

$$\mathcal{K}_{(3,2)}(\eta,\xi;\mathrm{x},\tau) = \mathcal{K}_3(\eta,\mathrm{x})\times R_2(\xi,\tau), \tag{8}$$

where $\mathcal{K}_3(\eta,\mathrm{x})$ and $R_2(\xi,\tau)$ are given in (7) and (6), respectively. Moreover, the inner product in this space is defined as

$$\begin{aligned} <\nu(\mathrm{x},\tau),\varpi(\mathrm{x},\tau)>_{\mathcal{W}^{(3,2)}_{2,\theta}(\Omega)} &= \sum_{i=1}^{2}\int_0^{\mathbb{T}}\frac{\partial^2}{\partial\tau^2}\frac{\partial^i}{\partial\mathrm{x}^i}\nu(0,\tau)\frac{\partial^2}{\partial\tau^2}\frac{\partial^i}{\partial\mathrm{x}^i}\varpi(0,\tau)\mathrm{d}\tau \\ &+\Big\langle\frac{\partial}{\partial\tau}\nu(\mathrm{x},0),\frac{\partial}{\partial\tau}\varpi(\mathrm{x},0)\Big\rangle_{\mathcal{W}^3_{2,0}} \\ &+\iint_{\Omega}\frac{\partial^3}{\partial\mathrm{x}^3}\frac{\partial^2}{\partial\tau^2}\nu(\mathrm{x},\tau)\frac{\partial^3}{\partial\mathrm{x}^3}\frac{\partial^2}{\partial\tau^2}\varpi(\mathrm{x},\tau)\mathrm{d}\mathrm{x}\mathrm{d}\tau \end{aligned}$$

Similarly, we can define $\mathcal{W}^{(1,1)}_{2}(\tilde{\Omega}) = \mathcal{W}^1_2[0,1]\otimes\mathcal{W}^1_2[0,\mathbb{T}]$, where $\tilde{\Omega} = [0,1]\times[0,\mathbb{T}]$. It is easy to show that the reproducing kernel space $\mathcal{W}^{(1,1)}_{2}(\tilde{\Omega})$ is complete. The reproducing kernel function for this space is given by

$$\mathcal{K}_{(1,1)}(\eta,\xi;\mathrm{x},\tau) = \mathcal{K}_1(\eta,\mathrm{x})\times\mathcal{K}_1(\xi,\tau) \tag{9}$$

where $\mathcal{K}_1(\eta,\mathrm{x})$ is defined in Eq (4) (See [26]).

3. Explanation of the method

Consider the linear differential operator $\mathcal{F}:\mathcal{W}^{(3,2)}_{2,\theta}(\Omega)\to\mathcal{W}^{(1,1)}_{2}(\tilde{\Omega})$ such that

$$\mathcal{F}\mathcal{U}(\mathrm{x},\tau) = \frac{\partial^\alpha\mathcal{U}(\mathrm{x},\tau)}{\partial\tau^\alpha} - \frac{\partial^2\mathcal{U}(\mathrm{x},\tau)}{\partial\mathrm{x}^2} - \mathrm{a}(\mathrm{x},\tau)\mathcal{U}(\mathrm{x},\tau) - \mathrm{b}(\mathrm{x},\tau)\mathcal{U}(\mathrm{x},\tau-\theta).$$

Using this operator, the time-fractional PDE (3) is rewritten as

$$\mathcal{F}\mathcal{U}(\mathrm{x},\tau) = \mathcal{G}(\mathrm{x},\tau),\quad (\mathrm{x},\tau)\in[0,1]\times[0,\mathbb{T}].$$

Since $\mathcal{U}(\mathrm{x},\tau)\in\mathcal{W}^{(3,2)}_{2,\theta}(\Omega)$, for $\tau\in[0,\mathbb{T}]$ we have

$$\mathcal{U}(\mathrm{x},0)=0,\quad \mathcal{U}(0,\tau)=0,\quad \mathcal{U}(1,\tau)=0,$$

and $\mathcal{U}(\mathrm{x},\tau)=0$ for $\tau\in[-\theta,0]$. In [6] it is proved that the linear operator $\mathcal{F}$ is bounded.

Now, let $\Big\{(\mathrm{x}_i,\tau_i)\Big\}_{i=1}^{\infty}$ is a countable and dense subset in $\tilde{\Omega}$ and $\mathcal{F}^*$ is the adjoint operator for $\mathcal{F}$ and define

$$\phi_i(\mathrm{x},\tau)=\mathcal{K}_{(1,1)}(\mathrm{x}_i,\tau_i;\mathrm{x},\tau),\quad \psi_i(\mathrm{x},\tau)=\mathcal{F}^*\phi_i(\mathrm{x},\tau),$$

where $\mathcal{K}_{(1,1)}$ is the reproducing kernel of $\mathcal{W}_2^{(1,1)}(\tilde{\Omega})$. The next theorem establishes the structure of $\psi_i(\mathrm{x},\tau)$.

Theorem 3.1 Let $\Big\{(\mathrm{x}_i,\tau_i)\Big\}_{i=1}^{\infty}$ be a countable dense subset in $\tilde{\Omega}$. Then the sequence $\Big\{\psi_i(\mathrm{x},\tau)\Big\}_{i=1}^{\infty}$ is a complete function system in $\mathcal{W}_{2,\theta}^{(3,2)}(\Omega)$ and

$$\begin{aligned}\psi_i(\mathrm{x},\tau)=&\frac{\partial^\alpha R_2(\xi,\tau)}{\partial\xi^\alpha}\mathcal{K}_3(\eta,\mathrm{x})-\frac{\partial^2\mathcal{K}_3(\eta,\mathrm{x})}{\partial\eta^2}R_2(\xi,\tau)\\&-\mathrm{a}(\mathrm{x},\tau)\mathcal{K}_3(\eta,\mathrm{x})R_2(\xi,\tau)-\mathrm{b}(\mathrm{x},\tau)\mathcal{K}_3(\eta,\mathrm{x})R_2(\xi,\tau-\theta)\Big|_{\eta=\mathrm{x}_i,\xi=\tau_i}\end{aligned}\tag{10}$$

Proof We have

$$\begin{aligned}\psi_i(\mathrm{x},\tau)&=\mathcal{F}^*\phi_i(\mathrm{x},\tau)=\Big\langle\mathcal{F}^*\phi_i(\mathrm{x},\tau),\mathcal{K}_{(3,2)}(\eta,\xi;\mathrm{x},\tau)\Big\rangle_{\mathcal{W}_{2,\theta}^{(3,2)}}\\&=\Big\langle\phi_i(\mathrm{x},\tau),\mathcal{F}\mathcal{K}_{(3,2)}(\eta,\xi;\mathrm{x},\tau)\Big\rangle_{\mathcal{W}_2^{(1,1)}}=\mathcal{F}\mathcal{K}_{(3,2)}(\eta,\xi;\mathrm{x},\tau)\Big|_{\eta=\mathrm{x}_i,\xi=\tau_i}\\&=\frac{\partial^\alpha R_2(\xi,\tau)}{\partial\xi^\alpha}\mathcal{K}_3(\eta,\mathrm{x})-\frac{\partial^2\mathcal{K}_3(\eta,\mathrm{x})}{\partial\eta^2}R_2(\xi,\tau)\\&\quad-\mathrm{a}(\mathrm{x},\tau)\mathcal{K}_3(\eta,\mathrm{x})R_2(\xi,\tau)-\mathrm{b}(\mathrm{x},\tau)\mathcal{K}_3(\eta,\mathrm{x})R_2(\xi,\tau-\theta)\Big|_{\eta=\mathrm{x}_i,\xi=\tau_i}.\end{aligned}$$

Clearly, $\psi_i(\mathrm{x},\tau)\in\mathcal{W}_{2,\theta}^{(3,2)}(\Omega)$. Now, let $\nu\in\mathcal{W}_{2,\theta}^{(3,2)}(\Omega)$ is fixed and $\Big\langle\nu(\mathrm{x},\tau),\psi_i(\mathrm{x},\tau)\Big\rangle_{\mathcal{W}_{2,\theta}^{(3,2)}}=0$, for $i=1,2,....$ Then

$$\begin{aligned}\Big\langle\nu(\mathrm{x},\tau),\psi_i(\mathrm{x},\tau)\Big\rangle_{\mathcal{W}_{2,\theta}^{(3,2)}}&=\Big\langle\nu(\mathrm{x},\tau),\mathcal{F}^*\phi_i(\mathrm{x},\tau)\Big\rangle_{\mathcal{W}_{2,\theta}^{(3,2)}}\\&=\Big\langle\mathcal{F}\nu(\mathrm{x},\tau),\phi_i(\mathrm{x},\tau)\Big\rangle_{\mathcal{W}_{2,\theta}^{(3,2)}}\\&=\mathcal{F}\nu(\mathrm{x}_i,\tau_i)=0.\end{aligned}$$

Moreover, according to the assumption of the theorem, $\Big\{(\mathrm{x}_i,\tau_i)\Big\}_{i=1}^{\infty}$ is dense in Ω. Hence, $\mathcal{F}\nu(\mathrm{x},\tau)=0$ and in this case $\nu=0$ and the desired result is obtained. □

For any n, $\Big\{\psi_i(\mathrm{x},\tau)\Big\}_{i=1}^{n}$ is linear independent [18]. By using the symmetric properties of conjugate operator $\mathcal{F}^*$, we obtain

$$\begin{aligned}\Big\langle\psi_i(\mathrm{x},\tau),\psi_j(\mathrm{x},\tau)\Big\rangle=&\frac{\partial^\alpha\psi_i(\mathrm{x}_j,\tau_j)}{\partial\tau_j^\alpha}-\frac{\partial^2\psi_i(\mathrm{x}_j,\tau_j)}{\partial\mathrm{x}_j^2}\\&-\mathrm{a}(\mathrm{x}_j,\tau_j)\psi_i(\mathrm{x}_j,\tau_j)-\mathrm{b}(\mathrm{x}_j,\tau_j)\psi_i(\mathrm{x}_j,\tau_j-\theta).\end{aligned}$$

Now, we will find an approximate solution for Eq.(3) in a subspace $\Psi_n=$

$\Big\{\psi_1, \psi_2, ..., \psi_n\Big\}$. Let $\mathcal{P}_n : \mathcal{W}_{2,\theta}^{(3,2)}(\Omega) \rightarrow \Psi_n$ is a orthogonal projection. Obviously, if $\mathcal{U}(\mathrm{x}, \tau)$ is the exact solution of time-fractional delay PDE (3), then $\mathcal{U}_n(\mathrm{x}, \tau) = \mathcal{P}_n\mathcal{U}(\mathrm{x}, \tau)$ is an approximate solution for the problem (3) and $\mathcal{U}_n(\mathrm{x}, \tau)$ can be displayed as follows:

$$\mathcal{U}_n(\mathrm{x}, \tau) = \sum_{j=1}^{n} a_j \psi_j(\mathrm{x}, \tau), \tag{11}$$

where $a_1, a_2, ..., a_n$ are unknown coefficients.

The following theorem establishes the inner product condition with which the approximate solution of problem (3) is obtained.

Theorem 3.2 Let $\mathcal{U}_n(\mathrm{x}, \tau)$ be an approximate solution of the time-fractional PDE (3). Then, $\mathcal{U}_n(\mathrm{x}, \tau)$ satisfies to the following equation:

$$\Big\langle \mathcal{U}_n, \psi_i \Big\rangle = \mathcal{G}(\mathrm{x}_i, \tau_i), \quad i = 1, 2, ..., n. \tag{12}$$

Proof Let $\mathcal{U}(\mathrm{x}, \tau) \in \mathcal{W}_{2,\theta}^{(3,2)}$ be the exact solution to the problem (3), then $\mathcal{F}\mathcal{U}(\mathrm{x}, \tau) = \mathcal{G}(\mathrm{x}, \tau)$. In addition,

$$\begin{aligned}
\Big\langle \mathcal{U}_n, \psi_i \Big\rangle_{\mathcal{W}_{2,\theta}^{(3,2)}} &= \Big\langle \mathcal{P}_n\mathcal{U}, \psi_i \Big\rangle_{\mathcal{W}_{2,\theta}^{(3,2)}} = \Big\langle \mathcal{U}, \mathcal{P}_n\psi_i \Big\rangle_{\mathcal{W}_{2,\theta}^{(3,2)}} = \Big\langle \mathcal{U}, \psi_i \Big\rangle_{\mathcal{W}_{2,\theta}^{(3,2)}} \\
&= \Big\langle \mathcal{U}, \mathcal{F}^*\mathcal{K}_{(1,1)}(\mathrm{x}_i, \tau_i; x, t) \Big\rangle_{\mathcal{W}_{2,\theta}^{(3,2)}} = \Big\langle \mathcal{F}\mathcal{U}, \mathcal{K}_{(1,1)}(\mathrm{x}_i, \tau_i; \mathrm{x}, \tau) \Big\rangle_{\mathcal{W}_{2,\theta}^{(3,2)}} \\
&= \mathcal{F}\mathcal{U}(\mathrm{x}_i, \tau_i) = \mathcal{G}(\mathrm{x}_i, \tau_i), \quad i = 1, 2, ..., n.\square
\end{aligned}$$

In order to find an approximation solution for problem (3), by substituting Eq.(11) into Eq.(12), we obtain

$$\sum_{j=1}^{n} a_j < \psi_j, \psi_i >= \mathcal{G}(\mathrm{x}_i, \tau_i), \quad i = 1, 2, ..., n. \tag{13}$$

The linear system of equations (13) can be rewritten as follows:

$$\mathcal{U}\mathbf{A} = \mathbb{G}, \tag{14}$$

where

$$\mathcal{U} = \begin{bmatrix} <\psi_1, \psi_1> & <\psi_1, \psi_2> & \cdots & <\psi_1, \psi_n> \\ <\psi_2, \psi_1> & <\psi_2, \psi_2> & \cdot & <\psi_2, \psi_n> \\ \vdots & \vdots & \ddots & \vdots \\ <\psi_n, \psi_1> & <\psi_n, \psi_2> & \cdot & <\psi_n, \psi_n> \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \mathbb{G} = \begin{bmatrix} \mathcal{G}(\mathrm{x}_1, \tau_1) \\ \mathcal{G}(\mathrm{x}_2, \tau_2) \\ \vdots \\ \mathcal{G}(\mathrm{x}_n, \tau_n). \end{bmatrix}$$

The set $\{\psi_i(\mathrm{x}, \tau)\}_{i=1}^{n}$ is a linearly independent subset of $\mathcal{W}_{2,\theta}^{(3,2)}(\omega)$, therefore $\mathbb{G}^{-1}$ is revertible. Solving the linear system of equations (14) by any method provides $\mathbf{A} = (a_1, a_2, ..., a_n)^T$. In fact, we have shown that Eq.(3) has a solution and it is unique.

Theorem 3.3 (See [18]) Both $\mathcal{U}_n(\mathrm{x}, \tau)$ and $\bigtriangledown\mathcal{U}_n(\mathrm{x}, \tau)$ uniformly converge to $\mathcal{U}(\mathrm{x}, \tau)$ and $\bigtriangledown\mathcal{U}(\mathrm{x}, \tau)$, respectively.

4. Numerical examples

This section deals with the numerical review of the method described in the previous sections. For this purpose, the obtained numerical results will be compared with the exact solutions using the following error function

$$E_N = |\mathcal{U}(\mathrm{x}, \tau) - \mathcal{U}_N(\mathrm{x}, \tau)|.$$

The package of Mathematica 12 and the command *NSolve* have been used to obtain the numerical results and the calculations have been implemented on a Intel Core i7-4790k and 4 GHz CPU and 4 GB RAM. The nodes $\{(\mathrm{x}_i, \tau_i)\}_{i=0}^{i=N}$ are distributed uniformly, $h = \frac{1}{N}$.

Example 4.1 Consider the time-fractional delay diffusion equation given by

$$\begin{cases} \frac{\partial^\alpha \mathcal{V}(\mathrm{x},\tau)}{\partial \tau^\alpha} - \frac{\partial^2 \mathcal{V}(\mathrm{x},\tau)}{\partial \mathrm{x}^2} = \mathcal{V}(\mathrm{x}, \tau - 1) + \mathrm{h}(\mathrm{x}, \tau), & (\mathrm{x}, \tau) \in [0,1] \times [0,1]; \\ \mathcal{V}(0, \tau) = 0, \quad \mathcal{V}(1, \tau) = 0, & 0 \leqslant \tau \leqslant \mathbb{T}; \\ \mathcal{V}(\mathrm{x}, \tau) = \tau^3 \mathrm{x}(\mathrm{x} - 1) & (\mathrm{x}, \tau) \in [0,1] \times [-1, 0]. \end{cases} \tag{15}$$

where

$$\mathrm{h}(\mathrm{x}, \tau) = -2\tau^3 - \mathrm{x}(\tau - 1)^3(\mathrm{x} - 1) - \frac{6\mathrm{x}\tau^{3-\alpha}(\mathrm{x} - 1)}{(\alpha^3 - 6\alpha^2 + 11\alpha - 6)\Gamma(1 - \alpha)}.$$

The exact solution to this problem is $\mathcal{V}(\mathrm{x}, \tau) = \tau^3 \mathrm{x}(\mathrm{x} - 1)$. The computational results for $N = 8$ and various α are recorded in Table 1.

Table 1. Numerical errors at different points for Example 4.1

x	τ	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
0.25	0.25	6.3682×10^{-5}	7.2741×10^{-5}	1.1085×10^{-4}
	0.5	3.2415×10^{-4}	2.9997×10^{-4}	2.6807×10^{-4}
	0.75	1.5799×10^{-3}	1.51663×10^{-3}	1.5410×10^{-3}
0.5	0.25	1.2753×10^{-4}	1.4114×10^{-4}	2.0719×10^{-4}
	0.5	1.7507×10^{-4}	1.4059×10^{-4}	7.0030×10^{-4}
	0.75	1.2868×10^{-3}	1.1997×10^{-3}	1.1911×10^{-3}
0.75	0.25	1.0842×10^{-4}	1.1782×10^{-4}	1.7476×10^{-4}
	0.5	2.6043×10^{-5}	2.7158×10^{-6}	6.1122×10^{-5}
	0.75	6.0129×10^{-4}	5.4032×10^{-4}	5.1938×10^{-4}

Approximate solutions $\mathcal{U}_4(\mathrm{x}, \tau)$ for various value of α are plotted in Fig. 1. Moreover, the corresponding logarithmic absolute errors are plotted in Fig. 2.

Example 4.2 Consider the following time-fractional delay PDE with non-homogeneous boundary conditions adopted from [14]:

$$\begin{cases} \frac{\partial^\alpha \varpi(\mathrm{x},\tau)}{\partial \tau^\alpha} - \frac{\partial^2 \varpi(\mathrm{x},\tau)}{\partial \mathrm{x}^2} = \varpi(\mathrm{x}, \tau) + \varpi(\mathrm{x}, \tau - \theta) + \mathrm{h}(\mathrm{x}, \tau), & (\mathrm{x}, \tau) \in [0,1] \times [0, \mathbb{T}]; \\ \varpi(0, \tau) = \varpi(1, \tau) = \tau^3, & 0 \leqslant \tau \leqslant \mathbb{T}; \\ \varpi(\mathrm{x}, \tau) = \tau^3 \cos(2\pi \mathrm{x}) & (\mathrm{x}, \tau) \in [0,1] \times [0, \mathbb{T}]. \end{cases} \tag{16}$$

with

$$\mathrm{h}(\mathrm{x}, \tau) = \cos(2\pi \mathrm{x}) \Big((\theta - \tau)^3 - \tau^3 + 4\pi^2 \tau^3 - \frac{6\tau^{3-\alpha}}{(-6 + 11\alpha - 6\alpha^2 + \alpha^3)\Gamma[1 - \alpha]} \Big).$$

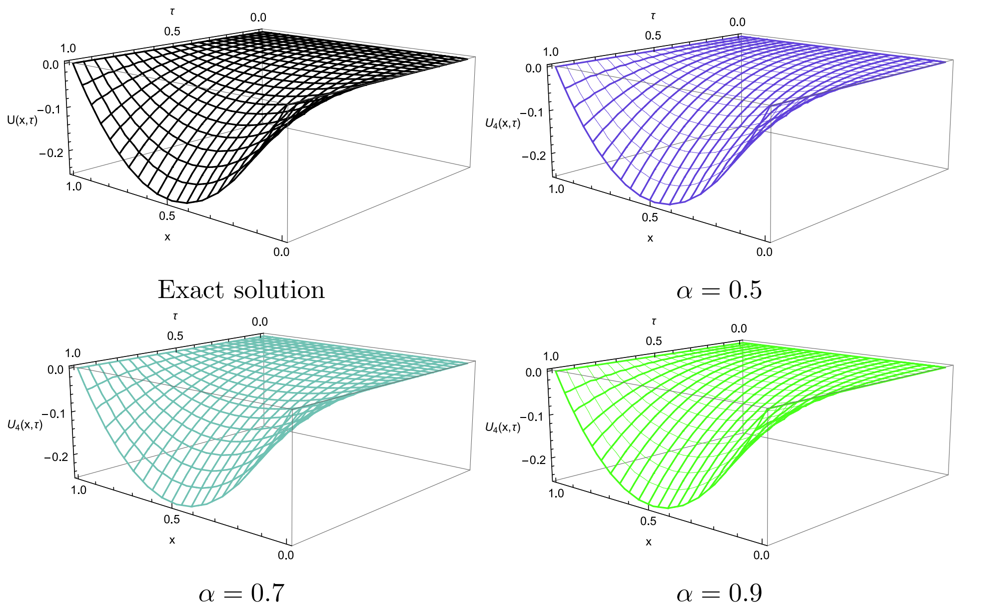

Figure 1. The exact solution and approximate solutions $\mathcal{U}_4(\mathrm{x},\tau)$ for different α in Example 4.1.

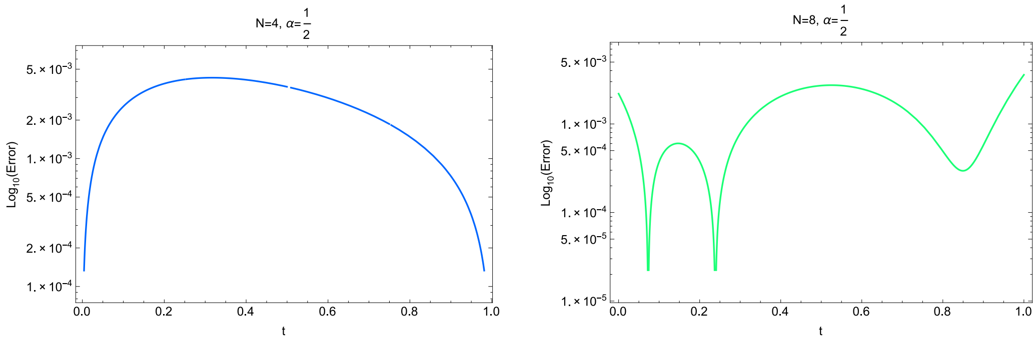

Figure 2. The error function graph for Example 4.1.

For $\theta = 1$ the exact solution is $\varpi(\mathrm{x},\tau) = \tau^3\cos(2\pi\mathrm{x})$. Using the procedure which is discussed in detail in the Section 1, Eq.(16) is transformed to the following homogeneous problem:

$$\begin{cases} \frac{\partial^\alpha \mathcal{U}(\mathrm{x},\tau)}{\partial \tau^\alpha} - \frac{\partial^2 \mathcal{U}(\mathrm{x},\tau)}{\partial \mathrm{x}^2} = \mathcal{U}(\mathrm{x},\tau) + \mathcal{U}(\mathrm{x},\tau-\theta) + \mathcal{G}(\mathrm{x},\tau), & (\mathrm{x},\tau) \in [0,1]\times[0,\mathbb{T}]; \\ \mathcal{U}(0,\tau) = \mathcal{U}(1,\tau) = 0, & 0 \leqslant \tau \leqslant \mathbb{T}; \\ \mathcal{U}(\mathrm{x},\tau) = 0 & (\mathrm{x},\tau) \in [0,1]\times[-\theta,0]. \end{cases}$$

where

$$\mathcal{G}(\mathrm{x},\tau) = \begin{cases} \mathrm{F}(\mathrm{x},\tau) + \Phi(\mathrm{x},\tau-\theta), & 0 < \tau \leqslant \tau; \\ \\ \mathrm{F}(\mathrm{x},\tau), & \theta < \tau \leqslant \mathbb{T}. \end{cases}$$

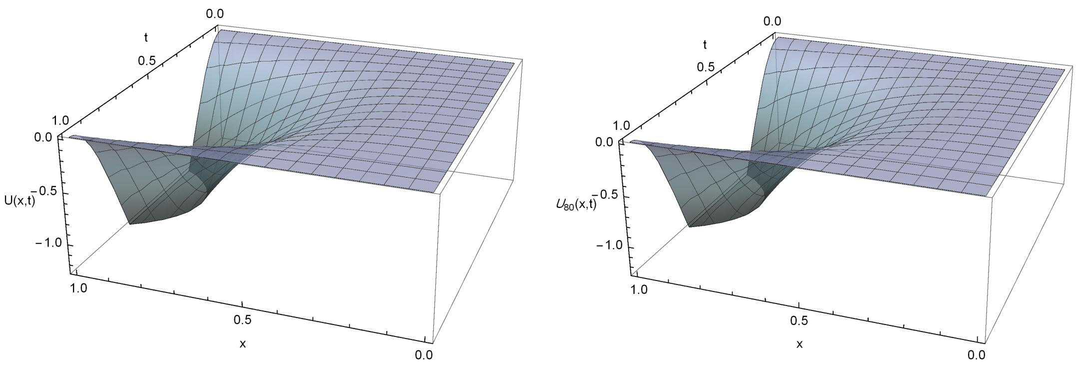

Figure 3. The form of the exact and approximate solutions, $\mathcal{U}_{80}(\mathrm{x},\tau)$ for Example 4.2.

and

$$\mathrm{F}(\mathrm{x},\tau) = \frac{12\tau^3 - \alpha\sin^2(\pi\mathrm{x})}{(-6+11\alpha-6\alpha^2+\alpha^3)\Gamma(1-\alpha)} + 4\pi^2\tau^3\cos(2\pi\mathrm{x}) + 2\Big(\tau^3 - (\theta-\tau)^3\Big)\sin^2(\pi\mathrm{x}).$$

Table 2 shows the errors obtained by the SRKM with $N = 80$ and the finite difference method [14]. The results show that the accuracy of the SRKM is similar to finite difference method. It is evident from the Table 2 and Fig. 3 that the approximate solutions are concurrent converge to the exact solution.

Table 2. Comparison of numerical results for Example 4.2.

h	Presented method $\alpha = 0.3$	Method in [14] $\alpha = 0.3$	Presented method $\alpha = 0.7$	Method in [14] $\alpha = 0.7$
$\frac{1}{20}$	3.4532×10^{-3}	2.4745×10^{-3}	2.2845×10^{-3}	3.9348×10^{-3}
$\frac{1}{40}$	7.5413×10^{-4}	6.2547×10^{-4}	1.4253×10^{-4}	9.8970×10^{-4}
$\frac{1}{80}$	1.3214×10^{-4}	1.5721×10^{-4}	2.1687×10^{-5}	2.4813×10^{-4}
$\frac{1}{160}$	1.2577×10^{-5}	3.9407×10^{-5}	3.3567×10^{-5}	6.2111×10^{-5}
$\frac{1}{320}$	3.6362×10^{-6}	9.8644×10^{-6}	1.1241×10^{-6}	1.5536×10^{-5}

Example 4.3 Next, we consider the following time-fractional reaction-diffusion equation with time delay:

$$\frac{\partial^\alpha \varpi(\mathrm{x},\tau)}{\partial\tau^\alpha} - \frac{\partial^2\varpi(\mathrm{x},\tau)}{\partial\mathrm{x}^2} = \varpi(\mathrm{x},\tau-1) + \mathrm{x}^2\left(\frac{\Gamma\left(\frac{8}{3}\right)\tau^{\frac{5}{3}-\alpha}}{\Gamma\left(\frac{8}{3}-\alpha\right)} + \frac{\Gamma\left(\frac{7}{3}\right)\tau^{\frac{4}{3}-\alpha}}{\Gamma\left(\frac{7}{3}-\alpha\right)}\right) - 2(\tau^{\frac{5}{3}}+\tau^{\frac{4}{3}}) - \mathrm{x}^2\Big((\tau-1)^{\frac{5}{3}} + (\tau-1)^{\frac{4}{3}}\Big),$$

$$(\mathrm{x},\tau) \in [0,1]\times[0,\mathbb{T}]$$

with the initial and boundary conditions

$$\varpi(0,\tau) = 0, \quad \varpi(1,\tau) = \tau^{\frac{5}{3}} + \tau^{\frac{4}{3}}, \qquad 0 \leqslant \tau \leqslant \mathbb{T}.$$

$$\varpi(\mathrm{x},\tau) = \mathrm{x}^2(\tau^{\frac{5}{3}} + \tau^{\frac{4}{3}}) \qquad (\mathrm{x},\tau) \in [0,1]\times[-1,0]$$

The given equation is solved using the method stated in this paper and the numerical results are shown in Table 3. The exact solution to this problem is unavailable. Hence, the numerical solution with $N = 100$ is considered as the benchmark for computing the errors.

Table 3. Numerical errors obtained for Example 4.3.

(x, τ)	$N = 40$	$N = 80$
$(0.0, 0.0)$	8.6676×10^{-4}	1.7271×10^{-4}
$(0.2, 0.2)$	4.4562×10^{-3}	2.4632×10^{-3}
$(0.4, 0.4)$	1.9565×10^{-3}	3.5691×10^{-3}
$(0.6, 0.6)$	2.4501×10^{-4}	3.7141×10^{-4}
$(0.8, 0.8)$	1.9121×10^{-3}	3.7411×10^{-5}

5. Conclusions

The numerical solution of the time delay and time-fractional reaction-diffusion equation has been investigated by using a new simplified reproducing kernel method. Primarily, a novel reproducing kernel space satisfying the time delay condition was introduced, and the approximate solution to the time-fractional delay PDE was represented in the form of series belonging to the proposed new reproducing kernel space. Ultimately, the effectiveness of the method was exhibited by various instances deciphered.

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