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Mutual Complexity in Hyperscaling Violating Background

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Abstract. In this paper, to investigate the concept of 'mutual complexity', in hyperscaling violating backgrounds, we employed the complexity equals action proposal.In order to describe holographic complexity for two subregions, we identify the definite bulk action inside the subregions, followed by the introduction of appropriate counterterms. We demonstrate that for two subregions, mutual complexity is subadditive. Furthermore, for three subregions, we introduce the concept of holographic 'tripartite complexity' and prove that this new quantity is superadditive.

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1. Introduction

one idea that has been the subject of discussion among physicists for years is quantum gravity. However, no universally accepted approach has been presented for this concept. On the other hand, for a better understanding of gravity and its interaction with matter, we need a quantum description of gravity. This is because, similar to how our perception of the interaction of light with matter was incomplete until electrodynamics was formulated quantum mechanically, our understanding of gravity will remain incomplete without a quantum framework.

Numerous attempts to access this theory have been unsuccessful. It is proposed that the idea of holography can create an effective connection between gravity and gauge theory, with the AdS/CFT correspondence proposal elucidating the main concept of holography [12].

An acceptable theory of gravity must describe the nature of space-time. To employ holographic methods, one should rely on explanations of bulk spacetime derived from the dual field theory description. Two quantities, entanglement holographic entropy and complexity, can serve as measures to understand the emergence of spacetime from the field theory perspective.

Ryu and Takayanagi's conjecture significantly aids in the holographic description of entanglement entropy [13]. This conjecture asserts that the entanglement entropy of a given region in a conformal field theory, situated on the AdS boundary, is fundamentally a geometric description. In fact, the entropy of holographic entanglement is obtained from the following relation: it is the minimum area of a surface in the AdS bulk that ends at the

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boundary of the region. Susskind and his collaborators conjectured that the region behind the horizon of an AdS black hole is related to the complexity of the dual boundary state [3, 4, 14, 15, 16, 17]. It's a new relation between the bulk geometry and the dual boundary state living in the conformal field theory. There is a proposal named the Complexity = Volume (CV) conjecture. According to this conjecture, the volume of a maximal spacelike slice should define the complexity of the state in the boundary dual field theory at a given time.

There is another conjecture in which the complexity is related to the Wheeler-DeWitt (WDW) patch action in a bulk bounded by a spacelike surface. This proposal is named the Complexity=Action (CA) conjecture. Additional explanations for these two conjectures can be found in [5,7,11].

The CV and CA conjectures state that the complexity of states increases linearly over time, corresponding to the region behind the black hole horizon.

The mass of the black hole and the time derivative are proportional to each other. Considering the AdS/CFT correspondence, the mass of the black hole can be interpreted as the energy state in the dual theory. In [1], the authors defined a quantity termed mutual complexity. This finite quantity can be considered a quantum measure of the correlation between two corresponding subsystems. And in reference [8], using the purification complexity for mixed states, the mutual complexity for both the thermal density matrix and the reduced density matrix is investigated for a subregion of the vacuum.

The purpose of this paper is to further investigate mutual complexity in a broader family of states that support anisotropic as well as hyperscaling-violating exponents. In theories of hyperscaling violation, the isotropy of space and time is described by introducing two parameters:

$$r \rightarrow \omega r, \quad x \rightarrow \omega x, \quad t \rightarrow \omega^{z} t, \quad ds \rightarrow \omega^{\frac{\theta}{d}} ds \tag{1}$$

where z is the anisotropic (Lifshitz) component and θ\theta is the hyperscaling-violating exponent. Note that with non-zero θ\theta, the distance within the concept of AdS/CFT reflects the violations of hyperscaling in the dual field theory.

In this paper, we will review the computation of holographic mutual complexity in hyperscaling-violating backgrounds. Additionally, we introduce and examine tripartite complexity and finally discuss the results in the conclusion section.

2. Holographic Mutual Complexity

In the field of quantum information, mutual information for two isolated systems, such as A_1 and A_2, can be used as a measure to quantify the amount of entanglement (or information) shared by these two systems. For two isolated systems, the mutual information is given by the following relationship [9]:

$$I^{[2]}(A_1: A_2) = S(A_1) + S(A_2) - S(A_1 \cup A_2) \tag{2}$$

where S (A_i) is the entanglement entropy of the region A_i and $S(A_1 \cup A_2)$ refers to the entanglement entropy for the union of the two entangled regions. Considering this definition, a new quantity called mutual complexity was defined in [6] for two subregions of spacetime enclosed by null boundaries.

in the right panel of figure (1) and similary to (2), For a certain subregions, denoted by l_1 and l_2, mutual complexity is introduced by the following relation:

$$C(l_1: l_2) = C(l_1) + C(l_2) - C(l_1 \cup l_2) \tag{3}$$

where C stands for the complexity of the given region evaluated using by CA proposal. By calculating the finite part of the on-shell action inside these regions, we derive the mutual complexity.

Mutual complexity obeys the subadditivity condition. By computing the on-shell action in the exterior regions, we can investigate its properties.

Additionally, under the exchange of l_1 and l_2, mutual complexity is always non-negative, finite, and symmetric [12]. The computation for black brane solutions of Einstein gravity has been completed. We now aim to investigate the properties of this quantity in a hyperscaling-violating background.

3. The Setup

For a given subregion denoted by l, at the time slice $t = 0$, the intersection of the WDW patch and the entanglement wedge creates a square subregion, as shown in the left panel of Figure 1. Our aim is to compute the on-shell action inside this square. To achieve this, we first need to find the on-shell action for the three triangles, denoted by r_1 ،r_2 and r_p.

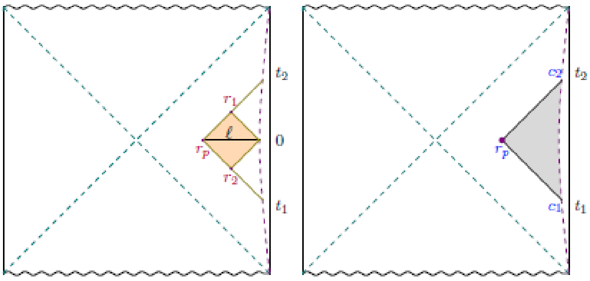

Figure 1. For a subregion, the orange region represents the intersection of the WDW patch and the entanglement wedge at the time slice t = 0 for half of an eternal black hole. The right panel shows the colored region in the shape of a triangle with two null sides and one timelike side. $\mathrm{r_p}$ is the joint point located at the crossing of the two null boundaries.

We obtain that the complexity for a subregion in the hyperscaling violating background (the shape of a triangle in the right panel of figure (2)) is given by the following relate:

$$I_{Tri}^{total} = \frac{V_d}{8\pi G_N}\left(\frac{z\tau^{\sim}}{2r_h^{d_e+z}} + \frac{1}{r_p^{d_e}} log\left(f(r_p)\right)\right) \quad (4)$$

where V_d is the d dimensional volume.
so, for the square subregion, by using the corresponding expression for three triangles and the equation (5),

$$S_A + S_B \geq S_{A\cup C} + S_{B\cup C} \quad (5)$$

so,

$$I_{s_q} = I_{r_p} - I_{r_1} - I_{r_2} = \frac{V_d}{8\pi G_N}\left(\frac{log|f(r_p)|}{r_p^d} - \frac{log|f(r_1)|}{r_1^d} - \frac{log|f(r_2)|}{r_2^d}\right) \quad (6)$$

Notic that, In writing the equation (6), all contributions obtain from the joint points because there is no timelike or spacelike boundaries.
In this case the most divergent term is positive as expected for an expression representing complexity.
To continue, we show two subregions l_1 وand l_2 on the right side of Figure 2.
Using the notation in the form and the relation $L^{\sim} = L_e$, it is obtained:

$$I_{l_1} = \frac{V_d}{8\pi G_N}\left(\frac{log|f(r_p)|}{r_p^{d_e}} - \frac{c_0}{r_h^{d_e}} - \frac{(d_e+z)r^*(r_p)}{r_h^{d_e+1}}\right) \quad (7)$$

$$f(r) = 1 - (\frac{r}{r_h})^{d+z-\theta} \quad (8)$$

Where $c_0 = \psi^{(0)}(1) - \psi^{(0)}(\frac{1}{d_e+1})$ is a positive number and $\psi^{(0)}(x) = \frac{\Gamma(x)}{\Gamma(x)}$ is the digamma function.

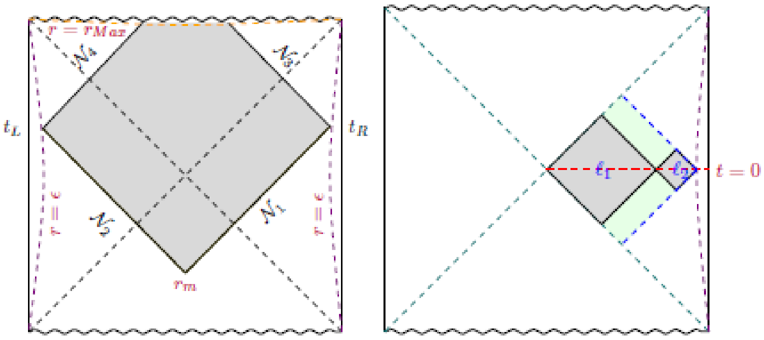

Figure 2. Left panel: The Penrose diagram of the WDW (Wheeler-DeWitt) patch of an eternal AdS black hole for $t_R = t_L$. The null boundaries are denoted by Ni. The infrared (IR) and ultraviolet (UV) cutoffs are represented by ε and r_{Max}.{\text{Max}}, respectively. To determine the complexity, one must compute the on-shell action inside this patch.
Right panel: The two subregions are labeled as l_1, l_2.

And for where $r^*(r)$:

$$r^*(r) = \int \frac{dr}{r^{1-z} f(r)} \tag{9}$$

And for l_2:

$$I_{l_2} = \frac{V_d}{8\pi G_N}\left(\frac{\log|f(r_p)|}{r_p^{d_e}} - \frac{\log|f(r_1)|}{r_1^{d_e}} - \frac{\log|f(r_2)|}{r_2^{d_e}}\right) \tag{10}$$

For the union part:

$$I_{l_1 \cup l_2} = -\frac{V_d}{8\pi G_N}\frac{c_0}{r_h^{d_e}} \tag{11}$$

For two subregions in a hyperscaling-violating background, using the definition of mutual complexity, we obtain:

$$A^{[2]} \equiv I_{l_1} + I_{l_2} - I_{l_1 \cup l_2} = \frac{V_d L^d}{8\pi G_N}\left(-\frac{2\log|f(r_p)|}{r_p^{d_e}} - \frac{\log|f(r_1)|}{r_1^{d_e}} - \frac{\log|f(r_2)|}{r_2^{d_e}} - \frac{(d_e+z) r^*(r_p)}{r_h^{d_e+1}}\right) \tag{12}$$

Now, to determine the sign of $A^{[2]}$ [10], we observe that in both limits $\{r_2, r_p, r_1\} \to r_h$ and $\{r_2, r_p, r_1\} \to 0$, $A^{[2]}$ vanishes. Additionally, the function $A^{[2]}$ approaches zero from above when at $\{r_2, r_p, r_1\} \approx 0$.

As a result, the on-shell action obeys subadditivity condition:

$$I_{l_1} + I_{l_2} \geq I_{l_1 \cup l_2} \tag{13}$$

so,

$$A^{[2]} \geq 0 \tag{14}$$

We know that for general quantum systems, the von Neumann entanglement entropies of the subsystems obey the subadditivity

$$S(A) + S(B) - S(A \cup B) \geq 0 \tag{15}$$

weak monotonicity:

$$S(A \cup B) + S(A \cup C) - S(B) - S(C) \geq 0 \tag{16}$$

and strong subadditivity:

$$S\,(A\,\cup B)+S\,(A\,\cup C)-S\,(A)-S\,(A\,\cup B\cup C)\geq\,0 \quad (17)$$

Moreover, the holographic entanglement entropies for holographic systems (the states of CFTs with classical holographic duals) obey a larger set of inequalities than those followed by generic quantum mechanical systems. The new inequality, provided by the tripartite information, is defined by the following relation:

$$S\,(A\,\cup B)+S\,(A\,\cup C)+S\,(B\,\cup C)\geq S\,(A)+S\,(B)+S\,(C)+S\,(A\,\cup B\cup C) \quad (18)$$

Scientists have explored these types of inequalities within the framework of quantum information theory and the quantum error correction interpretation of AdS/CFT in several significant studies [2]. Notably, the strong collectivity property of entanglement entropy, which must be upheld by any quantum system, has been confirmed through holographic methods.

This can serve as a validation of AdS/CFT. Monogamy, which asserts that quantum entanglement—unlike classical correlations—cannot be freely shared among multiple parties, is a property reflected in mutual information and is exclusively satisfied by holographic systems. As such, the ability of holographic theories to satisfy this relation provides a means to differentiate between CFTs with holographic dual potentials.

In this study, we focus on calculating the sub-region complexity within a hyperscaling-violating background. Following the principle of complexity being equivalent to action, it was demonstrated in [6] that the shell action of subregions satisfies a specific condition of collectivity. Building on this observation and recognizing the significance of inequalities in holographic systems, we define the concept of tripartite complexity and examine its sign within our model.

4. Holographic Tripartite Complexity

By extending the previous instruction to three subregions, labeled l_1، l_2 and l_3, (as illustrated in Figure 3), we can define a new quantity known as tripartite complexity. This concept can be explained as follows:

$$C(l_1\!:l_2\!:l_3)=C(l_1)+C(l_2)+C(l_3)-C(l_1\cup l_2)-C(l_1\cup l_3)-C(l_2\cup l_3)+C(l_1\cup l_2\cup l_3) \quad (19)$$

where $C(l_1\cup l_2\cup l_3)$ is the complexity of the union of three subregions.

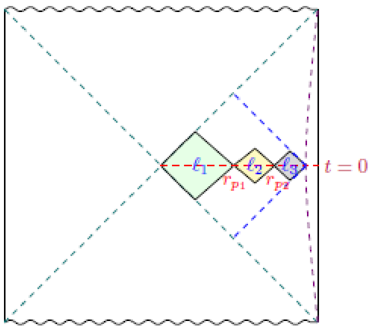

Figure 3. Three sub-regions showed by l_1, l_2, l_3

$$I_{l_1}=\frac{V_d}{8\pi G_N}\left(\frac{log|f(r_{p_1})|}{r_{p_1}^{d_e}}-\frac{c_0}{r_h^{d_e}}-\frac{(d_e+z)r^*(r_{p_1})}{r_h^{d_e+1}}\right) \quad (20)$$

$$I_{l_2}=\frac{V_d}{8\pi G_N}\left(\frac{\log|f(r_{p_1})|}{r_{p_1}^{d_e}}+\frac{\log|f(r_{p_2})|}{r_{p_2}^{d_e}}-\frac{\log|f(r_1)|}{r_1^{d_e}}-\frac{\log|f(r_2)|}{r_2^{d_e}}\right) \quad (21)$$

$$I_{l_3} = \frac{V_d}{8\pi G_N}\left(\frac{\log|f(r_{p_2})|}{r_{p_2}^{d_e}} - \frac{\log|f(r_3)|}{r_3^{d_e}} - \frac{\log|f(r_4)|}{r_4^{d_e}}\right) \tag{22}$$

For the union part of subregions l_1 and l_2:

$$I_{l_1\cup l_2} = \frac{V_d}{8\pi G_N}\left(\frac{\log|f(r_{p_1})|}{r_{p_1}^{d_e}} - \frac{c_0}{r_h^{d_e}} - \frac{(d_e+z)r^*(r_{p_1})}{r_h^{d_e+1}}\right) \tag{23}$$

similarly we can write for l_2 and l_3:

$$I_{l_2\cup l_3} = \frac{V_d}{8\pi G_N}\left(\frac{log|f(r_{p_2})|}{r_{p_2}^{d_e}} - \frac{log|f(r_3)|}{r_3^{d_e}} - \frac{log|f(r_4)|}{r_4^{d_e}}\right) \tag{24}$$

moreover, for the on-shell action for the union part of l_1 and l_2 ,one can write $I_{l_1\cup l_3} = I_{l_1}+I_{l_3}$.

Also,

$$I_{l_1\cup l_2\cup l_3} = -\frac{V_d}{8\pi G_N}\frac{c_0}{r_h^{d_e}} \tag{25}$$

Finally, one can write:

$$A^3 = I_{l_1} + I_{l_2} + I_{l_3} - I_{l_1\cup l_2} - I_{l_2\cup l_3} - I_{l_1\cup l_3} + I_{l_1\cup l_2\cup l_3} \tag{26}$$

And,
by approaching the joint points, to the horizon, namely, $\mathrm{r_{jnt}} \to \mathrm{r_h}$
it can be shown that:

$$A^{[3]} = \frac{V_d}{16\pi G_N}\frac{(d_e+z)\tau}{r_h^{d_e+1}} \tag{27}$$

Consequently, it was discovered that the on-shell action, calculated for the three subregions external to the black brane, conforms to the following relation:

$$A^{[3]} < 0 \tag{28}$$

5.Conclusions

In [8], the authors utilized the concept of mutual complexity to compare the complexity of the thermofield double state purification with that of a thermal mixed state.
In this approach, mutual complexity is introduced for a pure state $|\psi_{\mathrm{AB}}\rangle$ defined on an extended Hilbert space. By tracing out the degrees of freedom associated with B, we obtain the mixed state ρ_{A}.
Similarly, by integrating out the degrees of freedom associated with A, we derive ρ_{B}. By comparing the complexities of these three states, we arrive at the concept of mutual complexity.

$$\Delta C = c(\rho_A) + c(\rho_B) - C(|\psi_{AB}\rangle) \tag{29}$$

If $\Delta\mathrm{C} > 0$, this indicates that complexity is subadditive, demonstrating that the complexity of the state on the entire system is less than the sum of the complexities of the states on the two subsystems. For $\Delta\mathrm{C} < 0$, the complexity is superadditive, indicating that the complexity of the state on the entire system exceeds the sum of the complexities of the states on the two subsystems. The mutual complexity can be either positive or negative, depending on the temperature of the thermal state and the frequency of the reference state. Notably, for subregion complexity, it was demonstrated in [1] that mutual complexity is subadditive. In this study, we further explored this property and confirmed that our findings support the positivity of mutual complexity in a hyperscaling-violating background.

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