*International Journal of*
*Mathematical Modelling & Computations*
*Vol. 14, No. 04, 2024, 347- 361*
DOI: 10.71932/IJM.2024.1197994

# Fuzzy Efficiency Decomposition in Two–Stage Systems: A New Slack-Based Measure Approach

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**Abstract.** Data envelopment analysis (DEA) is a non-parametric tool for evaluating the relative efficiency of comparable entities referred to as Decision Making Units (DMUs). Conventional DEA models treat systems as black box and do not consider their internal structure. Network data envelopment analysis (NDEA) is a prominent method for assessing the efficiency of network systems based on radial and non-radial approaches. The special case of network systems are two-stage systems. Many real practices have two-stage structure where is divided into two processes. Conventional NDEA calculates the efficiency of these systems in presence of crisp data. But in real life applications, the observed values of data are often uncertain. In this paper, for the first time, a new non-radial approach (based on slack based measure) is introduced, which evaluates the efficiency of two-stage systems in the presence of triangular fuzzy numbers (TFNs) using$\alpha$ −cut technique and optimistic and pessimistic procedures. The properties of the suggested models will also be examined. Finally, a numerical example will be provided to illustrate the proposed models.

Received: 30 September 2024; Revised: 27 November 2024; Accepted: 01 December 2024.

**Keywords**: Data Envelopment Analysis, Efficiency, Two stage system, Triangular fuzzy number, Slack Based Measure.

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## 1. Introduction

Data envelopment analysis (DEA) introduced by Charnes, et al. [10] is a non-parametric tool for evaluating the relative efficiency of comparable entities referred to as Decision Making Units (DMUs). The first form of DEA was called CCR model. Then, the numerous studies have been presented by extending this model. Conventional DEA models treat systems as black box and do not consider their internal structure. As discussed in many DEA papers (researches), DMU$_s$ may have network structure. Hence, new DEA studies have been done that measure the efficiency of systems with a network structure and are

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called NDEA models. In many applications such as banks, hospitals, etc., the internal structure of systems can be considered in the form of two stages in which the intermediate measures (produced in the first stage) are consumed by the second stage in the role of input. In recent years, much attention has been paid to evaluate the performance of these two-stage systems and their developed structures. Researchers point out several procedures for measuring the efficiency of two-stage systems based upon geometric, additive mean and SBM efficiency. Firstly, Seifored and Zhu [40] suggested models for measuring the efficiency of two-stage systems, independently. A weakness of their proposed approach is that the first and second stages may not be efficient, but the whole system may be efficient. Therefore, considering the relationship between stages in performance evaluation, a new model based on the geometric mean of stages efficiency was introduced by Kao and Hwang [25] In fact, their proposed model calculates the efficiency of the system and the stages under the constant returns to scale (CRS), but is unable to evaluate the efficiency under the variable returns to scale (VRS). This problem was solved by Chen et al. [12] by introducing models based on the additive approach. Large number of authors focused on developed two-stage systems and have presented models to evaluate the performance of these systems. For example, the efficiency of two-stage systems in the presence of shared inputs and shared outputs by using the additive decomposition approach was calculated by Li et al. [30] (see [4, 8, 11, 15, 24]). Using slacks-based measure, a non-radial slacks-based measure (SBM) was proposed by Tone [43] calculates the efficiency of black box systems. Then, Tone and Tsutsui [44] suggested the network slacks-based measure (NSBM) model for evaluating the efficiency of systems with internal structures. Also, Ashrafi et al. [7] presented SBM models to measure the efficiency of two-stage systems. In this approach, the projected $DMU_s$ for inefficient DMUs are efficient. Akhther et al. [3] evaluated the efficiency of Bangladesh bank by using a network SBM model. And also, Kao [27] introduced the model for measuring the efficiency of systems with internal processes. Based on their model, the weighted average of the efficiency of processes is defined as the overall efficiency of whole system. Esfidani et al. [16] introduced a new NSBM model to measure the stages efficiency and overall efficiency of multi-period two-stage system, simultaneously (see [42, 45]).

All articles listed are formulated only when the data are accurately measured, while in practice, this is not always possible. Actually, in real environments, uncertainty often occurs in the form of fuzzy and random environments. When data is described inaccurately or stochastic, it becomes necessary to use fuzzy theory in order to represent this type of data.To handle such circumstances, many authors have developed models to evaluate the performance of systems (especially two-stage systems) in the presence of uncertain data. Jiang et al. [23] presented the new procedure to measuring the efficiency of two-stage network systems in presence of uncertain data by using uncertainty theory. The efficiency of two-stage systems with stochastic data proposed by Esfidani et.al [17]. In order to evaluate the efficiency of systems in the presence of fuzzy data, different fuzzy approaches (such as the possibility approach, the tolerance approach, the fuzzy ranking approach, the $\alpha$ −level approach, …) have been suggested. Lozano and Moreno [31] presented a well-known fuzzy DEA approach to measure the efficiency of two-stage serial system. Using the principle of expansion and the$\alpha$ −cut technique, Kao and Liu [26] proposed FNDEA models for evaluating the performance of two-stage systems in the presence of fuzzy data. $\alpha$ −cut efficiencies of two-stage systems is calculated by Lozano [32]. Liu [29] suggested a procedure to rank the fuzzy efficiency of two-stage systems. Soltani et al. [41] proposed two-stage fuzzy DEA model based on fuzzy arithmetic. Also, Nabahat [35] used a collective approach to evaluate the performance of two-stage systems using the$\alpha$ −cut technique. Arya and Singh [6] used the$\alpha$ −cut procedure and presented fuzzy models to evaluate the efficiency of two-stage parallel-series systems and calculate the lower and upper bound fuzzy efficiencies of systems. Peykani et al. [37] evaluated appraisal and

ranking of DMU$_s$ with two-stage network structure using three procedures two-stage DEA model, adjustable possibilistic programming (APP), and chance-constrained programming (CCP). And also, a new fuzzy two-stage DEA model was presented by Izadikhah [21] to measure the efficiency of 15 branches of Melli bank in Hamedan province. In recent years, many researchers have evaluated the performance of DMU$_s$ in the presence of fuzzy data based on Slacke-Based Measure. Agarwal [2] used possibility approach and proposed a fuzzy SBM DEA model in order to measure the efficiency given fuzzy input and output data. A new Fuzzy Network SBM model was presented by Momeni et al. [36] to survey the performance of supply chain networks with forward and reverse logistics. Afzalinejad and Abbasi [1] suggested a new dynamic slacks-based DEA model that reveals all sources of inefficiencies and provide more discrimination between DMU$_s$. In order to measure the cross efficiency in DEA, Kao and Liu [28] proposed a slacks-based measure model. When the input and output data are given as fuzzy sets, Arana-Jiménez et al. [5] suggested a well-known slacks-based additive inefficiency measure to survey the problem of efficiency assessment. Mahla and Agarwal [33] presented a fuzzy SBM model for measuring the efficiency of DMU$_s$ using credibility measure approach. (see [9, 13, 14, 18, 19, 20, 34, 39]). In the last decade, many network DEA models have been proposed to evaluate the performance of two-stage systems in the presence of fuzzy data, and most of them ignore the input, intermediate measure and output slacks, and this is not appropriate. Therefore, we will suggested fuzzy SBM model in order to measure the efficiency of two-stage systems. Among fuzzy data, we use triangular fuzzy numbers for simplicity in calculations. The proposed models can be generalized to total fuzzy data. In order to de-fuzzy, we will use the $\alpha-$cut approach and solve the created interval models with optimistic and pessimistic techniques. Then we will examine the properties of the proposed models. At the end, a numerical example is presented to illustrate the proposed approaches.
The paper is structures as follows. The second section provides a review of TFNs and the conventional SBM model, also, we present the fuzzy SBM model in presence of TFNs. Then, based on the $\alpha-$cut approach, the proposed model is converted to interval model. The optimistic and pessimistic procedure is applied to solve this interval model. In section 3, Properties of proposed models are also discussed. In section 4, we illustrate the suggested models by using the data of 10 Mellat bank branches in Tehran. Finally, in section 5 we present our conclusions and future research directions.

## 2. Model formulation

In this section, firstly, we reviewed the definitions of fuzzy set, fuzzy number and triangular fuzzy number and their alpha-cut sets. Also, conventional SBM DEA model of Tone [43] is presented. In the following, the fuzzy SBM model in presence of TFNs is presented.

### 2.1 *Triangular Fuzzy Number (TFN)*

Suppose $X$ is a global set. A fuzzy set $\tilde{A}$ is defined as $\tilde{A}=\{(x,\mu_{\tilde{A}}(x))|x\in X\}$ where $0<\mu_{A^*}(x)\leq 1$ shows the degree of membership of element $x\in X$ to the set $\tilde{A}\subset X$. And also, let $S(\tilde{A})$ is as $S(\tilde{A})=\{(x\in X|\mu_{\tilde{A}}(x)>0)\}$ that denote the support of $\tilde{A}$. The $\alpha-$cut set of $\tilde{A}$ is defined as $\tilde{A}_\alpha=\{x\in S(\tilde{A})|\mu_{\tilde{A}}(x)>\alpha\}$.

**Definition 2.1** Let $\tilde{A}\subset\mathbb{R}$ be a fuzzy set. If the following conditions are hold, $\tilde{A}$ is called FN:

1. $\tilde{A}$ is fuz zy set convex set if the membership function is fuzzy convex set: $\forall x_1,x_2\in R,\forall\lambda\in[0,1]$: $\mu_{\tilde{A}}(\lambda x_1+(1-\lambda)x_2)\geq min\{\mu_{\tilde{A}}(x_1),\mu_{\tilde{A}}(x_2)\}$
2. There is at least one $x'\in\mathbb{R}$ such that $\mu_{\tilde{A}}(x')=1$.
3. The membership $\mu_{\tilde{A}}(x)$ function is semi-continuous.

**Definition 2.2** A FN $\tilde{A}\subset\mathbb{R}$ is a TFN with membership function $\mu_{\tilde{A}}(x)$ of the following form:

$$\mu_{\tilde{A}}(x)=\begin{cases}\dfrac{x\text{-}x^m+x^l}{x^l}, & x^m-x^l\le x\le x^m\\ \dfrac{x^m+x^r-x}{x^r}, & x^m\le x\le x^m+x^r\end{cases}$$

Here, $x^m$ is called mean value and $x^r, x^l$ called the right and the left spreads of membership function, respectively. We denote the TFN by $\tilde{A}=(x^l,x^m,x^r)$. Moreover, $\alpha$ −cut set of TFN is defined as follows that is crisp subset of $\mathbb{R}$:

$$\tilde{A}_\alpha=[\tilde{A}^L_\alpha,\tilde{A}^U_\alpha]=[(x^m-x^l)+\alpha x^l,(x^m+x^r)-\alpha x^r].$$

### 2.2 *Slacks-Based Measure Model*

In this section, we review the Slacks-Based Measure (SBM) model that was presented by Tone [43]. Consider a set of DMUs that indexed by $DMU_j$. Also, assume that each $DMU_j (j=1,\ldots,n)$ has a black box structure with inputs $x_{ij} (i=1,\ldots,m)$ and outputs $y_{rj} (r=1,\ldots,s)$. Tone proposed the following model to measure the efficiency of the $DMU_o$ ($DMU$ under evaluation):

$$\begin{aligned}
E_o^s = max \quad & \frac{1-\frac{1}{m}(\sum_{i=1}^{m}\frac{s_i^-}{x_{io}})}{1+\frac{1}{s}(\sum_{r=1}^{s}\frac{s_r^+}{y_{ro}})}\\
s.t. \quad & \sum_{j=1}^{n}\lambda_j x_{ij}+s_i^-=x_{io}, \quad i=1,\ldots,m \qquad (1)\\
& \sum_{j=1}^{n}\lambda_j y_{rj}-s_r^+=y_{ro}, \quad r=1,\ldots,s\\
& \lambda_j, s_i^-, s_r^+\ge 0, j=1,\ldots,n,\ i=1,\ldots,m,\ r=1,\ldots,s
\end{aligned}$$

In this model, $s^-$, $s^+$, $\lambda$ are respectively the input slack, output slack and intensity vector associated with $DMU_j (j=1,\ldots,n)$. Also, it is assumed that $x_o>0$, $y_o>0$. If $x_o=0$, term $\frac{s_i^-}{x_{io}}$ will be removed from the model (1). And also, if $y_o=0$, term $\frac{s_r^+}{y_{ro}}$ will be substituted by a very small positive number. It is clear that the model (1) can be turned into a linear programming using Charnes-Cooper transformation.

**Definition 2.3** $DMU_o$ is said to be efficient if $E_o^s=1$.

**Definition 2.4** $E_o^s=1$ if and only if $s_i^{-*}=0 (i=1,\ldots,m), s_r^{+*}=0 (r=1,\ldots,s)$.

### 2.3 *Proposed Fuzzy SBM model*

In this section, by introducing the structure of two-stage systems, a model for evaluating the efficiency of these systems in the presence of TFNs is introduced. Consider a two-stage system shown in Figure1, wherein intermediate measure in stage1 is consumed by stage2. Suppose there are $n$, DMUs with two-stage structure. The input, output and intermediate measure vectors of $DMU_j (j=1,\ldots,n)$ are $x_{ij}, y_{rj}, z_{dj}$, respectively.

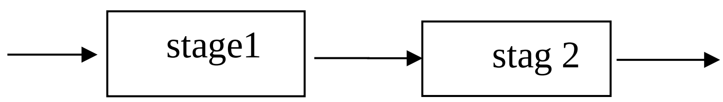

Figure1.Two-stage production system.

Now, we assume that there are systems where some observations are imprecise. Among these imprecise data, we intend to use TFNs. Note that a crisp number can be thought of (considered) as a TFN. Note that in this paper, due to the simplicity of the calculation procedure, it is assumed that inputs, intermediate measures and outputs are in the form of TFNs. Also, conventional radial models do not pay attention to slacks in evaluating the efficiency of systems, while in many cases it is necessary to identify excess inputs as well as lack of outputs, etc. Therefore, we introduce a non-radial model that solves these problems. Hence, it is assumed that all inputs, outputs and intermediate measures are TFNs. Therefore, we denote are $\tilde{x}_{ij}, \tilde{y}_{rj}, \tilde{z}_{dj}$ TFNs. It should be noted that ~ is a fuzzy symbol. The structure of $\tilde{x}_{ij}, \tilde{y}_{rj}, \tilde{z}_{dj}$ is as follows:

$$\tilde{x}_{ij} = (\tilde{x}^l{}_{ij}, \tilde{x}^m{}_{ij}, \tilde{x}^r{}_{ij}) \;,\; \tilde{y}_{rj} = (\tilde{y}^l{}_{rj}, \tilde{y}^m{}_{rj}, \tilde{y}^r{}_{rj}) \;,\; \tilde{z}_{dj} = (\tilde{z}^l{}_{dj}, \tilde{z}^m{}_{dj}, \tilde{z}^r{}_{dj})$$

We also note that intermediate measures in the first stage have an output role and therefore should increase and in the second stage in the input role, should decrease. Hence, the flexibility of intermediate measures in modeling issues may be problematic. Therefore, to evaluate the efficiency of two-stage systems in the presence of TFNs, we introduce the following model so that it overcomes this problem:

$$\begin{aligned}
\tilde{E}_o^s = max \quad & \frac{1 - \frac{1}{m+D}\left(\sum_{i=1}^{m} \frac{s_i^-}{(\tilde{x}^l{}_{io}, \tilde{x}^m{}_{io}, \tilde{x}^r{}_{io})} + \sum_{d=1}^{D} \frac{\pi_d}{(\tilde{z}^l{}_{do}, \tilde{z}^m{}_{do}, \tilde{z}^r{}_{do})}\right)}{1 + \frac{1}{s+D}\left(\sum_{r=1}^{s} \frac{s_r^+}{(\tilde{y}^l{}_{ro}, \tilde{y}^m{}_{ro}, \tilde{y}^r{}_{ro})} + \sum_{d=1}^{D} \frac{s_d}{(\tilde{z}^l{}_{do}, \tilde{z}^m{}_{do}, \tilde{z}^r{}_{do})}\right)} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j (\tilde{x}^l{}_{ij}, \tilde{x}^m{}_{ij}, \tilde{x}^r{}_{ij}) + s_i^- = (\tilde{x}^l{}_{io}, \tilde{x}^m{}_{io}, \tilde{x}^r{}_{io}), \quad i = 1, \dots, m \\
& \sum_{j=1}^{n} \lambda_j (\tilde{z}^l{}_{dj}, \tilde{z}^m{}_{dj}, \tilde{z}^r{}_{dj}) - s_d = (\tilde{z}^l{}_{do}, \tilde{z}^m{}_{do}, \tilde{z}^r{}_{do}), \; d = 1, \dots, D \qquad (2) \\
& \sum_{j=1}^{n} \lambda_j^{'} (\tilde{z}^l{}_{dj}, \tilde{z}^m{}_{dj}, \tilde{z}^r{}_{dj}) + \pi_d = (\tilde{z}^l{}_{do}, \tilde{z}^m{}_{do}, \tilde{z}^r{}_{do}), \; d = 1, \dots, D \\
& \sum_{j=1}^{n} \lambda_j^{'} (\tilde{y}^l{}_{rj}, \tilde{y}^m{}_{rj}, \tilde{y}^r{}_{rj}) - s_r^+ = (\tilde{y}^l{}_{ro}, \tilde{y}^m{}_{ro}, \tilde{y}^r{}_{ro}), \quad r = 1, \dots, s \\
& \lambda_j, \lambda_j^{'}, s_i^-, s_d, \pi_d, s_r^+ \geq 0, j = 1, \dots, n, i = 1, \dots, m, d = 1, \dots, D, r = 1, \dots, s
\end{aligned}$$

Wherein $s_i^-, s_r^+$ input slack vector and output slack vector, respectively. Also, intermediate measure slacks vectors are $s_d$ (as output for stage 1) and$\pi_d$ (as input for stage 2). $\lambda_j, \lambda_j^{'}$ are intensity vectors for stage 1 and stsge2, respectively. Saati [38] used the concept of $\alpha$ −cut and variable substitution to present an approach for solving fuzzy DEA models. Hence, we adopted this idea for calculating the fuzzy efficiency$\tilde{E}_o^{s*}$. Thus, we denote:

$$\begin{aligned}
(\tilde{x}_{ij})_\alpha &= [\tilde{x}^L{}_{ij\alpha}, \tilde{x}^U{}_{ij\alpha}] = [\tilde{x}^m{}_{ij} - \tilde{x}^l{}_{ij}(1-\alpha), \tilde{x}^m{}_{ij} + \tilde{x}^r{}_{ij}(1-\alpha)] \\
(\tilde{z}_{dj})_\alpha &= [\tilde{z}^L{}_{dj\alpha}, \tilde{z}^U{}_{dj\alpha}] = [\tilde{z}^m{}_{dj} - \tilde{z}^l{}_{dj}(1-\alpha), \tilde{z}^m{}_{dj} + \tilde{z}^r{}_{dj}(1-\alpha)] \qquad (3) \\
(\tilde{y}_{rj})_\alpha &= [\tilde{y}_{rj\alpha}{}^L, \tilde{y}^U{}_{rj\alpha}] = [\tilde{y}^m{}_{rj} - \tilde{y}^l{}_{rj}(1-\alpha), \tilde{y}^m{}_{rj} + \tilde{y}^r{}_{rj}(1-\alpha)]
\end{aligned}$$

As the$\alpha$ −cut of $\tilde{x}_{ij}, \tilde{y}_{rj}, \tilde{z}_{dj}$. Then, model (3) becomes:

$$\tilde{E}_o^s = max \quad \frac{1 - \frac{1}{m+D}\left(\sum_{i=1}^{m} \frac{s_i^-}{[\tilde{x}^L{}_{io\alpha}, \tilde{x}^U{}_{io\alpha}]} + \sum_{d=1}^{D} \frac{\pi_d}{[\tilde{z}^L{}_{do\alpha}, \tilde{z}^U{}_{do\alpha}]}\right)}{1 + \frac{1}{s+D}\left(\sum_{r=1}^{s} \frac{s_r^+}{[\tilde{y}_{ro\alpha}{}^L, \tilde{y}^U{}_{ro\alpha}]} + \sum_{d=1}^{D} \frac{s_d}{[\tilde{z}^L{}_{do\alpha}, \tilde{z}^U{}_{do\alpha}]}\right)}$$

$$\begin{aligned}
s.t. \quad & \sum_{j=1}^{n} \lambda_j [\tilde{x}^L{}_{ij\alpha}, \tilde{x}^U{}_{ij\alpha}] + s_i^- = [\tilde{x}^L{}_{io\alpha}, \tilde{x}^U{}_{io\alpha}],\ i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j [\tilde{z}^L{}_{dj\alpha}, \tilde{z}^U{}_{dj\alpha}] - s_d = [\tilde{z}^L{}_{do\alpha}, \tilde{z}^U{}_{do\alpha}], d = 1, \ldots, D \\
& \sum_{j=1}^{n} \lambda_j^{'} [\tilde{z}^L{}_{dj\alpha}, \tilde{z}^U{}_{dj\alpha}] + \pi_d = [\tilde{z}^L{}_{do\alpha}, \tilde{z}^U{}_{do\alpha}], d = 1, \ldots, D \\
& \sum_{j=1}^{n} \lambda_j^{'} [\tilde{y}_{rj\alpha}{}^L, \tilde{y}^U{}_{rj\alpha}] - s_r^+ = [\tilde{y}_{ro\alpha}{}^L, \tilde{y}^U{}_{ro\alpha}], r = 1, \ldots, s \\
& \lambda_j, \lambda_j^{'}, s_i^-, s_d, \pi_d, s_r^+ \geq 0, j = 1, \ldots, n, i = 1, \ldots, m, d = 1, \ldots, D, r = 1, \ldots, s.
\end{aligned} \tag{4}$$

Given that all coefficients in this model have interval form, hence this model represents an interval model. Thus, this model cannot be solved in its current form. Therefore, we use optimistic and pessimistic approaches for evaluation of model (4) and calculate the lower and the upper bounds of the$\alpha$ −cut of $DMU_o$ (i.e $\tilde{E}_o^{s(L)*}, \tilde{E}_o^{s(U)*}$). Firstly, we calculate the lower bound of the efficiency. In pessimistic approach, it is assumed that$DMU_o$is set to its worst situation (or $DMU_o$has the most unfavorable conditions) and other$DMUs$ have favorable conditions (or: the best condition). In other words, stage1 of $DMU_o$consumes input$\tilde{x}^U{}_{io\alpha}$, for producing the intermediate measure$\tilde{z}^L{}_{do\alpha}$. And also, the other$DMUs$, consume input $\tilde{x}^L{}_{io\alpha}$ to produce intermediate measure$\tilde{z}^U{}_{do\alpha}$. And also, in stage2, input and output of $DMU_o$ are$\tilde{z}^U{}_{do\alpha}$ and$\tilde{y}^L{}_{io\alpha}$, respectively. The other $DMUs$ also have input $\tilde{z}^L{}_{do\alpha}$and output$\tilde{y}^U{}_{io\alpha}$.Thus, for measuring the lower bound of the efficiency (i.e. $\tilde{E}_o^{s(L)*}$), the following model is proposed:

$$\tilde{E}_o^{s(L)} = max \quad \frac{1 - \frac{1}{m+D}\left(\sum_{i=1}^{m} \frac{s_i^-}{\tilde{x}^U{}_{io\alpha}} + \sum_{d=1}^{D} \frac{\pi_d}{\tilde{z}^U{}_{do\alpha}}\right)}{1 + \frac{1}{s+D}\left(\sum_{r=1}^{s} \frac{s_r^+}{\tilde{y}^L{}_{ro\alpha}} + \sum_{d=1}^{D} \frac{s_d}{\tilde{z}^L{}_{do\alpha}}\right)}$$

$$\begin{aligned}
s.t. \quad & \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j \tilde{x}^L{}_{ij\alpha} + \lambda_o \tilde{x}^U{}_{io\alpha} + s_i^- = \tilde{x}^U{}_{io\alpha},\ i = 1, \ldots, m \\
& \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j \tilde{z}^U{}_{dj\alpha} + \lambda_o^{'} \tilde{z}^L{}_{do\alpha} - s_d = \tilde{z}^L{}_{do\alpha}, d = 1, \ldots, D \\
& \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j^{'} \tilde{z}^L{}_{dj\alpha} + \lambda_o^{'} \tilde{z}^U{}_{do\alpha} + \pi_d = \tilde{z}^U{}_{do\alpha}, d = 1, \ldots, D \\
& \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j^{'} \tilde{y}^U{}_{ro\alpha} + \lambda_o^{'} \tilde{y}^L{}_{ro\alpha} - s_r^+ = \tilde{y}^L{}_{io\alpha},\ r = 1, \ldots, s
\end{aligned} \tag{5}$$

$$\lambda_j, \lambda_j^{'}, s_i^-, s_d, \pi_d, s_r^+ \geq 0, j = 1, \dots, n, i = 1, \dots, m, d = 1, \dots, D, r = 1, \dots, s$$

By using the Charnes-Cooper transformation, model (5) can be transformed into the linear model. For this purpose, we let:

$$t = \frac{1}{1 + \frac{1}{s+D}\left(\sum_{r=1}^{s} \frac{s_r^+}{\tilde{y}^L{}_{ro\alpha}} + \sum_{d=1}^{D} \frac{s_d}{\tilde{z}^L{}_{do\alpha}}\right)}$$

And set $\eta_j = t\lambda_j, \eta_j^{'} = t\lambda_j^{'}, \delta_i^- = ts_i^-, \gamma_r^+ = ts_r^+, \gamma_d = t\pi_d, \delta_d = ts_d$. Then, the model (5) converted to the following model:

$$\begin{aligned}
\tilde{E}_o^{s(L)} = max \quad & t - \frac{1}{m+D}\left(\sum_{i=1}^{m} \frac{\delta_i^-}{\tilde{x}^U{}_{io\alpha}} + \sum_{d=1}^{D} \frac{\gamma_d}{\tilde{z}^U{}_{do\alpha}}\right) \\
s.t. \qquad & t + \frac{1}{s+D}\left(\sum_{r=1}^{s} \frac{\gamma_r^+}{\tilde{y}^L{}_{ro\alpha}} + \sum_{d=1}^{D} \frac{\delta_d}{\tilde{z}^L{}_{do\alpha}}\right) = 1 \\
& \sum_{\substack{j=1\\ j\neq o}}^{n} \eta_j \tilde{x}^L{}_{ij\alpha} + \eta_o \tilde{x}^U{}_{io\alpha} + \delta_i^- = t\tilde{x}^U{}_{io\alpha}, \quad i = 1, \dots, m \\
& \sum_{\substack{j=1\\ j\neq o}}^{n} \eta_j \tilde{z}^U{}_{dj\alpha} + \eta_o \tilde{z}^L{}_{do\alpha} - \delta_d = t\tilde{z}^L{}_{do\alpha}, \quad d = 1, \dots, D \qquad (6) \\
& \sum_{\substack{j=1\\ j\neq o}}^{n} \eta_j^{'} \tilde{z}^L{}_{dj\alpha} + \eta_o^{'} \tilde{z}^U{}_{do\alpha} + \gamma_d = t\tilde{z}^U{}_{do\alpha}, \quad d = 1, \dots, D \\
& \sum_{\substack{j=1\\ j\neq o}}^{n} \eta_j^{'} \tilde{y}^U{}_{rj\alpha} + \eta_o^{'} \tilde{y}^L{}_{ro\alpha}, - \gamma_r^+ = t\tilde{y}^L{}_{ro\alpha}, \quad r = 1, \dots, s \\
& \eta_j, \eta_j^{'}, \delta_i^-, \delta_d, \gamma_d, \gamma_r^+ \geq 0, j = 1, \dots, n, i = 1, \dots, m, d = 1, \dots, D, r = 1, \dots, s, \\
& t > 0.
\end{aligned}$$

Suppose $(t^{l^*}, \eta_j^{l^*}, \eta^{'l^*}_j, \delta_i^{l-^*}, \delta_d^{l^*}, \gamma_d^{l^*}, \gamma_r^{l+^*})$ is an optimal solution of model (6). Then the optimal solution of the model (5) will be as follows:

$$\lambda_j^{l^*} = \frac{\eta_j^{l^*}}{t^{l^*}}, \lambda_j^{'l^*} = \frac{\eta^{'l^*}_j}{t^{l^*}}, s_i^{l-^*} = \frac{\delta_i^{l-^*}}{t^{l^*}}, s_r^{l+^*} = \frac{\gamma_r^{l+^*}}{t^{l^*}}, \pi_d^{l^*} = \frac{\gamma_d^{l^*}}{t^{l^*}}, s_d^{l^*} = \frac{\delta_d^{l}}{t^{l^*}}$$

Therefore, the overall efficiency ($\tilde{E}_{o\alpha}^{s(L)}$) and efficiency of stages ($\tilde{E}_{o\alpha}^{I(L)}, \tilde{E}_{o\alpha}^{II(L)}$) are as follows:

$$\tilde{E}_{o\alpha}^{s(L)} = \frac{1 - \frac{1}{m+D}\left(\sum_{i=1}^{m} \frac{s_i^{l-^*}}{x_{io}^U} + \sum_{d=1}^{D} \frac{\pi_d^{l^*}}{z_{do}^U}\right)}{1 + \frac{1}{s+D}\left(\sum_{r=1}^{s} \frac{s_r^{l+^*}}{y_{ro}^L} + \sum_{d=1}^{D} \frac{s_d^{l^*}}{z_{do}^L}\right)}$$

$$\tilde{E}_{o\alpha}^{I(L)} = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^{l-^*}}{x_{io}^U}}{1 + \frac{1}{D}\sum_{d=1}^{D} \frac{s_d^{l^*}}{z_{do}^L}}, \qquad \tilde{E}_{o\alpha}^{II(L)} = \frac{1 - \frac{1}{D}\sum_{d=1}^{D} \frac{\pi_d^{l^*}}{z_{do}^U})}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^{l+^*}}{y_{ro}^L})} \qquad (7)$$

**Definition 2.5** $DMU_o$ is lower overall efficient if and only if $\tilde{E}_{o\alpha}^{s(L)} = 1$.

**Definition 2.6** In stage1, $DMU_o$ is lower efficient in the lower bound if and only if $\tilde{E}_{o\alpha}^{I(L)} = 1$.

**Definition 2.7** In stage2, $DMU_o$ is efficient in the lower bound if and only if $\tilde{E}_{o\alpha}^{II(L)} = 1$.

Now, we consider the optimistic approach to calculate the upper bound of the efficiency. In this procedure, $DMU_o$ has the most favorable conditions and the remaining $DMU_s$ have unfavorable conditions. Therefore, $\tilde{x}^L{}_{io\alpha}, \tilde{z}^L{}_{do\alpha}$ are inputs of stage1 and stage2, respectively. And also, the output of the first stage is $\tilde{z}^U{}_{do\alpha}$ and the output of the second stage is $\tilde{y}^U{}_{ro\alpha}$. Therefore, according to the above, model (8) is proposed to calculate the upper bound of the efficiency:

$$
\begin{aligned}
\tilde{E}_o^{s(U)} = max \quad & \frac{1 - \frac{1}{m+D}\left(\sum_{i=1}^m \frac{s_i^-}{\tilde{x}^L{}_{io\alpha}} + \sum_{d=1}^D \frac{\pi_d}{\tilde{z}^L{}_{do\alpha}}\right)}{1 + \frac{1}{s+D}\left(\sum_{r=1}^s \frac{s_r^+}{\tilde{y}^U{}_{ro\alpha}} + \sum_{d=1}^D \frac{s_d}{\tilde{z}^U{}_{do\alpha}}\right)} \\
s.t. \quad & \sum_{\substack{j=1 \\ j\neq o}}^n \lambda_j \tilde{x}^U{}_{ij\alpha} + \lambda_o \tilde{x}^L{}_{io\alpha} + s_i^- = \tilde{x}^L{}_{io\alpha}, \ \ i = 1,\dots,m \\
& \sum_{\substack{j=1 \\ j\neq o}}^n \lambda_j \tilde{z}^L{}_{dj\alpha} + \lambda_o^{'} \tilde{z}^U{}_{do\alpha} - s_d = \tilde{z}^U{}_{do\alpha}, \qquad d = 1,\dots,D \\
& \sum_{\substack{j=1 \\ j\neq o}}^n \lambda_j^{'} \tilde{z}^U{}_{dj\alpha} + \lambda_o^{'} \tilde{z}^L{}_{do\alpha} + \pi_d = \tilde{z}^L{}_{do\alpha}, d = 1,\dots,D \\
& \sum_{\substack{j=1 \\ j\neq o}}^n \lambda_j^{'} \tilde{y}^L{}_{ro\alpha} + \lambda_o^{'} \tilde{y}^U{}_{ro\alpha} - s_r^+ = \tilde{y}^U{}_{io\alpha}, \ \ r = 1,\dots,s \\
& \lambda_j, \lambda_j^{'}, s_i^-, s_d, \pi_d, s_r^+ \geq 0, \ j = 1,\dots,n, i = 1,\dots,m, d = 1,\dots,D, r = 1,\dots,s
\end{aligned} \tag{8}
$$

By using the Charnes-Cooper transformation, model (8) can be transformed into the linear model. For this purpose, we let:

$$
t^{'} = \frac{1}{1 + \frac{1}{s+D}\left(\sum_{r=1}^s \frac{s_r^+}{\tilde{y}^U{}_{ro\alpha}} + \sum_{d=1}^D \frac{s_d}{\tilde{z}^U{}_{do\alpha}}\right)}
$$

And set $\eta_j = t^{'}\lambda_j, \eta_j^{'} = t^{'}\lambda_j^{'}, \delta_i^- = t^{'}s_i^-, \gamma_r^+ = t^{'}s_r^+, \gamma_d = t^{'}\pi_d, \delta_d = t^{'}s_d$. Then, the model (8) converted to the following model:

$$
\begin{aligned}
\tilde{E}_o^{s(U)} = max \quad & t^{'} - \frac{1}{m+D}\left(\sum_{i=1}^m \frac{\delta_i^-}{\tilde{x}^L{}_{io\alpha}} + \sum_{d=1}^D \frac{\gamma_d}{\tilde{z}^L{}_{do\alpha}}\right) \\
s.t. \quad & t^{'} + \frac{1}{s+D}\left(\sum_{r=1}^s \frac{\gamma_r^+}{\tilde{y}^U{}_{ro\alpha}} + \sum_{d=1}^D \frac{\delta_d}{\tilde{z}^U{}_{do\alpha}}\right) = 1
\end{aligned}
$$

$$\sum_{\substack{j=1\\j\neq o}}^{n}\eta_j\tilde{x}^U{}_{ij\alpha}+\eta_o\tilde{x}^L{}_{io\alpha}+\delta_i^-=t'\tilde{x}^L{}_{io\alpha},\ i=1,\dots,m$$

$$\sum_{\substack{j=1\\j\neq o}}^{n}\eta_j\tilde{z}^L{}_{dj\alpha}+\eta_o\tilde{z}^U{}_{do\alpha}-\delta_d=t'\tilde{z}^U{}_{do\alpha},\ d=1,\dots,D \tag{9}$$

$$\sum_{\substack{j=1\\j\neq o}}^{n}\eta'_j\tilde{z}^U{}_{dj\alpha}+\eta'_o\tilde{z}^L{}_{do\alpha}+\gamma_d=t'\tilde{z}^L{}_{do\alpha},\ \ d=1,\dots,D$$

$$\sum_{\substack{j=1\\j\neq o}}^{n}\eta'_j\tilde{y}^L{}_{rj\alpha}+\eta'_o\tilde{y}^U{}_{ro\alpha}-\gamma_r^+=t'\tilde{y}^U{}_{ro\alpha},\ \ r=1,\dots,s$$

$$\eta_j,\eta'_j,\delta_i^-,\delta_d,\gamma_d,\gamma_r^+\geq 0,j=1,\dots,n,i=1,\dots,m,d=1,\dots,D,r=1,\dots,s$$
$$t'>0.$$

Suppose $(t'^{u*},\eta_j^{u*},\eta'^{u*}_j,\delta_i^{u-*},\delta_d^{u*},\gamma_d^{u*},\gamma_r^{u+*})$ is an optimal solution of model (9). Then the optimal solution of the model (8) will be as follows:

$$\lambda_j^{u*}=\frac{\eta_j^{u*}}{t'^{u*}},\lambda'^{u*}_j=\frac{\eta'^{u*}_j}{t'^{u*}},s_i^{u-*}=\frac{\delta_i^{u-*}}{t'^{u*}},s_r^{u+*}=\frac{\gamma_r^{u+*}}{t'^{u*}},\pi_d^{u*}=\frac{\gamma_d^{u*}}{t'^{u*}},s_d^{u*}=\frac{\delta_d^{u*}}{t'^{u*}}$$

Therefore, the overall efficiency ($\tilde{E}_{o\alpha}^{s(U)}$) and efficiency of stages ($\tilde{E}_{o\alpha}^{I(U)},\tilde{E}_{o\alpha}^{II(U)}$) are as follows:

$$\tilde{E}_{o\alpha}^{s(U)}=\frac{1-\frac{1}{m+D}\left(\sum_{i=1}^{m}\frac{s_i^{u-*}}{x_{io}^L}+\sum_{d=1}^{D}\frac{\pi_d^{u*}}{z_{do}^L}\right)}{1+\frac{1}{s+D}\left(\sum_{r=1}^{s}\frac{s_r^{u+*}}{y_{ro}^U}+\sum_{d=1}^{D}\frac{s_d^{u*}}{z_{do}^U}\right)}$$

$$E_{o\alpha}^{I(U)}=\frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^{u-*}}{x_{io}^L}}{1+\frac{1}{D}\sum_{d=1}^{D}\frac{s_d^{u*}}{z_{do}^U}}\quad,\qquad E_{o\alpha}^{II(U)}=\frac{1-\frac{1}{D}\sum_{d=1}^{D}\frac{\pi_d^{u*}}{z_{do}^L})}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_r^{u+*}}{y_{ro}^U})} \tag{10}$$

**Definition 2.8** $DMU_o$ is overall efficient in the upper bound if and only if $\tilde{E}_{o\alpha}^{s(U)}=1$.
**Definition 2.9** In stage 1, $DMU_o$ is efficient in the upper bound 1 if and only if $\tilde{E}_{o\alpha}^{I(U)}=1$.
**Definition 2.1** In stage2, $DMU_o$ is efficient in the upper bound if and only if $\tilde{E}_{o\alpha}^{II(U)}=1$.

## 3. Mathematical analysis of the model

In this section, we will investigate the properties of the suggested models.
**Theorem 3.1** Models (5) and (9) are unit invariant. Actually, if $\tilde{x}_{ij}=(\tilde{x}^l{}_{ij},\tilde{x}^m{}_{ij},\tilde{x}^r{}_{ij})$, $\tilde{y}_{rj}=(\tilde{y}^l_{rj},\tilde{y}^m_{rj},\tilde{y}^r_{rj})$, $\tilde{z}_{dj}=(\tilde{z}^l{}_{dj},\tilde{z}^m{}_{dj},\tilde{z}^r{}_{dj})$ are replaced by $\beta_i\tilde{x}_{ij}=(\beta_i\tilde{x}^l{}_{ij},\beta_i\tilde{x}^m{}_{ij},\beta_i\tilde{x}^r{}_{ij})$, $\omega_r\tilde{y}_{rj}=(\omega_r\tilde{y}^l_{rj},\omega_r\tilde{y}^m_{rj},\omega_r\tilde{y}^r_{rj})$, $\rho_d\tilde{z}_{dj}=(\rho_d\tilde{z}^l{}_{dj},\rho_d\tilde{z}^m{}_{dj},\rho_d\tilde{z}^r{}_{dj})$, the efficiency does not be changed.

***Proof.*** Based on the constraints of the model (5):

$$\beta_i s_i^- = \beta_i \tilde{x}^U{}_{io\alpha} - \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j(\beta_i \tilde{x}^L{}_{ij\alpha}) - \lambda_o(\beta_i \tilde{x}^U{}_{io\alpha})$$

$$= \beta_i(\tilde{x}^U{}_{io\alpha} - \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_{j_i} \tilde{x}^L{}_{ij\alpha} - \lambda_o \tilde{x}^U{}_{io\alpha}), i = 1,\dots,m$$

$$\rho_d s_d = \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j(\rho_d \tilde{z}^U{}_{dj\alpha}) + \lambda'_o(\rho_d \tilde{z}^L{}_{do\alpha}) - (\rho_d \tilde{z}^L{}_{do\alpha})$$

$$= \rho_d \left( \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \tilde{z}^U{}_{dj\alpha} + \lambda'_o \tilde{z}^L{}_{do\alpha} - \tilde{z}^L{}_{do\alpha} \right), d = 1,\dots,D$$

$$\rho_d \pi_d = \rho_d \tilde{z}^U{}_{do\alpha}$$

$$- \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda'_j(\rho_d \tilde{z}^L{}_{dj\alpha}) - \lambda'_o(\rho_d \tilde{z}^U{}_{do\alpha})$$

$$= \rho_d \left( \tilde{z}^U{}_{do\alpha} - \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda'_j \tilde{z}^L{}_{dj\alpha} - \lambda'_o \tilde{z}^U{}_{do\alpha} \right), d = 1,\dots,D$$

$$\omega_r s_r^+ = \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda'_j(\omega_r \tilde{y}^U{}_{ro\alpha}) + \lambda'_o(\omega_r \tilde{y}^L{}_{ro\alpha}) - (\omega_r \tilde{y}^L{}_{io\alpha})$$

$$= \omega_r \left( \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda'_j \tilde{y}^U{}_{ro\alpha} + \lambda'_o \tilde{y}^L{}_{ro\alpha} - \tilde{y}^L{}_{io\alpha} \right), r = 1,\dots,s.$$

Hence,

$$\frac{\beta_i s_i^-}{\beta_i \tilde{x}^U{}_{io\alpha}} = \frac{s_i^-}{\tilde{x}^U{}_{io\alpha}}, \frac{\rho_d \pi_d}{\rho_d \tilde{z}^U{}_{do\alpha}} = \frac{\pi_d}{\tilde{z}^U{}_{do\alpha}}, \frac{\omega_r s_r^+}{\omega_r \tilde{y}^L{}_{ro\alpha}} = \frac{s_r^+}{\tilde{y}^L{}_{ro\alpha}}, \ \frac{\rho_d s_d}{\rho_d \tilde{z}^L{}_{do\alpha}} = \frac{s_d}{\tilde{z}^L{}_{do\alpha}}$$

It is clear that the efficiency of whole system and stages will not change. The similar proof for the model (9) is hold and the proof is complete.

**Theorem 3.2.** For each$DMU_o$: the upper bounds of the efficiency of systems and stages are in range(0,1]. (i.e.$0 < \tilde{E}_{o\alpha}^{S(L)} \leq 1, 0 < \tilde{E}_{o\alpha}^{I(L)} \leq 1, 0 < \tilde{E}_{o\alpha}^{II(L)} \leq 1$).

***Proof.*** According to the constraints of the model (5): $\frac{s_i^-}{\tilde{x}^U{}_{io\alpha}} \leq 1, \frac{\pi_d}{\tilde{z}^U{}_{do\alpha}} \leq 1$ .

Hence, $\sum_{i=1}^{m} \frac{s_i^-}{\tilde{x}^U{}_{io\alpha}} + \sum_{d=1}^{D} \frac{\pi_d}{\tilde{z}^U{}_{do\alpha}} \leq \mathrm{m} + \mathrm{D}$. Then, given that

$1 + \frac{1}{s+D}(\sum_{r=1}^{s} \frac{s_r^+}{\tilde{y}^L{}_{ro\alpha}} + \sum_{d=1}^{D} \frac{s_d}{\tilde{z}^L{}_{do\alpha}}) \geq 1$ and $0 < 1 - \frac{1}{m+D}(\sum_{i=1}^{m} \frac{s_i^-}{\tilde{x}^U{}_{io\alpha}} + \sum_{d=1}^{D} \frac{\pi_d}{\tilde{z}^U{}_{do\alpha}}) \leq 1$

It results that $0 < \tilde{E}_{o\alpha}^{S(L)} \leq 1$. The other efficiencies are proved similarly.

**Theorem 3.3.** For each$DMU_o$: the upper bounds of the efficiency of systems and stages are in range(0,1]. (i.e.$0 < \tilde{E}^{s(U)}_{o\alpha} \leq 1, 0 < \tilde{E}^{I(U)}_{o\alpha} \leq 1, 0 < \tilde{E}^{II(U)}_{o\alpha} \leq 1$).
***Proof.*** The proof is similar to the proof Theorem 2.

**Theorem 3.4.** Suppose $(t^{l^*}, \lambda_j^{l^*}, \lambda'^{l^*}_j, s_i^{l-^*}, s_r^{l+^*}, \pi_d^{l^*}, s_d^{l^*})$ is an optimal solution of the model (5). In this case, $\tilde{E}^{s(L)}_{o\alpha} = 1$ if and only if $s_i^{l-^*} = 0, s_r^{l+^*} = 0, \pi_d^{l^*} = 0, s_d^{l^*} = 0$. (A similar situation can be written for the efficiency of the stages).
***Proof.*** If $s_i^{l-^*} = 0, s_r^{l+^*} = 0, \pi_d^{l^*} = 0, s_d^{l^*} = 0$, it is obvious that $\tilde{E}^{s(L)}_{o\alpha} = 1$. Conversely, if $\tilde{E}^{s(L)}_{o\alpha} = 1$, we have:

$$1 - \frac{1}{m+D}\left(\sum_{i=1}^{m} \frac{s_i^{u-^*}}{x_{io}^L} + \sum_{d=1}^{D} \frac{\pi_d^{u^*}}{z_{do}^L}\right) = 1 + \frac{1}{s+D}\left(\sum_{r=1}^{s} \frac{s_r^{u+^*}}{y_{ro}^U} + \sum_{d=1}^{D} \frac{s_d^{u^*}}{z_{do}^U}\right)$$

This give the result that $s_i^{l-^*} = 0, s_r^{l+^*} = 0, \pi_d^{l^*} = 0, s_d^{l^*} = 0$.

**Theorem 3.5.** Suppose $(t'^{u^*}, \lambda_j^{u^*}, \lambda'^{u^*}_j, s_i^{u-^*}, s_r^{u+^*}, \pi_d^{u^*}, s_d^{u^*})$ is an optimal solution of the model (5). In this case, $\tilde{E}^{s(U)}_{o\alpha} = 1$if and only if$s_i^{u-^*} = 0, s_r^{u+^*} = 0, \pi_d^{u^*} = 0, s_d^{u^*} = 0$. (A similar situation can be written for the efficiency of the stages.
***Proof.*** The proof is similar to the proof Theorem 4.

**Theorem 3.6.** $\tilde{E}^{s(L)}_{o\alpha} = 1$ if and only if $\tilde{E}^{I(L)}_{o\alpha} = 1, \tilde{E}^{II(L)}_{o\alpha} = 1$.
***Proof.*** If $\tilde{E}^{s(L)}_{o\alpha} = 1$, then $s_i^{l-^*} = 0, s_r^{l+^*} = 0, \pi_d^{l^*} = 0, s_d^{l^*} = 0$. In this case we have$\tilde{E}^{\mathrm{I}(L)}_{o\alpha} = 1, \tilde{E}^{\mathrm{II}(L)}_{o\alpha} = 1$. Conversely, if $\tilde{E}^{\mathrm{I}(L)}_{o\alpha} = 1, \tilde{E}^{\mathrm{II}(L)}_{o\alpha} = 1$, it is obvious that $\tilde{E}^{s(L)}_{o\alpha} = 1$.

**Theorem 3.7.** $\tilde{E}^{s(U)}_{o\alpha} = 1$ if and only if $\tilde{E}^{I(U)}_{o\alpha} = 1, \tilde{E}^{II(U)}_{o\alpha} = 1$.
***Proof.*** The proof is similar to the proof Theorem 6.

**Theorem 3.8.** Let $\hat{x}_{ij} \in [\tilde{x}^L{}_{ij\alpha}, \tilde{x}^U{}_{ij\alpha}], \hat{z}_{dj} \in [\tilde{z}^L{}_{dj\alpha}, \tilde{z}^U{}_{dj\alpha}], \hat{y}_{rj} \in [\tilde{y}_{rj\alpha}{}^L, \tilde{y}^U{}_{rj\alpha}]$. And suppose $(\lambda_j^*, \lambda'^*_j, s_i^{-^*}, s_d^*, \pi_d^*, s_r^{+^*})$ is an optimal solution of the model (4). Hence, it can be concluding that: "each optimal solution of the model (4) is a feasible solution of the model (5)".
***Proof.*** Given that $\tilde{x}^L{}_{ij\alpha} \leq \hat{x}_{ij} \leq \tilde{x}^U{}_{ij\alpha}$, $\tilde{z}^L{}_{dj\alpha} \leq \hat{z}_{dj} \leq \tilde{z}^U{}_{dj\alpha}$, $\tilde{y}^L{}_{rj\alpha} \leq \hat{y}_{rj} \leq \tilde{y}^U{}_{rj\alpha}$, using rewriting the constraints of model the model (4), we have:

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^* \tilde{x}^L{}_{ij\alpha} \leq \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^* x_{ij} \leq (\lambda_o^* + 1) x_{io} \leq (\lambda_o^* + 1) \tilde{x}^U{}_{io\alpha}, \qquad i = 1, \dots, m$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^* \tilde{z}^U{}_{dj\alpha} \geq \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^* z_{dj} \geq (\lambda_o^* - 1) z_{do} \geq (\lambda_o^* - 1) \tilde{z}^L{}_{do\alpha}, \qquad d = 1, \dots, D$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda'^*_j \tilde{z}^L{}_{dj\alpha} \leq \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda'^*_j z_{dj} \leq (\lambda'^*_o + 1) z_{do} \leq (\lambda'^*_o + 1)\, \tilde{z}^U{}_{do\alpha}, \qquad d = 1, \dots, D$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda'^*_j \tilde{y}^U{}_{ro\alpha} \geq \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda'^*_j y_{rj} \geq (\lambda'^*_o - 1) y_{ro} \geq (\lambda'^*_o - 1)\, \tilde{y}^L{}_{io\alpha}, \qquad r = 1, \dots, s$$

Therefore:

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^* \tilde{x}^L{}_{ij\alpha} \le (\lambda_o^* + 1)\tilde{x}^U{}_{io\alpha}, \qquad i = 1,\dots,m$$
$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^* \tilde{z}^U{}_{dj\alpha} \ge (\lambda_o^* - 1)\tilde{z}^L{}_{do\alpha}, \qquad d = 1,\dots,D$$
$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda'^*_j \tilde{z}^L{}_{dj\alpha} \le (\lambda'^*_o + 1)\tilde{z}^U{}_{do\alpha}, \qquad d = 1,\dots,D$$
$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda'^*_j \tilde{y}^U{}_{ro\alpha} \ge (\lambda'^*_o - 1)\tilde{y}^L{}_{io\alpha}, \qquad r = 1,\dots,s.$$

And so, the proof is complete.

**Theorem 3.9.** Let $\hat{x}_{ij} \in [\tilde{x}^L{}_{ij\alpha}, \tilde{x}^U{}_{ij\alpha}], \hat{z}_{dj} \in [\tilde{z}^L{}_{dj\alpha}, \tilde{z}^U{}_{dj\alpha}], \hat{y}_{rj} \in [\tilde{y}_{rj\alpha}{}^L, \tilde{y}^U{}_{rj\alpha}]$. And suppose$(\lambda_j^*, \lambda'^*_j, s_i^{-*}, s_d^*, \pi_d^*, s_r^{+*})$ is an optimal solution of the model (4). Hence, it can be concluding that: each optimal solution of the model (8) is a feasible solution of the model (4).

***Proof.*** The proof is similar to the proof Theorem 8.

## 4. Result and discussion

To illustrate the proposed models in this paper, we use the data of 10 Mellat bank branches in Tehran for the year 2013[16]. Note that the proposed models (5), (8) calculate the lower and the upper the bound of the efficiency, respectively. In recently years, evaluating the efficiency of bank branches is an important topic. Each bank branch is composed of two-stage. Actually, each bank branch is considered as a DMU with a two-stage structure. "Personal score", "Paid profit" are inputs of stage1. Also, we use two intermediate measures" Total of four deposits", "Other resources" and 4 outputs "Facilities", "Received handling fee", "Earned profit", "Deferred claims". Note that undesirable output "Deferred claims" is considered as its inverse. It must be noted that each real number $x$can be considered as a TFN (i.e. $(x^l, x^m, x^r)$). Hence, we consider data have TFN structure. Thus, we firstly, calculate the $\alpha -$ cut intervals for inputs and outputs and intermediate measures. For TFN $x$ , the $\alpha -$cut interval is $[x^m - x^l(1-\alpha), x^m + x^r(1-\alpha)]$. Suppose that$\alpha = 0.25$. For$DMU_2$, the $\alpha -$cut intervals of personal score and paid profit are [6.07, 8.445] and [523081701, 5233176187], respectively. These intervals are calculated following similarly for other$DMUs$.

Finally, by applying these intervals to models 5 and 8, the results are obtained.

Table 1.The upper and the lower bound of the efficiencies.

| DMU | $[\tilde{E}_o^{s(L)}, \tilde{E}_o^{s(U)}]$ | $[\tilde{E}_o^{I(L)}, \tilde{E}_o^{I(U)}]$ | $[\tilde{E}_o^{II(L)}, \tilde{E}_o^{II(U)}]$ |
|---|---|---|---|
| 1 | [0.5000,0.6326] | [1,1] | [0.3960,0.5344] |
| 2 | [0.1116,0.1523] | [0.0125,0.3460] | [0.7081,1] |
| 3 | [0.1956,0.4876] | [1,1] | [0.2960,0.3788] |
| 4 | [0.1650,0.3242] | [0.0260,0.1340] | [0.1138,0.2579] |
| 5 | [0.2520,0.2652] | [0.0852,0.1248] | [0.2197,0.2688] |
| 6 | [0.0523,0.1547] | [0.1037,0.2690] | [0.1329,0.6987] |
| 7 | [0.2610,0.6123] | [0.1420,0.5250] | [0.1120,0.5642] |
| 8 | [0.1780,2472] | [0.1892,0.3283] | [0.2639,0.5176] |
| 9 | [0.1917,0.2587] | [0.4363,0.4578] | [0.1735,0.3528] |
| 10 | [0.2361,0.7230] | [0.5856,0.6913] | [0.1842,0.4549] |

Now, we illustrate these obtained results. The second to fourth columns report intervals of the overall efficiency and the efficiency of stages, respectively. Based on this table, all of $DMUs$ are inefficient in whole system.$DMU_1$and $DMU_3$are the upper (and the lower) efficient in stage 1. In stage 2, $DMU_2$ is efficient at the upper bound. In the lower bound of the efficiencies, between inefficient $DMUs$ of whole system and stages 1, 2,$DMU_1$,$DMU_{10}$ and $DMU_2$have the best efficiency with scores 0.5856, 0.7081 and 0.5000, respectively. Also, the lowest efficiency of the lower bound belongs to $DMU_6$, $DMU_2$ and $DMU_7$, respectively. In upper bound, $DMU_{10}$ has the highest efficiency in whole system and stage1. And also, $DMU_5$ has the worst efficiency in the stages 1 and 2 with scores 0.1248 and 0.2688, respectively. A similar interpretation can be written for other$DMUs$.

## 5. Conclusion

DEA is a useful technique for evaluating the efficiency of systems. Conventional DEA consider $DMUs$as a black box. But in practice many systems may have a network structure and this structure of these systems is ignored in evaluating the performance. Hence, to solve this problem, NDEA models are proposed to measure the efficiency of these systems. Note that in practice, many $DMUs$ (such as banks, etc.) can be considered as a two-stage system. Thus, two-stage systems are very importance among network systems. So far, many models are introduced to calculate the efficiency of this systems in presence of certain and uncertain data. In many manufacturing processes, uncertain data can be expressed by FN. Hence, in this paper, we focused on TFNs and presented a novel approach based on the non-redial models to evaluate the efficiency of two-stage systems in presence of TFNs. Actually, we considered all fuzzy inputs, intermediate measures and outputs are TFNs and suggested a non-redial model to evaluate the efficiency. For solving the proposed model, we used $\alpha -$cut approach and calculated$\alpha -$cut intervals of the inputs, intermediate measures and outputs. By applying these intervals to the proposed model, an interval model was obtained. Then, an optimistic and pessimistic procedure was used to solve the obtained interval model. Finally, some of the properties of the proposed model was described. It must be noted that we have shown that the whole system is efficient if and only if its stages are efficient. For future study, this technique can be extend the fuzzy DEA models to the multi–period slacks–based DEA models.

## Acknowledgment

The authors would like to appreciate the anonymous referees, whom their comment are valuable in enriching the manuscript.

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