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# Improvements of Some Inequalities via Steffensen-Popoviciu Measure and it's Dual

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**Abstract.** In this paper we establish new inequalities for convex and strongly convex defined on intervals in framework of Steffensen-Popoviciu and Dual Steffensen-Popoviciu measures are introduced. Some inequalities in this setting are also involved. Suitable examples are also involved are given.

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## 1. Introduction

Convexity of sets and function with their generalizations plays a crucial role in mathematical analysis and related topics as inequalities, optimization and other related fields and inequalities are important in mathematical analysis and its applications. There have been several works in the literature which are devoted to investigating generalizations of this topic, see [4, 10, 13] and references therein. On the other hand several important inequalities are derived and improved by using the Steffensen-Popoviciu measure ( SP ) and Dual Steffensen-Popoviciu measure ( DSP ) see for example [5–8]. The integral of a nonnegative convex or concave functions, in terms of an arbitrary measure, may not be necessarily nonnegative,

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but their integral in terms of SP and DSP measures are nonnegative. Characterization of SP measure investigated by T. Popoviciu in [11] and A. M. Fink in [2]. Additionally, C. P. Niculescu in [7] obtained new Jensen-type inequalities using these measures and in [8] introduced the notion of DSP measure on interval and obtained sufficient condition for DSP measure.

Assume that $I$ is an interval of real numbers. The set of all continuous real valued functions on $I$ will denoted by $C(I)$, the set of all continuous real valued convex and concave functions on $I$, will denoted by $C$ and $-C$, respectively, and the set of all continuous real valued affine functions on $I$ will represented by $A(I)$.

We recall some definitions and results in convex analysis which we need throughout the paper (see for example from [8, 10]).

**Definition 1.1** Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$, be a real valued function. Then,
(a) $f$ is said to be convex if for every $x, y \in I$,

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y), \quad \text{for all } t \in [0,1],$$

(b) if for some $c > 0$ and for every $x, y \in I$,

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2, \text{ for all } t \in [0,1],$$

then $f$ is said to be strongly convex with modulus $c$.

Clearly every strongly convex function is convex but the converse is not true as we see in the following example.

**Example 1.2** Define the function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} (x+2)^2, & x < -2 \\ 0, & -2 \leqslant x \leqslant 2 \\ (x-2)^2, & x > 2. \end{cases}$$

It is clear that the function $f$ is a convex function while function $f$ is not strongly convex with some modulus $c > 0$. Indeed, by chossing $x := 2, y := -2$ and $t := \frac{1}{2}$ we have:

$$f\left(\frac{1}{2}(2) + \frac{1}{2}(-2)\right) \leqslant \frac{1}{2}f(2) + \frac{1}{2}f(-2) - 4c,$$

so $c \leqslant 0$, and this is contradictory to the strong convexity of $f$.

The following characterization of strongly convex functions introduced in Proposition 1.1.2 from [3]

**Lemma 1.3** *A function $f : I \to \mathbb{R}$ is strongly convex with modulus c if and only if the function $g : I \to \mathbb{R}$ defined by $g(x) = f(x) - cx^2$ is convex.*

The SP and DSP measures are specific Borel measure wich are related to convex and concave functions. See [5, 8].

**Definition 1.4** A real Borel measure $\mu$ defined on interval $I = [a, b]$ referred to as

(1) Steffensen-Popoviciu measure provided that,
 i) $\mu(I) > 0$,

ii) $\int_a^b f(x)d\mu(x) \geqslant 0$ for every nonnegative $f \in C$.

(2) dual Steffensen-Popoviciu measure provided that,
 i) $\mu(I) > 0$,
 ii) $\int_a^b f(x)d\mu(x) \geqslant 0$ for any nonnegative $f \in -C$.

It can be seen that every finite positive Borel measure is both a SP and DSP measures. Although we don't have a criterion to identiy the DSP measures but Popoviciu [11] and Fink [2] introduce the following lemma to identify the SP measures. See also [6, p177], for details.

**Lemma 1.5** Assume that $\mu$ is a real Borel measure defined on the interval $[a, b]$ such that $\mu([a, b]) > 0$. Then $\mu$ qualifies as a SP measure iff, it satisfies the following conditions

$$\int_a^x (x-t)d\mu(t) \geqslant 0 \quad \text{and} \quad \int_x^b (t-x)d\mu(t) \geqslant 0,$$

for any $x \in [a, b]$.

The measures

$$(x^2 - 1/6)^3 dx \quad on \quad [-1, 1]$$

$$\left[ (\frac{2x-a-b}{b-a})^2 + \alpha \right] dx \quad on \quad [a, b] \quad (\alpha \geqslant -\frac{1}{4})$$

$$\left[ (\frac{2x-a-b}{b-a})^2 - \alpha(\frac{2x-a-b}{b-a}) \right] dx \quad on \quad [a, b] \quad (|\alpha| \leqslant \frac{2}{3}),$$

are examples of SP measure, see [8]. Moreover we recall the following examples of DSP measures over the interval $[a, b]$ (see [5] and [8]).

**Example 1.6** The three measures

$$-\delta_a + \delta_{\frac{3a+b}{2}} + \delta_{\frac{a+b}{2}} + \delta_{\frac{a+3b}{2}} - \delta_b$$

$$\left[ 6\left(x - \frac{a+b}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2 \right] dx$$

$$\left[ \left(\frac{2x-a-b}{b-a}\right)^2 + \alpha \right] dx \quad for \ \alpha \geqslant -\frac{1}{6},$$

are DSP measures on interval $[a, b]$.

We also need the notion of concave on the coordinates function, see [1].

**Definition 1.7** The function $f : [a, b] \times [c, d] \to \mathbb{R}$ is known as concave with respect to the coordinates (coordinated concave) if maps $f_x : [c, d] \to \mathbb{R}$, defined by $f_x(v) = f(x, v)$ and $f_y : [a, b] \to \mathbb{R}$, defined by $f_y(u) = f(u, y)$, are both concave for any $x \in [a, b]$ and $y \in [c, d]$ respectively.

It is clear that every two variable concave function on a rectangle in $\mathbb{R}^2$, is concave on the coordinates function, See [1]. Motivated by the above studies we establish some new inequalities for convex and strongly convex functions defined on intervals via SP and DSP measures. Then some inequalities in this setting are also involved for two variables functions.

## 2. Main results

Jensens inequality is an important tool in convex analysis, revealing property of continuous convex functions by a Borel measure(mass distribution). Let $f$ is a continuous convex function on interval $I$ and $\mu$ is a real Borel measure on $I$. Then $\mu$ have barycenter

$$b_\mu = \frac{1}{\mu(I)} \int_I x d\mu(x),$$

and

$$f(b_\mu) \leqslant \frac{1}{\mu(I)} \int_I f(x) d\mu(x).$$

For more details see [6] and [9]. This inequality also holds for SP measures, see [6], Theorem 4.2.1. The analogue of this inequality for strongly convex functions is as follows:

**Theorem 2.1** Let $\mu$ be a $SP$ measure on interval $I$ and $f : I \to R$ be a continuous strongly convex function with modulus $c$. Then

$$f(b_\mu) \leqslant \frac{1}{\mu(I)} \int_I f d\mu(x) - \frac{c}{\mu(I)} \int_I x^2 d\mu(x) + cb_\mu^2.$$

***Proof*** By Lemma 1.3, $f(x) - cx^2$ is a continuous convex function and we have

$$f(b_\mu) - cb_\mu^2 \leqslant \frac{1}{\mu(I)} \int_I \left(f(x) - cx^2\right) d\mu(x),$$

therefore

$$f(b_\mu) \leqslant \frac{1}{\mu(I)} \int_I f(x) d\mu(x) - \frac{c}{\mu(I)} \int_I x^2 d\mu(x) + cb_\mu^2,$$

hence

$$f(b_\mu) \leqslant \frac{1}{\mu(I)} \int_I f(x) d\mu - c\left(\frac{1}{\mu(I)} \int_I x^2 d\mu\right) + cb_\mu^2.$$

■

**Remark 2.2** A function $f : I \to \mathbb{R}$ is strongly convex with modulus $c$ if and only if for every $x_0 \in \text{int } I$ there exists an $a \in \mathbb{R}$ such that

$$f(x) \geqslant c(x - x_0)^2 + a(x - x_0) + f(x_0), \quad x \in I$$

i.e. $f$ has a quadratic support at $x_0$. For a twice differentiable $f$, $f$ is strongly convex with modulus $c$ if and only if $f'' \geqslant 2c$, see [12].

Recall that a function $f : I \to \mathbb{R}$, is said to be quasiconvex provided that for every $x, y \in I$,

$$f(tx + (1-t)y) \leqslant \max\{f(x), f(y)\}, \quad \text{for all, } t \in [0, 1].$$

Now we drive a new inequality for strongly convex function, by using a specific SP mesure:

**Theorem 2.3** Let $f : [a,b] \longrightarrow R$ be a twice differentiable strongly convex function with modulus $c$, whose second derivative is an absolutely continuous quasiconvex function. Then

$$\frac{1}{b-a}\int_a^b f(x)dx \geqslant \frac{f(a)+f(b)}{2} - \frac{3(b-a)}{32}\left(f'(b)-f'(a)\right) + \frac{c(b-a)^2}{48}.$$

***Proof*** Taking into account that $f'' \geqslant 2c$ we get $f''-2c$ is a nonnegative absolutely continuous quasiconvex function. In the other hand

$$g(x)dx = \left(\left(\frac{2x-a-b}{b-a}\right)^2 - \frac{1}{4}\right)dx,$$

satisfy in conditions Theorem 2 in [8], therefore

$$\begin{aligned}
&0 \leqslant \int_a^b (f''(x)-2c)\left(\left(\frac{2x-a-b}{b-a}\right)^2 - \frac{1}{4}\right)dx \\
&\Rightarrow 2c\int_a^b \left(\left(\frac{2x-a-b}{b-a}\right)^2 - \frac{1}{4}\right)dx \leqslant \int_a^b f''(x)\left(\left(\frac{2x-a-b}{b-a}\right)^2 - \frac{1}{4}\right)dx \\
&\Rightarrow 2c\left(\frac{b-a}{12}\right) \leqslant \left[\left(\frac{2x-a-b}{b-a}\right)^2 - \frac{1}{4}\right] f'(x)\Big|_a^b - \frac{4}{b-a}\int_a^b \frac{2x-a-b}{b-a} f'(x)dx \\
&\Rightarrow \frac{c(b-a)}{6} \leqslant \frac{3}{4}\left(f'(b)-f'(a)\right) - \left[\frac{4}{b-a}\frac{2x-a-b}{b-a}f(x)\right]_a^b + \frac{8}{(b-a)^2}\int_a^b f(x)dx \\
&\Rightarrow \frac{c(b-a)}{6} \leqslant \frac{3}{4}\left(f'(b)-f'(a)\right) - \frac{4}{b-a}(f(a)+f(b)) + \frac{8}{(b-a)^2}t\int_a^b f(x)dx \\
&\Rightarrow \frac{1}{b-a}\int_a^b f(x)dx \geqslant \frac{f(a)+f(b)}{2} - \frac{3(b-a)}{32}\left(f'(b)-f'(a)\right) + \frac{c(b-a)^2}{48}.
\end{aligned}$$

■

At this point, we present the notion of DSP measure in plane and derive several new inequalities based on these measure. Let $D$ be a non-empty convex compact subset of $\mathbb{R}^2$. Assume that $C(D)$ is the space of all two variable continuous real valued functions on $D$ and let $-C$ represent the set of all two variable continuous and real valued concave functions over $D$.

**Definition 2.4** A DSP measure on $D$ is characterized as a real Borel measure $\mu$ on $D$ which meets the following criteria:

i) $\mu(D) = \int\int_D d\mu(x,y) > 0$,
ii) $\int\int_D f(x,y)d\mu(x,y) \geqslant 0$ for any nonnegative $f \in -C$.

In what follows we introduce some results and properties of DSP measures in this setting.

**Theorem 2.5** Let $p(x)dx$ and $q(x)dx$ denote two DSP measures defined over intervals $[a, b]$ and $[c, d]$, respectively. Then

$$\mu(x, y) = (p(x) + q(y))dxdy,$$

represents a DSP measure on the rectangle $[a, b] \times [c, d]$.

***Proof*** At first

$$\begin{aligned}
\mu([a, b] \times [c, d]) &= \int_c^d \int_a^b (p(x) + q(y))dxdy \\
&= \int_c^d \int_a^b p(x)dxdy + \int_c^d \int_a^b q(y)dxdy \\
&= (d - c) \int_a^b p(x)dx + (b - a) \int_c^d q(y)dy > 0,
\end{aligned}$$

expressed as a sum of non-negative terms. Suppose that $f : [a, b] \times [c, d] \to \mathbb{R}$ is a non-negative continuous real valued concave function, so $f$ is concave on coordinates. Thus

$$f(tx_1 + (1 - t)x_2, y) \geqslant tf(x_1, y) + (1 - t)f(x_2, y),$$

for every $x_1, x_2 \in [a, b]$, $y \in [c, d]$ and for any $t$ in $[0, 1]$. By integrating the above inequality we have

$$\int_c^d f(tx_1 + (1 - t)x_2, y)dy \geqslant t \int_c^d f(x_1, y)dy + (1 - t) \int_c^d f(x_2, y)dy.$$

Thus the map

$$x \to \int_c^d f(x, y)dy,$$

is a nonnegative concave function. Similarly the map

$$y \to \int_a^b f(x, y)dx,$$

is also a nonnegative concave function. Therefore

$$\begin{aligned}
&\int_a^b \int_c^d f(x, y)(p(x) + q(y))dydx \\
&\qquad = \int_a^b \left( \int_c^d f(x, y)dy \right) p(x)dx + \int_c^d \left( \int_a^b f(x, y)dx \right) q(y)dy \geqslant 0,
\end{aligned}$$

as a sum of nonnegative numbers. ■

**Remark 2.6** If $p$ and $q$ are the same ones in Theorem 2.5, one can readily see that the measure $(\alpha p(x) + \beta q(y))dxdy$ also qualifies as a DSP measure over $[a, b] \times [c, d]$ for any $\alpha, \beta \geqslant 0$.

**Example 2.7** According to Example 1.6, $\left[(\frac{2x-a-b}{b-a})^2+\alpha\right]dx$ is a DSP measure over $[a,b]$ for any $\alpha \geqslant -\frac{1}{6}$. Therefore, by Theorem 2.5

$$\left[(\frac{2x-a-b}{b-a})^2+(\frac{2y-c-d}{d-c})^2+\gamma\right]dxdy,$$

is a DSP measure over rectangle $[a,b]\times[c,d]$ for any $\gamma \geqslant -\frac{1}{3}$.

**Theorem 2.8** Let $p(x)dx$ and $q(y)dy$ be DSP measures defined over $[a,b]$ and $[c,d]$ respectively. Thus

$$\mu(x,y)=p(x)q(y)dxdy,$$

is a DSP measure over the rectangle $[a,b]\times[c,d]$ provided that at least one of the functions $p(x)$ or $q(y)$ is non-negative.

***Proof***

$$\begin{aligned}\mu([a,b]\times[c,d]) &= \int_c^d\int_a^b p(x)q(y)dxdy\\ &= \int_c^d\left(\int_a^b p(x)dx\right)q(y)dy = \left(\int_a^b p(x)dx\right)\left(\int_c^d q(y)dy\right)>0,\end{aligned}$$

as a product of positive numbers. Consider $f:[a,b]\times[c,d]\to\mathbb{R}$ as a non negative continuous concave function. Suppose that $p$ is a non-negative function defined over interval $[a,b]$. It is evident that the mapping

$$y\to\int_a^b f(x,y)p(x)dx,$$

is a concave and nonnegative function on the interval $[c,d]$. Thus

$$\int_c^d\int_a^b f(x,y)p(x)q(y)dxdy=\int_c^d\left(\int_a^b f(x,y)p(x)dx\right)q(y)dy\geqslant 0.$$

Therefore the $\mu(x,y)=p(x)q(y)dxdy$ is a DSP measure over $[a,b]\times[c,d]$. ■

These types of measures are also applicable to triangular domains of the form $\{(x,y): x\geqslant 0, y\geqslant 0, x+y\leqslant c\}$ for $c>0$.

**Example 2.9** According to Example 1.6 and by applying Theorem 2.8 it is easy to see that

$$\left(\frac{2x-a-b}{b-a}\right)^2\left(\left(\frac{2y-c-d}{d-c}\right)^2+\alpha\right)dxdy,$$

is a DSP measure over $[a,b]\times[c,d]$ which $\alpha\geqslant-\frac{1}{6}$.

**Remark 2.10** Consider $p(x)dx$ and $q(y)dy$ as two DSP measures as established in Theorem 2.8 and $f:[a-d,b-c]\to\mathbb{R}$ is a nonnegative concave function of class

$C^2$ with a concave second derivative. Then we have

$$\int_a^b \int_c^d f(x-y)p(x)q(y)dydx \geqslant 0.$$

In next theorem we obtain a DSP measure under certain conditions.

**Theorem 2.11** Let $p(x)dx$ and $q(y)dy$ be DSP measures defined over $[a,b]$ and $[c,d]$, respectively. Thus

$$\mu(x,y) = \frac{p(x)}{q(y)}dxdy,$$

is a DSP measure on the rectangle $[a,b]\times[c,d]$, provided that $q(y)$ is a positive function on the interval $[c,d]$.

***Proof***

$$\begin{aligned}\mu([a,b]\times[c,d]) &= \int_c^d \int_a^b \frac{p(x)}{q(y)}dxdy \\ &= \int_c^d \left(\int_a^b p(x)dx\right)\frac{1}{q(y)}dy = \left(\int_a^b p(x)dx\right)\left(\int_c^d \frac{1}{q(y)}dy\right) > 0,\end{aligned}$$

as a product of positive numbers. Now let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a non-negative continuous concave function. Hence, the map

$$x \to \int_c^d \frac{f(x,y)}{q(y)}dy,$$

is a concave and non-negative function over the interval $[a,b]$. Indeed,

$$\begin{aligned}\int_c^d \frac{f(tx_1+(1-t)x_2,y)}{q(y)}dy &\geqslant \int_c^d \frac{tf(x_1,y)+(1-t)f(x_2,y)}{q(y)}dy \\ &= t\int_c^d \frac{f(x_1,y)}{q(y)}dy + (1-t)\int_c^d \frac{f(x_2,y)}{q(y)}dy,\end{aligned}$$

for every $x_1, x_2 \in [a,b]$ and $t\in[0,1]$.

Since $p(x)dx$ is a DSP measure on $[a,b]$ we have

$$\int_c^d \int_a^b f(x,y)\frac{p(x)}{q(y)}dxdy = \int_a^b \left(\int_c^d \frac{f(x,y)}{q(y)}dy\right)p(x)dx \geqslant 0.$$

Therefore $\mu(x,y) = \frac{p(x)}{q(y)}dxdy$ is a DSP measure on the rectangle $[a,b]\times[c,d]$. ■

By using the Example 1.6, Remark 2.6, Theorem 2.11 and along with an appropriate selection of

$$p(x)dx := \left(\left(\frac{2x-a-b}{b-a}\right)^2 + \alpha\right)dx,$$

and

$$q(y)dy := \left(\frac{2y-c-d}{d-c}\right)^2 dy,$$

it is easy to show that

$$\left(\left(\frac{2x-a-b}{2y-c-d}\right)^2 + \alpha\left(\frac{b-a}{2y-c-d}\right)^2\right) dxdy,$$

is a DSP measure on the rectangle $[a,b]\times[c,d]$ for every $\alpha \geqslant -\frac{1}{6}$.

Now we are in position to investigate DSP measure on a compact disc.

**Proposition 2.12** Let $p(x)dx$ represent a DSP measure defined on the interval $[0,R]$, then $\frac{p(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}dxdy$ is a DSP measure on the compact disc $\bar{D}_R(0) = \{(x,y) : x^2+y^2 \leqslant R\}$.

***Proof*** By apply converting from rectangular to polar coordinates we have

$$\iint\limits_{\bar{D}_R(0)} \frac{p(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}dxdy = \int_0^{2\pi}\int_0^R \frac{p(r)}{r} rdrd\theta = 2\pi\int_0^R p(r)dr > 0.$$

Now let $f$ be a nonnegative real valued continuous concave function on compact disc $\bar{D}_R(0)$. In the other hand $p(r)dr$ and $1d\theta$ are two DSP measures on intervals $[0,R]$ and $[0,2\pi]$ respectively wich satisfy in Theorem 2.8. Hence $p(r)drd\theta$ is a DSP measure on $\bar{D}_R(0)$. Therefore

$$\iint\limits_{\bar{D}_R(0)} f(x,y)\frac{p(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}dxdy = \int_0^{2\pi}\int_0^R f(r,\theta)p(r)drd\theta \geqslant 0,$$

which is required. ■

**Example 2.13** By choosing $a := 0$ and $b := 1$ in Example 1.6, $[(2x-1)^2+\alpha]dx$ is a DSP measure on the interval $[0,1]$ for every $\alpha \geqslant -\frac{1}{6}$ and by applying Proposition 2.12 we have

$$\frac{(2\sqrt{x^2+y^2}-1)^2+\alpha}{\sqrt{x^2+y^2}}dxdy,$$

is a DSP measure on unit compact disc $\bar{D}_1(0)$.

## 3. Conclusion

In this paper, we improve several inequalities for convex and strongly convex functions by using Steffensen-Popoviciu and dual Steffensen-Popoviciu measures. In one hand the characterizations of these two kinds of interesting measures in the several variables case, and on the other hand, generalizations of Jensen and Hermite-Hadamard inequalities in this setting remain open for future researches.

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