Using PG Elements for Solving Fredholm Integro-Differential Equations

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Abstract. In this paper, we use Petrov-Galerkin elements such as continuous and discontinuous Lagrange-type k-0 elements and Hermite-type 3-1 elements to find an approximate solution for linear Fredholm integro-differential equations on $[0, 1]$. Also we show the efficiency of this method by some numerical examples.

Index to information contained in this paper

1 **Introduction**
2 **Petrov-Galerkin method**
3 **Continuous lagrange-type $k - 0$ elements**
4 **Discontinuous lagrange-type $k - 0$ elements**
5 **Hermite-type 3-1 elements**
6 **Numerical solution of the equation**
7 **Numerical results**
8 **Conclusion**

1. Introduction

A linear Fredholm integro-differential equation is

$$\begin{cases} u'(t) - u(t) - (Ku)(t) = f(t) \ ; \ t \in [0, 1] \\ u(0) = u_0 \end{cases} \tag{1}$$

where

$$(Ku)(t) = \int_0^1 k(t, s)u(s)ds.$$

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There are different numerical methods to solve the equation (1) including least square, collocation, Galerkin. The analysis of these methods can be found in, for example, [1–3, 5]. Petrov Galerkin method is established for linear Fredholm integral equations in [4]. In that paper, it is shown that Petrov-Galerkin method includes the Galerkin, collocation and least square methods. One of the advantages of Petrov-Galerkin method is that it allows us to achieve the same order of convergence as the Galerkin method with much less computational cost by choosing the test spaces to be spaces of piecewise polynomials of lower degree. In [6–9], authors used Lagrange-type $k - 0$ elements, Hermite-type 3-1 elements and multi-wavelets bases for solving linear Fredholm integral equations and Hammerstein integral equations on $[0, 1]$. Also, wavelet Petrov - Galerkin method for solving integro-differential equations has been used in [10].

The rest of this paper is organized as follows: In section 2 we review petrov-Galerkin method and illustrate it for equation (1). In sections 3, 4 and 5 we review some kinds of Petrov-Galerkin elements such as continuous and discontinuous Lagrange-type k-0 elements and Hermite-type 3-1 elements. In section 6 we explain how to use this method for solving equation (1) and finally, in section 7 we use some numerical examples to show the efficiency of this method.

2. Petrov-Galerkin method

In this section, following [4], we explain Petrov-Galerkin method.

Suppose X is a Banach space and X^* is the dual space of X consisting of continuous linear functionals on X. Assume that X_n and Y_n, for each positive integer n, are finite dimensional vector spaces such that $X_n \subset X$, $Y_n \subset X^*$ and

$$\dim X_n = \dim Y_n, \quad n = 1, 2, \cdots. \tag{2}$$

Also suppose that X_n and Y_n satisfy in the following condition:
(H) : For each $x \in X$ and $y \in X^*$, there exist $x_n \in X_n$ and $y_n \in Y_n$ such that $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ as $n \to \infty$.

DEFINITION 2.1 *For $x \in X$, an element $P_n x \in X_n$ is called the generalized best approximation from X_n to X with respect to Y_n whenever*

$$(x - P_n x, y_n) = 0 \quad for\ all\ y_n \in Y_n. \tag{3}$$

In [4], it is proved that for each $x \in X$, there exists a unique generalized best approximation from X_n to X with respect to Y_n if and only if

$$Y_n \cap X_n^\perp = \{0\}. \tag{4}$$

and in this case, P_n is a projection, i.e., $P_n^2 = P_n$.

Now let for each positive integer n, there exist a linear operator $\Pi_n : X_n \to Y_n$ with $\Pi_n X_n = Y_n$ satisfying two following conditions:

(H-1) for all $x_n \in X_n$, $\|x_n\| \leqslant C_1 (x_n, \Pi_n x_n)^{1/2}$
(H-2) for all $x_n \in X_n$, $\|\Pi_n x_n\| \leqslant C_2 \|x_n\|$.

If a pair of space sequences $\{X_n\}$ and $\{Y_n\}$ satisfy $(H - 1)$ and $(H - 2)$, we call $\{X_n, Y_n\}$ a regular Pair. It is shown that, if a regular Pair $\{X_n, Y_n\}$ satisfies

$\dim X_n = \dim Y_n$ and condition (H), then the corresponding generalized projection P_n satisfies:

(1) for all $x \in X$, $\|P_n x - x\| \to 0$ as $n \to \infty$

(2) there is a constant $C > 0$ such that, $\|P_n\| < C$, $n = 1, 2, \cdots$

(3) for some constant $C > 0$, independent of n , we have $\|P_n x - x\| \leqslant C\|Q_n x - x\|$

where $Q_n x$ is the best approximation from X_n to x.

Petrov-Galerkin method for equation (1) is a numerical method to find $u_n \in X_n$ such that

$$(u'_n - u_n - Ku_n, y_n) = (f, y_n) \quad for \ all \ y_n \in Y_n. \tag{5}$$

Now assume $u_n \in X_n$ and $\{b_i\}_{i=1}^n$ is a basis for X_n and $\{b_j^*\}_{j=1}^n$ is a basis for Y_n. Therefore Petrov-Galerkin method on $[0, 1]$ for equation (1) is

$$(u'_n - u_n - Ku_n, b_j^*) = (f, b_j^*) \quad j = 1, \cdots, n. \tag{6}$$

Petrov-Galerkin method using regular pairs $\{X_n, Y_n\}$ of piecewise polynomial spaces are called Petrov-Galerkin elements. If we use piecewise polynomials of degrees k and k' for spaces X_n and Y_n respectively, we call the corresponding Petrov-Galerkin elements $k - k'$ elements.

3. Continuous lagrange-type $k - 0$ elements

Let $0 = t_0 < t_1 < \cdots < t_n = 1$. We divide the interval $[0, 1]$ into n subinterval $I_i = [t_{i-1}, t_i]$. Let $h_i = t_i - t_{i-1}$ for $i = 1, \cdots, n$ and X_n be the space of continuous piecewise polynomials of degree $\leqslant k$ with knots at $t_i, i = 1, 2, \cdots, n - 1$. We construct a basis for X_n as follows:

let $\tau_j = \frac{j}{k}, j = 0, 1, \cdots, k$ and define

$$t_j^{(i)} = t_{i-1} + \tau_j h_i \quad j = 0, 1, \cdots, k \ , \ i = 1, \cdots, n.$$

Clearly $t_{i-1} = t_0^{(i)} < \cdots < t_k^{(i)} = t_i$. Now let we define $nk + 1$ functions $\Phi_j^{(i)}(t)$ by

$$\Phi_j^{(i)}(t) = \begin{cases} \displaystyle\prod_{\substack{\ell=0 \\ \ell \neq i}}^k \frac{t - t_\ell^{(i)}}{t_j^{(i)} - t_\ell^{(i)}} & t \in I_i \\ \\ 0 & t \notin I_i \end{cases} \quad \begin{bmatrix} \begin{cases} i = 1, 2, \cdots, n \\ j = 1, 2, \cdots, k - 1 \end{cases} \\ i = 1, j = 0 \\ i = n, j = k \end{bmatrix}$$

$$\Phi_k^{(i)}(t) = \begin{cases} \displaystyle\prod_{\ell=0}^{k-1} \frac{t - t_\ell^{(i)}}{t_k^{(i)} - t_\ell^{(i)}} & t \in I_i \\[2em] \displaystyle\prod_{\ell=1}^{k} \frac{t - t_\ell^{(i+1)}}{(t_0^{(i+1)} - t_\ell^{(i)})} & t \in I_i + 1 \\[2em] 0 & t \notin I_i \cup I_i + 1 \end{cases} \qquad i = 1, 2, \cdots, n - 1.$$

Note that for any $x_n \in X_n$, we have

$$x_n(t) = \sum_{j=0}^{k} x_n(t_j^{(i)}) \Phi_j^{(i)}(t) \quad t \in I_i, i = 1, \cdots, n$$

To construct the test space Y_n, we define

$$\Psi_0^{(i)}(t) = \begin{cases} 1 & t_{i-1} \leqslant t \leqslant t_{i-1} + \frac{1}{2k} \\ 0 & otherwise \end{cases} \quad i = 1, \cdots, n$$

$$\Psi_j^{(i)}(t) = \begin{cases} 1 & t_{i-1} + \frac{2j-1}{2k} \leqslant t \leqslant t_{i-1} + \frac{2j+1}{2k} \\ 0 & otherwise \end{cases} \quad \begin{array}{l} i = 1, \cdots, n \\ j = 1, 2, \cdots, k-1 \end{array}$$

$$\Psi_k^{(i)}(t) = \begin{cases} 1 & t_{i-1} + \frac{2k-1}{2k} \leqslant t \leqslant t_i \\ 0 & otherwise \end{cases} \quad i = 1, 2, \cdots, n$$

Then $\dim X_n = \dim Y_n = nk + 1$ and in [4] it is proved that for $1 \leqslant k \leqslant 5$ these two space sequences form a regular pair.

4. Discontinuous lagrange-type $k - 0$ elements

As before we divide the interval $[0, 1]$ into n subinterval $I_i = [t_{i-1}, t_i]$ by a sequence of points $0 = t_0 < t_1 < \cdots < t_n = 1$. Let $h_i = t_i - t_{i-1}$ for $i = 1, \cdots, n$ and X_n be the space of piecewise polynomials of degree $\leqslant k$ with knots at $t_i, i = 1, \cdots, n - 1$. Let $\tau_j = \frac{2j+1}{2k+2}, j = 0, 1 \cdots, k$ and define

$$t_j^{(i)} = t_{i-1} + \tau_j h_i \qquad\qquad j = 0, 1, \cdots, k \ , \ i = 1, \cdots, n$$

we define $n(k + 1)$ functions $\Phi_j^{(i)}(t)$ by

$$\Phi_j^{(i)}(t) = \begin{cases} \displaystyle\prod_{\substack{\ell=0 \\ \ell \neq j}}^{k} \frac{t - t_\ell^{(i)}}{t_j^{(i)} - t_\ell^{(i)}} & t \in I_i \\[2em] 0 & t \notin I_i \end{cases} \quad \begin{array}{l} i = 1, \cdots, n \\ j = 0, 1, \cdots, k. \end{array}$$

So for each $x_n \in X_n$, we have

$$x_n(t) = \sum_{j=0}^{k} x_n(t_j^{(i)}) \Phi_j^{(i)}(t), \quad t \in I_i, \ \ i = 1, \cdots, n.$$

Now we construct the test space Y_n by

$$\psi_j^{(i)}(t) = \begin{cases} 1 & t_{i-1} + \frac{jh_i}{k+1} \leqslant t \leqslant t_{i-1} + \frac{(j+1)h_i}{k+1} & j = 0, 1, \cdots, k \\ 0 & otherwise & i = 1, \cdots, n \end{cases}$$

Then $\dim X_n = \dim Y_n = n(k+1)$ and in [4] it is proved that for $1 \leqslant k \leqslant 5$ these two spaces sequences form a regular pair.

5. Hermite-type 3-1 elements

Again let $0 = t_0 < t_1 < \cdots < t_n = 1$ and define I_i and h_i as the previous section. Let X_n be the space of piecewise Hermite-type cubic polynomials,that is:

$$X_n = \{ \ x_n \in C^1[0,1] : x_n|_{I_i} \ is \ a \ cubic \ polynomial \ determined \ by$$
$$x_n^l(t_{i-1}), x_n^l(t_i), l = 0, 1, i = 1, \cdots, n\}$$
$$= span\{b_1(t), b_2(t), \cdots, b_{2n+2}(t)\}$$

Using Hermite interpolation, we can show that for each $x_n \in X_n$ the following relation is satisfied

$$x_n(t) = \sum_{j=1}^{n+1} \{x_n(t_{j-1})b_{2j-1}(t) + x_n'(t_{j-1})b_{2j}(t)\},$$

where

$$b_j(t) = \begin{cases} \phi_j(\tau)(h_1)^{j-1} & \tau = \frac{t-t_0}{h_1} \ , \ t \in I_1 \\ \\ 0 & t \notin I_1 \end{cases} \quad j = 1, 2$$

$$b_{2i+j}(t) = \begin{cases} \phi_{j+2}(\tau)(h_i)^{j-1} & \tau = \frac{t-t_{i-1}}{h_i} \ , \ t \in I_i \\ \phi_j(\tau)(h_{i+1})^{j-1} & \tau = \frac{t-t_i}{h_{i+1}} \ , \ t \in I_{i+1} \\ 0 & t \notin I_i \bigcup I_{i+1} \end{cases} \begin{cases} i = 1, \cdots, n-1 \\ j = 1, 2 \end{cases}$$

$$b_{2n+j}(t) = \begin{cases} \phi_{j+2}(\tau)(h_n)^{j-1} & \tau = \frac{t-t_{n-1}}{h_n} \ , \ t \in I_n \\ \\ 0 & t \notin I_n \end{cases} \quad j = 1, 2$$

and

$$\phi_1(\tau) = (1-\tau)^2(2\tau+1)$$
$$\phi_2(\tau) = \tau(1-\tau)^2$$
$$\phi_3(\tau) = \tau^2(3-2\tau)$$
$$\phi_4(\tau) = (\tau-1)\tau^2.$$

Now suppose Y_n is the space of piecewise linear polynomials, that is:

$$Y_n = span\{b_1^*(t), b_2^*(t), \cdots, b_{2n+2}^*(t)\}$$

where

$$b_{2i+1}^*(t) = \begin{cases} 1 & t \in [t_i - \frac{h_i}{2}, t_i + \frac{h_{i+1}}{2}] \\ \\ 0 & otherwise \end{cases} \qquad i = 0, 1, \cdots, n$$

$$b_{2i+2}^*(t) = \begin{cases} t - t_i & t \in [t_i - \frac{h_i}{2}, t_i + \frac{h_{i+1}}{2}] \\ \\ 0 & otherwise \end{cases} \qquad i = 0, 1, \cdots, n$$

$$h_0 = h_{n+1} = 0$$

Then $\dim X_n = \dim Y_n = 2n + 2$ and in [4] it is proved that $\{X_n, Y_n\}$ form a regular pair.

6. Numerical solution of the equation

Now for solving equation(1), note that $u_n' \in X_n$ can be written as:

$$u_n'(t) = \sum_{j=1}^{N} c_j b_j(t), \qquad (7)$$

where $N = nk + 1$ for continuous Lagrange-type k-0 elements and $N = n(k + 1)$ for discontinuous Lagrange-type k-0 elements and $N = 2n + 2$ for Hermite-type 3-1 elements. Therefore

$$u_n(t) = \int_0^t u_n'(\eta)d\eta + u(0)$$

$$= \sum_{j=1}^{N} c_j \int_0^t b_j(\eta)d\eta + u_0. \qquad (8)$$

From (6), Petrov-Galerkin method for equation(1) is

$$(u_n'(t) - u_n(t) - \int_0^1 k(t,s)u_n(s)ds, b_i^*(t) =$$

$$(f(t), b_i^*(t)) \ , \quad i = 1, \cdots, N. \qquad (9)$$

If we substitute (7) and (8) in (9), this leads to determin $\{c_1, c_2, \cdots, c_N\}$ as the

Table 1. Error of the solution of example 7.1, by using continuous Lagrange-type k-0 elements

k\n	1	2	4	10
1	0.0711171	0.0188107	0.054749	0.005467
2	0.0318487	0.0217291	0.00369031	$3.76589*10^{-5}$
3	0.00718256	0.00482825	0.00011322	$1.34267*10^{-5}$
4	0.00379042	$5.01533*10^{-4}$	$1.43578*10^{-5}$	$8.91439*10^{-8}$
5	0.000644195	$2.11348*10^{-4}$	$2.25054*10^{-6}$	$1.39802*10^{-10}$

Table 2. Error of the solution of example 7.1, by using discontinuous Lagrange-type k-0 elements

k\n	1	2	4	10
1	0.0511045	0.0224796	0.071885	0.0193661
2	0.00360519	0.00812911	0.000710505	$2.86684*10^{-4}$
3	0.00187936	0.00216985	0.000183828	$7.35154*10^{-5}$
4	0.00138635	$3.43556*10^{-4}$	$3.07522*10^{-5}$	$6.93185*10^{-8}$
5	0.000358096	$1.48186*10^{-5}$	$2.78763*10^{-6}$	$1.14221*10^{-10}$

solution of linear system

$$\sum_{j=1}^{N} c_j \{ (b_j(t), b_i^*(t)) - (\int_0^t b_j(\eta)d\eta, b_i^*(t)) -$$

$$(\int_0^1 k(t,s)\int_0^s b_j(\eta)d\eta ds, b_i^*(t)) \} = (f(t), b_i^*(t)) +$$

$$(u_0, b_i^*(t)) + (u_0 \int_0^1 k(t,s)ds, b_i^*(t)) \ , \ i = 1, \cdots, N. \tag{10}$$

7. Numerical results

Example 7.1 Consider integro-differential equation

$$\begin{cases} u'(t) - u(t) - \int_0^1 e^{st}u(s)ds = \frac{1-e^{t+1}}{t+1} & 0 \leqslant t \leqslant 1 \\ u(0) = 1 \end{cases}$$

with the exact solution $u(t) = e^t$. In table 1 and table 2 we computed $\|u_n(t_j^{(i)}) - u(t_j^{(i)})\|_2$ by using continuous Lagrange-type k-0 elements and discontinuous Lagrange-type k-0 elements, respectively, and in table 5, we computed $\|u_n(t_i) - u(t_i)\|_2$ by using Hermite-type 3-1 elements for $n = 1, 2, 4, 10$ with equally spaced points.

Example 7.2 Consider integro-differential equation

$$\begin{cases} u'(t) - u(t) - \frac{1}{(\ln 2)^2} \int_0^1 \frac{t}{s+1}u(s)ds = \frac{1}{t+1} - \frac{t}{2} - \ln(t+1) & 0 \leqslant t \leqslant 1 \\ u(0) = 0 \end{cases}$$

with the exact solution $u(t) = \ln(t + 1)$. In table 3 and table 4 we computed $\|u_n(t_j^{(i)}) - u(t_j^{(i)})\|_2$ by using continuous Lagrange-type k-0 elements and discontinuous Lagrange-type k-0 elements, respectively, and in table 5, we computed $\|u_n(t_i) - u(t_i)\|_2$ by using Hermite-type 3-1 elements for $n = 1, 2, 4, 10$ with equally spaced points.

Table 3. Error of the solution of example 7.2 by using continuous Lagrange-type k-0 elements

k\n	1	2	4	10
1	0.281718	0.0981084	0.0143986	0.0198249
2	0.00615015	0.000388461	$3.63723*10^{-4}$	$8.69589*10^{-6}$
3	0.00189197	0.000289194	$4.30127*10^{-4}$	$8.68816*10^{-6}$
4	$5.64184*10^{-7}$	$7.56978*10^{-6}$	$2.08519*10^{-8}$	$1.85981*10^{-9}$
5	$3.14611*10^{-7}$	$4.61389*10^{-6}$	$5.34152*10^{-8}$	$2.07412*10^{-9}$

Table 4. Error of the solution of example 7.2 by using discontinuous Lagrange-type k-0 elements

k\n	1	2	4	10
1	0.0116123	0.00483185	0.017779	0.004548
2	0.000668994	0.0079569	$8.47232*10^{-5}$	$3.51021*10^{-6}$
3	0.00128848	0.000119047	$1.09386*10^{-5}$	$4.31958*10^{-6}$
4	$3.01056*10^{-5}$	$7.61518*10^{-6}$	$2.17294*10^{-7}$	$1.41735*10^{-9}$
5	$4.10969*10^{-5}$	$5.16155*10^{-7}$	$2.58529*10^{-7}$	$2.67806*10^{-9}$

Table 5. Error of the solution of examples 7.1 and 7.2 by using Hermite-type 3-1 elements

n	Example 7.1	Example 7.2
1	0.00660823	0.01902821
2	0.00028836	0.00261158
4	$1.17916*10^{-5}$	0.000334349
10	$4.89515*10^{-7}$	$1.552805*10^{-6}$

8. Conclusion

When using piecewise polynomials as basis functions we face with the following problems:

1) Increasing in the polynomials's degree increases calculation's errors;

2) Numerical solution of equations by the Galerkin method is so difficult.

In this paper, we used Petrov-Galerkin elements for solving linear Fredholm integro-differential equations. Using this method, by choosing the test space to be space of piecewise polynomials of lower degree, we practically showed that the above problems could be removed, consequently, we are able to solve equation (1) with less computational cost. In section 7 we showed , by two examples , that increasing in the degree of polynomials causes decreasing of the calculation's errors.

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