International Journal of
Mathematical Modelling & Computations
Vol. 02, No. 04, 2012, 283 - 298

Numerical Solution of the Most General Nonlinear Fredholm Integro-Differential-Difference Equations by using Taylor Polynomial Approach

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Received: 27 February 2012; Accepted: 25 July 2012.

Abstract. In this study, a Taylor method is developed for numerically solving the high-order most general nonlinear Fredholm integro-differential-difference equations in terms of Taylor expansions. The method is based on transferring the equation and conditions into the matrix equations which leads to solve a system of nonlinear algebraic equations with the unknown Taylor coefficients. Also, we test the method by numerical examples.

Keywords: Nonlinear Fredholm integro-differential-difference equations; Taylor Series

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1. Introduction

In recent years, many authors have presented a wide range of numerical methods and developed them for solving integro-differential-difference equation such as Legendre polynomial method [11] ,Legendre multiwavelets method [6], trigonometric wavelets method[7], Chebyshev polynomial method [10], Chebyshev collocation method [4], Hermite collocation Method [2], Bessel polynomial method [12], continuous Runge-Kutta method [1] and Runge-Kutta-Pouzet method [5] ,

A Taylor expansion approach for solving linear Fredholm integro-difference and integro-differential-difference equations has been presented by Mehmet Sezer and Mustafa Gulsu [3, 14, 15]. Also, Y. Ordokhani and M. J. Mohtashami have used the method for nonlinear Fredholm integro-differential-difference equations [9].

In this paper, previous studies are developed and applied to solve the high-order

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nonlinear Fredholm integro-differential-difference equations as

$$\sum_{k=0}^{m}\sum_{j=0}^{n} P_{kj}(x)y^{(k)}(x-\tau_{kj}) = f(x) + \int_a^b \sum_{p=1}^{q}\sum_{l=0}^{s} K_{pl}(x,t)y^p(t-\tau_{pl})\mathrm{d}t; \quad \tau_{kj} \geqslant 0, \tau_{pl} \geqslant 0 \tag{1}$$

with the mixed conditions

$$\sum_{k=0}^{m-1}\sum_{i=0}^{R} a_{lk}^{i} y^{(k)}(a_i) = \lambda_l, \quad l = 0,1,...,m-1, a \leqslant a_i \leqslant b, \tag{2}$$

where $P_{kj}(x)$, $K_{pl}(x,t)$ and $f(x)$ are given functions that have suitable derivatives and a_{ik}^{l}, λ_i, i and l are given real constants and represented by truncated Taylor expansion of degree N at x = c and the unknown function $y(x)$ is expressed in the form

$$y(x) = \sum_{n=0}^{N} \frac{y^{(n)}(c)}{n!}(x-c)^n, \quad a \leqslant c \leqslant b, N \geqslant m. \tag{3}$$

2. Fundamental matrix relations

Let us convert the equation (1) into matrix form. We consider the solution $y(x)$ by the truncated Taylor series defined in (3) and express it in the matrix form

$$[y(x)] = \mathbf{X}\mathbf{M_0}\mathbf{Y}, \tag{4}$$

where

$$\mathbf{X} = [1 \quad (x-c) \quad (x-c)^2 \quad ... \quad (x-c)^N],$$

$$\mathbf{M_0} = \begin{bmatrix} \frac{1}{0!} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{1!} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2!} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{N!} \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} y^{(0)}(c) \\ y^{(1)}(c) \\ y^{(2)}(c) \\ \vdots \\ y^{(N)}(c) \end{bmatrix}.$$

By substituting $(x - \tau_{kj})$ instead of x in (3) and differentiating it N times with respect to x, we will have

$$y^{(0)}(x-\tau_{kj}) = \sum_{n=0}^{N} \frac{y^{(n)}(c)}{n!}(x-\tau_{kj}-c)^n$$

$$y^{(1)}(x-\tau_{kj}) = \sum_{n=1}^{N} \frac{y^{(n)}(c)}{(n-1)!}(x-\tau_{kj}-c)^{n-1}$$

$$y^{(2)}(x-\tau_{kj})=\sum_{n=2}^{N}\frac{y^{(n)}(c)}{(n-2)!}(x-\tau_{kj}-c)^{n-2}$$

$$\vdots$$

$$y^{(N)}(x-\tau_{kj})=\sum_{n=N}^{N}\frac{y^{(n)}(c)}{(n-N)!}(x-\tau_{kj}-c)^{n-N}.$$

and

$$[\mathbf{Y}(\tau_{\mathbf{kj}})]=\mathbf{X}(\tau_{\mathbf{kj}})\mathbf{Y}, \tag{5}$$

where

$$\mathbf{X}(\tau_{\mathbf{kj}})=\begin{bmatrix} \frac{1}{0!} & \frac{(-\tau_{kj})^1}{1!} & \frac{(-\tau_{kj})^2}{2!} & \cdots & \frac{(-\tau_{kj})^N}{N!} \\ 0 & \frac{1}{0!} & \frac{(-\tau_{kj})^1}{1!} & \cdots & \frac{(-\tau_{kj})^{(N-1)}}{(N-1)!} \\ 0 & 0 & \frac{1}{0!} & \cdots & \frac{(-\tau)^{(N-2)}}{(N-2)!} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!} \end{bmatrix},$$

$$\mathbf{Y}(\tau_{\mathbf{kj}})=[y^{(0)}(c-\tau_{kj}) \quad y^{(1)}(c-\tau_{kj}) \quad \ldots \quad y^{(N)}(c-\tau_{kj})]^T,$$

$$\mathbf{Y}=[y^{(0)}(c) \quad y^{(1)}(c) \quad \ldots \quad y^{(N)}(c)]^T.$$

Moreover, we consider the term $P_{kj}(x)y^{(k)}(x-\tau_{kj})$ of difference part of equation (1) as the truncated Taylor expansions of degree N at $x=c$ in the form

$$P_{kj}(x)y^{(k)}(x-\tau_{kj})=\sum_{n=0}^{N}\frac{1}{n!}[P_{kj}(x)y^{(k)}(x-\tau_{kj})]^{(n)}_{x=c}(x-c)^n. \tag{6}$$

By using the Leibnitzs rule

$$[P_{kj}(x)y^{(k)}(x-\tau_{kj})]^{(n)}_{x=c}=\sum_{i=0}^{n}\binom{n}{i}P_{kj}^{(n-i)}(c)y^{(k+i)}(c-\tau_{kj}),$$

and substituting in the expression (6), we have

$$P_{kj}(x)y^{(k)}(x-\tau_{kj})=\sum_{n=0}^{N}\sum_{i=0}^{n}\frac{1}{n!}\binom{n}{i}P_{kj}^{(n-i)}(c)y^{(k+i)}(c-\tau_{kj})(x-c)^n$$

and

$$[\mathbf{P}_{\mathbf{kj}}(x)y^{(k)}(x-\tau_{kj})]=\mathbf{X}\mathbf{P}_{\mathbf{kj}}\mathbf{Y}(\tau_{\mathbf{kj}}). \tag{7}$$

Or from (5)

$$[\mathbf{P_{kj}}(x)y^{(k)}(x-\tau_{kj})] = \mathbf{X}\mathbf{P_{kj}}\mathbf{X}_{(}\tau_{\mathbf{kj}})\mathbf{Y}, \tag{8}$$

where

$$\mathbf{P_{kj}} = \begin{bmatrix} 0\dots 0 & \frac{P_{kj}^{(0)}(c)}{0!0!} & 0 & 0 & \cdots & 0 & 0 \\ 0\dots 0 & \frac{P_{kj}^{(1)}(c)}{1!0!} & \frac{P_{kj}^{(0)}(c)}{0!1!} & 0 & \cdots & 0 & 0 \\ 0\dots 0 & \frac{P_{kj}^{(2)}(c)}{2!0!} & \frac{P_{kj}^{(1)}(c)}{1!1!} & \frac{P_{kj}^{(0)}(c)}{0!2!} & \cdots & 0 & 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0\dots 0 & \frac{P_{kj}^{(N-k)}(c)}{N!0!} & \frac{P_{kj}^{(N-k-1)}(c)}{(N-k-1)!1!} & \frac{P_{kj}^{(N-k-2)}(c)}{(N-k-2)!2!} & \cdots & \frac{P_{kj}^{(1)}(c)}{1!(N-k-1)!} & \frac{P_{kj}^{(0)}(c)}{0!(N-k)!} \\ 0\dots 0 & \frac{P_{kj}^{(N-k+1)}(c)}{(N-k+1)!0!} & \frac{P_{kj}^{(N-k)}(c)}{(N-k)!1!} & \frac{P_{kj}^{(N-k-1)}(c)}{(N-k-1)!2!} & \cdots & \frac{P_{kj}^{2}(c)}{2!(N-k-1)!} & \frac{P_{kj}^{(1)}(c)}{0!(N-k)!} \\ \vdots\ \vdots\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0\dots 0 & \frac{P_{kj}^{(N-1)}(c)}{(N-1)!0!} & \frac{P_{kj}^{(N-2)}(c)}{(N-2)!1!} & \frac{P_{kj}^{(N-3)}(c)}{(N-3)!2!} & \cdots & \frac{P_{kj}^{(k)}(c)}{k!(N-k-1)!} & \frac{P_{kj}^{(k-1)}(c)}{(k-1)!(N-k)!} \\ 0\dots 0 & \frac{P_{kj}^{(N)}(c)}{(N)!0!} & \frac{P_{kj}^{(N-1)}(c)}{(N-1)!1!} & \frac{P_{kj}^{(N-2)}(c)}{(N-2)!2!} & \cdots & \frac{P_{kj}^{(k+1)}(c)}{(k+1)!(N-k-1)!} & \frac{P_{kj}^{(k)}(c)}{(k)!(N-k)!} \end{bmatrix}$$

Clearly

$$\mathbf{P_{0j}} = \begin{bmatrix} \frac{P_{0j}^{(0)}(c)}{0!0!} & 0 & 0 & \cdots & 0 \\ \frac{P_{0j}^{(1)}(c)}{1!0!} & \frac{P_{0j}^{(0)}(c)}{0!1!} & 0 & \cdots & 0 \\ \frac{P_{0j}^{(2)}(c)}{2!0!} & \frac{P_{0j}^{(1)}(c)}{1!1!} & \frac{P_{0j}^{(0)}(c)}{0!2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{P_{0j}^{(N)}(c)}{N!0!} & \frac{P_{0j}^{(N-1)}(c)}{(N-1)!1!} & \frac{P_{0j}^{(N-2)}(c)}{(N-2)!2!} & \cdots & \frac{P_{0j}^{(0)}(c)}{(0)!N!} \end{bmatrix}.$$

Now, we approximate $f(x)$ by truncated Taylor series and represent it as the following matrix form

$$[f(x)] = \mathbf{X}\mathbf{M_0}\mathbf{F}, \tag{9}$$

where

$$\mathbf{F} = [f^{(0)}(c) \quad f^{(1)}(c) \quad \cdots \quad f^{(N)}(c)]^T.$$

2.1 *Matrix relation for integral part*

Let us approximate The kernel functions $K_{pl}(x,t)$, $p = 1, ..., q$; $l = 0, 1, ..., s$ by the truncated Taylor series of degree N about $x = c$, $t = c$ as

$$K_{pl}(x,t) = \sum_{n=0}^{N}\sum_{m=0}^{N} k_{nm}^{pl}(x-c)^n(t-c)^m, \tag{10}$$

where

$$k_{nm}^{pl} = \frac{1}{n!m!}\frac{\partial^{n+m}K_{pl}(c,c)}{\partial x^n \partial t^m}, \quad n, m = 0, 1, ..., N.$$

The expression (10) can be expressed in the following matrix form

$$[\mathbf{K}_{pl}(x,t)] = \mathbf{X}\mathbf{K}_{pl}\mathbf{T}^T, \tag{11}$$

where

$$\mathbf{K}_{pl} = [k_{nm}^{pl}], \quad n,m = 0,1,...,N$$

$$\mathbf{T} = [1 \quad (t-c) \quad (t-c)^2 \quad ... \quad (t-c)^N].$$

Now, we use the Cauchy product of p series and express $y(x)$ in the following matrix form

$$y^p(x) = \bar{\mathbf{X}}_p\bar{\mathbf{Y}}_p \tag{12}$$

so that

$$\bar{\mathbf{Y}}_{\mathbf{p}} = \begin{bmatrix} \bar{y}_0^p \\ \bar{y}_1^p \\ \vdots \\ \bar{y}_{nN}^p \end{bmatrix} = \begin{bmatrix} \sum_{k_1=0}^{0}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{p-1}=0}^{k_{p-2}} \frac{y_{k_{p-1}}y_{k_{p-2}-k_{p-1}}\cdots y_{k_1-k_2}y_{0-k_1}}{k_{p-1}!(k_{p-2}-k_{p-1})!...(k_1-k_2)!(0-k_1)!} \\ \sum_{k_1=0}^{1}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{p-1}=0}^{k_{p-2}} \frac{y_{k_{p-1}}y_{k_{p-2}-k_{p-1}}\cdots y_{k_1-k_2}y_{1-k_1}}{k_{p-1}!(k_{p-2}-k_{p-1})!...(k_1-k_2)!(1-k_1)!} \\ \vdots \\ \sum_{k_1=0}^{pN}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{p-1}=0}^{k_{p-2}} \frac{y_{k_{p-1}}y_{k_{p-2}-k_{p-1}}\cdots y_{k_1-k_2}y_{pN-k_1}}{k_{p-1}!(k_{p-2}-k_{p-1})!...(k_1-k_2)!(pN-k_1)!} \end{bmatrix},$$

where $y_m = 0$ for $m > N$ and

$$\bar{\mathbf{X}}_{\mathbf{p}} = [1 \quad (x-c) \quad (x-c)^2... \quad (x-c)^{pN}].$$

Thereby, we have the matrix form

$$[y^p(t-\tau_{pl})] = \bar{\mathbf{T}}_p(\tau_{pl})\bar{\mathbf{Y}}_p \tag{13}$$

where

$$\bar{\mathbf{T}}_{\mathbf{p}}(\tau_{\mathbf{pl}}) = [1 \quad (t-\tau_{pl}-c) \quad (t-\tau_{pl}-c)^2 \quad ... \quad (t-\tau_{pl}-c)^{pN}].$$

Substituting (11) and (13) into the integral part of Eq. (1), we get the following matrix form

$$\begin{aligned} \int_a^b \sum_{p=1}^{q}\sum_{l=0}^{s} \mathbf{K}_{pl}(x,t)y^p(t-\tau_{pl})\mathrm{d}t &= \int_a^b \sum_{p=1}^{q}\sum_{l=0}^{s} \mathbf{X}\mathbf{K}_{pl}\mathbf{T}^T\bar{\mathbf{T}}_p(\tau_{pl})\bar{\mathbf{Y}}_p\mathrm{d}t \\ &= \mathbf{X}\sum_{p=1}^{q}\sum_{l=0}^{s}\mathbf{K}_{pl}\{\int_a^b \mathbf{T}^T\bar{\mathbf{T}}_p(\tau_{pl})\mathrm{d}t\}\bar{\mathbf{Y}}_p \\ &= \mathbf{X}\sum_{p=1}^{q}\sum_{l=0}^{s}\mathbf{K}_{pl}\mathbf{H}_{pl}\bar{\mathbf{Y}}_p, \end{aligned} \tag{14}$$

where

$$\mathbf{H}_{pl} = \int_a^b \mathbf{T}^T\bar{\mathbf{T}}_p(\tau_{pl})\mathrm{d}t = [h_{nm}^{pl}];$$

if $\tau_{pl} \neq 0$,

$$h_{nm}^{pl} = \sum_{k=0}^{n} \binom{n}{k} (\tau_{pl})^k \frac{(b-\tau_{pl}-c)^{n+m-k+1} - (a-\tau_{pl}-c)^{n+m-k+1}}{n+m-k+1}$$

and if $\tau_{pl} = 0$,

$$h_{nm}^{pl} = \frac{(b-c)^{n+m+1} - (a-c)^{n+m+1}}{n+m+1}; \quad n = 0,1,...,N, m = 0,1,...,pN.$$

Finally, substituting (8), (9) and (14) into the equation (1) we have

$$\sum_{k=0}^{m}\sum_{j=0}^{n} \mathbf{P}_{kj}\mathbf{X}(\tau_{kj})\mathbf{Y} - \sum_{p=1}^{q}\sum_{l=0}^{s} \mathbf{K}_{pl}\mathbf{H}_{pl}\bar{\mathbf{Y}}_p = \mathbf{M}_0\mathbf{F}, \quad p,q,s < m \tag{15}$$

2.2 *Matrix relation for the mixed conditions*

Now we obtain the matrix representation of the mixed conditions (2). We can express The expression (3) and its derivatives as

$$[y^{(k)}(a_i)] = \mathbf{A}_i\mathbf{M}_k\mathbf{Y}, \tag{16}$$

where

$$\mathbf{A}_i = [1 \quad (a_i-c) \quad (a_i-c)^2 \quad ... \quad (a_i-c)^N],$$

$$\mathbf{M_k} = \begin{bmatrix} 0 \; 0 \ldots & \frac{1}{0!} & 0 & \ldots & 0 \\ 0 \; 0 \ldots & 0 & \frac{1}{1!} & \ldots & 0 \\ \vdots \; \vdots & \vdots & \vdots & & \vdots \\ 0 \; 0 \ldots & 0 & 0 & \ldots & \frac{1}{(N-k)!} \\ 0 \; 0 \ldots & 0 & 0 & \ldots & 0 \\ \vdots \; \vdots & \vdots & \vdots & & \vdots \\ 0 \; 0 \ldots & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

By replacing (16) into conditions (2) and then simplifying, we have

$$\mathbf{U}_l\mathbf{Y} = [\lambda_l], \quad l = 0,1,...,m-1 \tag{17}$$

where

$$\mathbf{U}_l = \sum_{k=0}^{m-1}\sum_{i=0}^{R} \{a_{lk}^i \mathbf{A}_i\mathbf{M}_k\} = [u_{l0} \quad u_{l1} \quad \ldots \quad u_{lN}]$$

3. Method of solution

The relation matrix (15) corresponds to a system of $(N+1)$ algebraic equations for the $(N+1)$ unknown coefficients $y^{(n)}(c), n = 0,1,...,N$ and we can write it in the form

$$\mathbf{W}\mathbf{Y} - \mathbf{V}\bar{\mathbf{Y}}_p = \mathbf{M}_0\mathbf{F}, \quad p = 1, ..., q \quad or \quad [\mathbf{W}; \mathbf{V} : \mathbf{M}_0\mathbf{F}] \tag{18}$$

where

$$\mathbf{W} = [w_{hi}] = \sum_{k=0}^{m}\sum_{j=0}^{n} \mathbf{P}_{kj}\mathbf{X}(\tau_{kj}), \quad h, i = 0, 1, ..., N,$$

$$\mathbf{V} = [v_{st}] = \sum_{p=1}^{q}\sum_{l=0}^{s} K_{pl}H_{pl}, \quad s, t = 0, 1, ..., N.$$

Consider the matrix form of condition (2), represented into (17). We can write

$$[U_l; \lambda_l] = [u_{l0} \quad u_{l1} \quad ... \quad u_{lN}; \lambda_l], \quad l = 0, 1, ..., m-1. \tag{19}$$

To obtain the approximate solution of equation (1) under the mixed conditions (2) in terms of Taylor polynomials, we substitute m rows matrices (19) by the last m rows of the matrix (18), thus, we have the augmented matrix as [1-4]

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{V}} : \tilde{\mathbf{F}}] = \left[\begin{array}{cccccccccccccccccc}
w_{0,0} & w_{0,1} & \cdots & w_{0,N} & ; & v_{0,0} & v_{0,1} & \cdots & v_{0,1N} & ; \cdots ; & v_{0,0} & v_{0,1} & \cdots & v_{0,pN} & ; & \frac{f^{(0)}(c)}{0!} \\
w_{1,0} & w_{1,1} & \cdots & w_{1,N} & ; & v_{1,0} & v_{1,1} & \cdots & v_{1,1N} & ; \cdots ; & v_{1,0} & v_{1,1} & \cdots & v_{1,pN} & ; & \frac{f^{(1)}(c)}{1!} \\
\vdots & \vdots & \vdots & \vdots & ; & \vdots & \vdots & \vdots & \vdots & ; \vdots ; & \vdots & \vdots & \vdots & ; & \vdots \\
w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & v_{N-m,0} & v_{N-m,1} & \cdots & v_{N-m,1N} & ; \cdots ; & v_{N-m,0} & v_{N-m,1} & \cdots & v_{N-m,pN} & ; & \frac{f^{(N-m)}(c)}{(N-m)!} \\
u_{0,0} & u_{0,1} & \cdots & u_{0,N} & ; & 0 & 0 & \cdots & 0 & ; \cdots ; & 0 & 0 & ... & 0 & ; & \lambda_0 \\
u_{1,0} & u_{1,1} & \cdots & u_{1,N} & ; & 0 & 0 & \cdots & 0 & ; \cdots ; & 0 & 0 & ... & 0 & ; & \lambda_1 \\
\vdots & \vdots & \vdots & \vdots & ; & \vdots & \vdots & \vdots & \vdots & ; \cdots ; & \vdots & \vdots & \vdots & \vdots & ; & \vdots \\
u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & 0 & 0 & \cdots & 0 & ; \cdots ; & 0 & 0 & \cdots & 0 & ; & \lambda_{m-1}
\end{array}\right]. \tag{20}$$

By using this nonlinear system, we can obtain the unknown Taylor coefficients $y^{(n)}(x), n = 0, 1, ..., N$. Substituting this coefficient into the expression (3) we find the approximation solution $y(x)$.

Also, we can approximate the error for the integro-differential-difference equations. The Taylor expansion (3) is an approximate solution of equation (1). When the function $y(x)$ and its derivatives are replaced into the equation (1), the resuming equation must be satisfied approximately, that is, for $x = x_i$, $(i = 0, 1, ..., N)$

in [a, b]

$$E(x_i) = |\sum_{k=0}^{m}\sum_{j=0}^{n} P_{kj}(x_i)y^{(k)}(x_i - \tau_{kj}) - \int_a^b \sum_{p=1}^{q}\sum_{l=0}^{s} K_{pl}(x_i,t)y^p(t-\tau_{pl})\mathrm{d}t - f(x_i)| \cong 0 \tag{21}$$

or

$$E(X_i) \leqslant 10^{-k_i}(k_i \quad \text{is any positive integer}).$$

If $\max|10^{-k_i}| = 10^{-i}$ (k is any positive integer) is determined, then until the difference $|D(x_i)|$ at each of the points becomes smaller than the prescribed 10^{-i}, the truncation limit N is increased [8, 13].

4. Illustrations

The method of this paper is useful in obtaining the solutions of nonlinear Fredholm integro-differential-difference equations in terms of Taylor polynomials. In this section, we will show some examples to get appropriate approximations to the solution of integro-differential-difference equations. The approximate solutions are generated by means of the Computer Algebra System Maple 11.

Example 1. Let us first consider the following nonlinear Fredholm integro-differential-difference equation

$$y''(x)+xy'(x-2)+2y(x-1) = f(x)+\int_0^2 \{(x+t)[y(t-1)]^2+[y(t-2)]^2-t[y(t-1)]^4\}\mathrm{d}t$$

with the conditions

$$y(0) = 0, \quad y'(0) = 0$$

and approximate the solution $y(x)$ by the Taylor polynomial

$$y(x) = \sum_{n=0}^{3} \frac{y^{(n)}(0)}{n!}(x-c)^n,$$

so that, $N = 3$, $a = c = 0$, $b = 2$, $\tau_{00} = 1$, $\tau_{10} = 2$, $\tau_{20} = 0$, $P_{00}(x) = 2$, $P_{10}(x) = x$, $P_{20}(x) = x$, $K_{20}(x,t) = t$, $K_{21}(x,t) = 1$, $K_{40}(x,t) = 1$, $f(x) = -5x^3 + 22x^2 - \frac{1144}{35}x - \frac{257458}{6435}$. The required matrices are

$$P_{00} = \begin{bmatrix} 2&0&0&0\\ 0&2&0&0\\ 0&0&1&0\\ 0&0&0&\frac{1}{3} \end{bmatrix}, \quad P_{10} = \begin{bmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&\frac{1}{2} \end{bmatrix}, \quad P_{20} = \begin{bmatrix} 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix},$$

$$X_{00} = \begin{bmatrix} 1&-1&\frac{1}{2}&-\frac{1}{6}\\ 0&1&-1&\frac{1}{2}\\ 0&0&1&-1\\ 0&0&0&1 \end{bmatrix}, \quad X_{10} = \begin{bmatrix} 1&-2&2&-\frac{4}{3}\\ 0&1&-2&2\\ 0&0&1&-2\\ 0&0&0&1 \end{bmatrix}, \quad X_{20} = \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix},$$

$$M_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{6} \end{bmatrix}, \quad F = \begin{bmatrix} -\frac{257458}{6435} \\ -\frac{1144}{35} \\ 44 \\ -30 \end{bmatrix},$$

$$K_{20} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_{40} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H_{20} = \begin{bmatrix} 2 & 0 & \frac{2}{3} & 0 & \frac{2}{5} & 0 & \frac{2}{7} \\ 2 & \frac{2}{3} & \frac{2}{3} & \frac{2}{5} & \frac{2}{5} & \frac{2}{7} & \frac{2}{7} \\ \frac{8}{3} & \frac{4}{3} & \frac{16}{15} & \frac{4}{10} & \frac{24}{35} & \frac{4}{7} & \frac{32}{63} \\ 4 & \frac{36}{15} & \frac{44}{30} & \frac{38}{35} & \frac{44}{35} & \frac{68}{63} & \frac{20}{21} \end{bmatrix},$$

$$H_{21} = \begin{bmatrix} 2 & -2 & \frac{8}{3} & -4 & \frac{32}{5} & -\frac{64}{6} & \frac{128}{7} \\ 2 & -\frac{4}{3} & \frac{4}{3} & -\frac{8}{5} & \frac{32}{15} & -\frac{64}{21} & \frac{32}{7} \\ \frac{8}{3} & -\frac{4}{3} & \frac{16}{15} & -\frac{16}{15} & \frac{128}{105} & -\frac{32}{21} & \frac{128}{63} \\ 4 & -\frac{8}{5} & \frac{16}{15} & -\frac{32}{35} & \frac{32}{35} & -\frac{64}{63} & \frac{128}{105} \end{bmatrix},$$

$$H_{40} = \begin{bmatrix} 2 & 0 & \frac{2}{3} & 0 & \frac{2}{5} & 0 & \frac{2}{7} & 0 & \frac{2}{9} & 0 & \frac{2}{11} & 0 & \frac{2}{13} \\ 2 & \frac{2}{3} & \frac{2}{3} & \frac{2}{5} & \frac{2}{5} & \frac{2}{7} & \frac{2}{7} & \frac{2}{9} & \frac{2}{9} & \frac{2}{11} & \frac{2}{11} & \frac{2}{13} & \frac{2}{13} \\ \frac{8}{3} & \frac{4}{3} & \frac{16}{15} & \frac{4}{10} & \frac{24}{35} & \frac{4}{7} & \frac{32}{63} & \frac{4}{9} & \frac{40}{99} & \frac{4}{11} & \frac{48}{143} & \frac{4}{13} & \frac{56}{195} \\ 4 & \frac{36}{15} & \frac{44}{30} & \frac{38}{35} & \frac{44}{35} & \frac{68}{63} & \frac{20}{21} & \frac{28}{33} & \frac{76}{99} & \frac{100}{143} & \frac{92}{143} & \frac{116}{195} & \frac{36}{65} \end{bmatrix}.$$

The matrices for conditions are computed as

$$U_0 = [1 \quad 0 \quad 0 \quad 0], \quad \lambda_0 = 0$$
$$U_1 = [0 \quad 1 \quad 0 \quad 0], \quad \lambda_1 = 0.$$

Substituting the above matrices into the fundamental matrix equation (18) and solving the system, Taylor coefficients are obtained as

$$y^{(0)}(0) = 0, \quad y^{(1)}(0) = 0, \quad y^{(2)}(0) = 2, \quad y^{(3)}(0) = -6,$$

which gives the exact solution $y(x) = x^2 - x^3$.

Example 2. Let us consider the integro-differential-difference equation as

$$y'''(x) - xy''(x-2) + y(x-1) = -6x^3 + 36x^2 - \frac{12402}{35}x - \frac{43547}{55} + \int_0^1 \{-ty(t-2) + x[y(t-1)]^2 + t[y(t)]^3\} \mathrm{d}t$$

with the conditions

$$y(0) = 0, \quad y'(0) = 0,$$

where $N = 5$, $a = c = 0$, $b = 1$, $\tau_{00} = 0$, $\tau_{20} = 1$, $\tau_{30} = 2$, $P_{00}(x) = 1$, $P_{10}(x) = -1$, $P_{20}(x) = 1$, $K_{10}(x,t) = -t$, $K_{20}(x,t) = x$, $K_{30}(x,t) = t$, $f(x) = -6x^3 + 36x^2 - \frac{12402}{35}x - \frac{43547}{55}$. So, the required matrices are

$$P_{00} = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&\frac{1}{2}&0&0&0 \\ 0&0&0&\frac{1}{6}&0&0 \\ 0&0&0&0&\frac{1}{24}&0 \\ 0&0&0&0&0&\frac{1}{120} \end{bmatrix}, \quad P_{10} = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix},$$

$$P_{20} = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&-1&0&0&0 \\ 0&0&0&-1&0&0 \\ 0&0&0&0&-\frac{1}{2}&0 \\ 0&0&0&0&0&-\frac{1}{6} \\ 0&0&0&0&0&0 \end{bmatrix}, \quad P_{30} = \begin{bmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&\frac{1}{2} \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix},$$

$$X_{00} = \begin{bmatrix} 1&-1&\frac{1}{2}&-\frac{1}{6}&\frac{1}{24}&-\frac{1}{120} \\ 0&1&-1&\frac{1}{2}&-\frac{1}{6}&\frac{1}{24} \\ 0&0&1&-1&\frac{1}{2}&-\frac{1}{6} \\ 0&0&0&1&-1&\frac{1}{2} \\ 0&0&0&0&1&-1 \\ 0&0&0&0&0&1 \end{bmatrix}, \quad X_{10} = \begin{bmatrix} 1&-2&2&-\frac{8}{6}&\frac{16}{24}&-\frac{32}{120} \\ 0&1&-2&2&-\frac{8}{6}&\frac{16}{24} \\ 0&0&1&-2&2&-\frac{8}{6} \\ 0&0&0&1&-2&2 \\ 0&0&0&0&1&-2 \\ 0&0&0&0&0&1 \end{bmatrix},$$

$$X_{20} = \begin{bmatrix} 1&-2&2&-\frac{8}{6}&\frac{16}{24}&-\frac{32}{120} \\ 0&1&-2&2&-\frac{8}{6}&\frac{16}{24} \\ 0&0&1&-2&2&-\frac{8}{6} \\ 0&0&0&1&-2&2 \\ 0&0&0&0&1&-2 \\ 0&0&0&0&0&1 \end{bmatrix}, \quad X_{30} = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix},$$

$$K_{10} = \begin{bmatrix} 0&-1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}, \quad K_{20} = \begin{bmatrix} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix},$$

$$K_{30} = \begin{bmatrix} 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&\frac{1}{2}&0&0&0 \\ 0&0&0&\frac{1}{6}&0&0 \\ 0&0&0&0&\frac{1}{24}&0 \\ 0&0&0&0&0&\frac{1}{120} \end{bmatrix}, \quad F = \begin{bmatrix} -\frac{43547}{55} \\ -\frac{12402}{35} \\ 156 \\ -36 \\ 0 \\ 0 \end{bmatrix}$$

$$H_{10} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{3} & -\frac{15}{4} & \frac{31}{5} & -\frac{63}{6} \\ -\frac{7}{2} & \frac{16}{3} & -\frac{101}{12} & \frac{137}{10} & -\frac{229}{10} & \frac{274}{7} \\ \frac{37}{3} & -\frac{229}{12} & \frac{458}{15} & -\frac{503}{10} & \frac{2973}{35} & -\frac{8201}{56} \\ -\frac{175}{4} & \frac{687}{10} & -\frac{3341}{30} & \frac{6494}{35} & -\frac{88573}{280} & \frac{138535}{252} \\ \frac{61}{5} & -\frac{623}{30} & \frac{3781}{105} & -\frac{17757}{280} & \frac{35642}{315} & -\frac{128579}{630} \\ -\frac{3367}{6} & \frac{19022}{21} & -\frac{252667}{168} & \frac{644425}{252} & -\frac{2793991}{630} & \frac{5417618}{693} \end{bmatrix},$$

$$H_{20} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{3} & -\frac{1}{4} & \frac{1}{5} & -\frac{1}{6} & \frac{1}{7} & -\frac{1}{8} & \frac{1}{9} & -\frac{1}{10} & \frac{1}{11} \\ \frac{1}{2} & -\frac{1}{6} & \frac{1}{12} & -\frac{1}{20} & \frac{1}{30} & -\frac{1}{42} & \frac{1}{56} & -\frac{1}{72} & \frac{1}{90} & -\frac{1}{110} & \frac{1}{132} \\ \frac{1}{3} & -\frac{1}{12} & \frac{1}{30} & -\frac{1}{60} & \frac{1}{105} & -\frac{1}{168} & \frac{1}{252} & -\frac{1}{360} & \frac{1}{495} & -\frac{1}{660} & \frac{1}{858} \\ \frac{1}{4} & -\frac{1}{20} & \frac{1}{60} & -\frac{1}{140} & \frac{1}{280} & -\frac{1}{504} & \frac{1}{840} & -\frac{1}{1320} & \frac{1}{1980} & -\frac{1}{2860} & \frac{1}{4004} \\ \frac{1}{5} & -\frac{1}{30} & \frac{1}{105} & -\frac{1}{280} & \frac{1}{630} & -\frac{1}{1260} & \frac{1}{2310} & -\frac{1}{3960} & \frac{1}{6435} & -\frac{1}{1001} & \frac{1}{5005} \\ \frac{1}{6} & -\frac{1}{42} & \frac{1}{168} & -\frac{1}{504} & \frac{1}{1260} & -\frac{1}{2772} & \frac{1}{5544} & -\frac{1}{10296} & \frac{1}{18018} & -\frac{1}{2002} & \frac{1}{16016} \end{bmatrix},$$

$$H_{30} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} & \frac{1}{19} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} & \frac{1}{19} & \frac{1}{20} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} & \frac{1}{19} & \frac{1}{20} & \frac{1}{21} \end{bmatrix},$$

Also, the matrices for conditions are

$$U_0 = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \quad \lambda_0 = 0$$
$$U_1 = [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0], \quad \lambda_1 = 0.$$

By substituting the above matrices in the matrix equation (18) and solving the system, we will have

$$y^{(0)}(0) = 0, \quad y^{(1)}(0) = 0, \quad y^{(2)}(0) = 48, \quad y^{(3)}(0) = -36, y^{(4)}(0) = 0, \quad y^{(5)}(0) = 0,$$

By putting into (3), we obtain the exact solution as

$$y(x) = 24x^2 - 6x^3. \tag{22}$$

Example 3. Now consider the following nonlinear equation

$$y'''(x-\frac{\pi}{2})+y''(x-\pi)-xy'(x-\pi)-xy(x-\frac{\pi}{2}) = f(x)+\int_0^{\pi} \{x[y(t-\pi)]^3+t[y(t-\frac{\pi}{2})]^2\}\mathrm{d}t,$$

with the conditions

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1,$$

and approximate the solution $y(x)$ by the Taylor polynomial

$$y(x) = \sum_{n=0}^{7} \frac{y^{(n)}(0)}{n!}(x-c)^n,$$

Where as $a = 0$, $b = \pi$, $c = 0$, $P_{00}(x) = -x$, $P_{10}(x) = -x$, $P_{20}(x) = 1$, $P_{30}(x) = 1$, $K_{20}(x,t) = t$, $K_{30}(x,t) = x$,$\tau_{10} = \tau_{20} = \pi$, $\tau_{30} = \frac{\pi}{2}$ and $f(x) = -2x\sin(x) - \frac{\pi^2}{4}$. So, the required matrices are

$$P_{00} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{120} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{720} & 0 \end{bmatrix},$$

$$P_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{120} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{720} \end{bmatrix},$$

$$P_{20} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{120} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_{30} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$X_{00} = X_{30} = \begin{bmatrix} 1 & -\frac{\pi}{2} & \frac{\pi^2}{8} & -\frac{\pi^3}{48} & \frac{\pi^4}{384} & -\frac{\pi^5}{3840} & \frac{\pi^6}{46080} & -\frac{\pi^7}{645120} \\ 0 & 1 & -\frac{\pi}{2} & \frac{\pi^2}{8} & -\frac{\pi^3}{48} & \frac{\pi^4}{384} & -\frac{\pi^5}{3840} & \frac{\pi^6}{46080} \\ 0 & 0 & 1 & -\frac{\pi}{2} & \frac{\pi^2}{8} & \frac{\pi^3}{48} & -\frac{\pi^4}{384} & \frac{\pi^5}{3840} \\ 0 & 0 & 0 & 1 & -\frac{\pi}{2} & \frac{\pi^2}{8} & \frac{\pi^3}{48} & -\frac{\pi^4}{384} \\ 0 & 0 & 0 & 0 & 1 & -\frac{\pi}{2} & \frac{\pi^2}{8} & \frac{\pi^3}{48} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{\pi}{2} & \frac{\pi^2}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{\pi}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$X_{10} = X_{20} = \begin{bmatrix} 1 & -\pi & \frac{\pi^2}{2} & -\frac{\pi^3}{6} & \frac{\pi^4}{24} & -\frac{\pi^5}{120} & \frac{\pi^6}{720} & -\frac{\pi^7}{5040} \\ 0 & 1 & -\pi & \frac{\pi^2}{2} & -\frac{\pi^3}{6} & \frac{\pi^4}{24} & -\frac{\pi^5}{120} & \frac{\pi^6}{720} \\ 0 & 0 & 1 & -\pi & \frac{\pi^2}{2} & -\frac{\pi^3}{6} & \frac{\pi^4}{24} & -\frac{\pi^5}{120} \\ 0 & 0 & 0 & 1 & -\pi & \frac{\pi^2}{2} & -\frac{\pi^3}{6} & \frac{\pi^4}{24} \\ 0 & 0 & 0 & 0 & 1 & -\pi & \frac{\pi^2}{2} & -\frac{\pi^3}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\pi & \frac{\pi^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\pi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$K_{20} = \begin{bmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}, \quad K_{30} = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$M_0 = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&\frac{1}{2}&0&0&0&0&0 \\ 0&0&0&\frac{1}{6}&0&0&0&0 \\ 0&0&0&0&\frac{1}{24}&0&0&0 \\ 0&0&0&0&0&\frac{1}{120}&0&0 \\ 0&0&0&0&0&0&\frac{1}{720}&0 \\ 0&0&0&0&0&0&0&\frac{1}{5040} \end{bmatrix}, \quad F = \begin{bmatrix} -2.46 \\ 0 \\ -4 \\ 0 \\ 8 \\ 0 \\ -12 \\ 0 \end{bmatrix}$$

$$H_{20} = \left[\begin{matrix} \pi & 0 & \frac{\pi^3}{12} & 0 & \frac{\pi^5}{80} & 0 & \frac{\pi^7}{448} & 0 & \frac{\pi^9}{2304} & 0 \\ \frac{\pi^2}{2} & \frac{\pi^3}{12} & \frac{\pi^4}{24} & \frac{\pi^5}{80} & \frac{\pi^6}{160} & \frac{\pi^7}{448} & \frac{\pi^8}{896} & \frac{\pi^9}{2304} & \frac{\pi^{10}}{4608} & \frac{\pi^{11}}{11264} \\ \frac{\pi^3}{3} & \frac{\pi^4}{12} & \frac{\pi^5}{30} & \frac{\pi^6}{80} & \frac{3\pi^7}{560} & \frac{\pi^8}{448} & \frac{4915\pi^9}{16128} & \frac{\pi^{10}}{2304} & \frac{5\pi^{11}}{25344} & \frac{\pi^{12}}{11264} \\ \frac{\pi^4}{4} & \frac{3\pi^5}{40} & \frac{7\pi^6}{240} & \frac{13\pi^7}{1120} & \frac{11\pi^8}{2240} & \frac{17\pi^9}{8064} & \frac{5\pi^{10}}{5376} & \frac{7\pi^{11}}{16896} & \frac{19\pi^{12}}{101376} & \frac{25\pi^{13}}{292864} \\ \frac{\pi^5}{5} & \frac{\pi^6}{15} & \frac{11\pi^7}{420} & \frac{3\pi^8}{280} & \frac{\pi^9}{5040} & \frac{23\pi^{10}}{504} & \frac{13\pi^{11}}{14784} & \frac{5\pi^{12}}{12672} & \frac{59\pi^{13}}{329472} & \frac{3\pi^{14}}{36608} \\ \frac{\pi^6}{6} & \frac{5\pi^7}{84} & \frac{\pi^8}{42} & \frac{5\pi^9}{504} & \frac{43\pi^{10}}{10080} & \frac{83\pi^{11}}{44352} & \frac{37\pi^{12}}{44352} & \frac{31\pi^{13}}{82368} & \frac{113\pi^{14}}{658944} & \frac{173\pi^{15}}{2196480} \\ \frac{\pi^7}{7} & \frac{3\pi^8}{56} & \frac{11\pi^9}{504} & \frac{31\pi^{10}}{3360} & \frac{37\pi^{11}}{9240} & \frac{157\pi^{12}}{88704} & \frac{305\pi^{13}}{384384} & \frac{79\pi^{14}}{219648} & \frac{271\pi^{15}}{1647360} & \frac{111\pi^{16}}{1464320} \\ \frac{\pi^8}{8} & \frac{7\pi^9}{144} & \frac{29\pi^{10}}{1440} & \frac{91\pi^{11}}{10560} & \frac{239\pi^{12}}{63360} & \frac{533\pi^{13}}{329472} & \frac{1163\pi^{14}}{1537536} & \frac{2269\pi^{15}}{6589440} & \frac{2083\pi^{16}}{13178880} & \frac{3637\pi^{17}}{49786880} \end{matrix} \right.$$

$$\left. \begin{matrix} \frac{\pi^{11}}{11264} & 0 & \frac{\pi^{13}}{53248} & 0 & \frac{\pi^{15}}{245760} \\ \frac{\pi^{12}}{22528} & \frac{\pi^{13}}{53248} & \frac{\pi^{14}}{106496} & \frac{\pi^{15}}{245760} & \frac{\pi^{16}}{491520} \\ \frac{3\pi^{13}}{73216} & \frac{\pi^{14}}{53248} & \frac{7\pi^{15}}{798720} & \frac{\pi^{16}}{245760} & \frac{\pi^17}{522240} \\ \frac{23\pi^{14}}{585728} & \frac{29\pi^{15}}{1597440} & \frac{9\pi^{16}}{1064960} & \frac{11\pi^{17}}{2785280} & \frac{31\pi^{18}}{16711680} \\ \frac{83\pi^{15}}{2196480} & \frac{7\pi^{16}}{399360} & \frac{37\pi^{17}}{4526080} & \frac{\pi^{18}}{261120} & \frac{143\pi^{19}}{79380480} \\ \frac{\pi^{16}}{27456} & \frac{23\pi^{17}}{1357824} & \frac{43\pi^{18}}{5431296} & \frac{59\pi^{19}}{15876096} & \frac{139\pi^{20}}{79380480} \\ \frac{175\pi^{17}}{4978688} & \frac{89\pi^{18}}{5431296} & \frac{33\pi^{19}}{4299776} & \frac{191\pi^{20}}{52920320} & \frac{631\pi^{21}}{370442240} \\ \frac{2029\pi^{18}}{59744256} & \frac{3275\pi^{19}}{206389248} & \frac{5123\pi^{20}}{687964160} & \frac{2599\pi^{21}}{740884480} & \frac{351\pi^{22}}{211681280} \end{matrix} \right].$$

Table 1. Error analysis of Example 3 for the x value

x_i	Exact solution	Present method					
		N=6	$E(x_i)$	N=7	$E(x_i)$	N=8	$E(x_i)$
0	1.000000	1.000000	0.000000	1.000000	0.000000	1.000000	0.000000
$\pi/10$	0.951059	0.951067	0.000007	0.951056	0.000002	0.951058	0.000001
$2\pi/10$	0.809028	0.809475	0.000447	0.809377	0.000348	0.809363	0.000334
$3\pi/10$	0.587809	0.590421	0.002611	0.590075	0.002266	0.5900482	0.002238
$2\pi/5$	0.309054	0.317454	0.008399	0.316541	0.007486	0.316101	0.007046
$\pi/2$	0.000000	0.020008	0.020008	0.017565	0.017565	0.015993	0.015993
$3\pi/5$	-0.308960	-0.269804	0.039155	-0.276757	0.032203	-0.275786	0.03317
$7\pi/10$	-0.587728	-0.521129	0.066598	-0.540468	0.047260	-0.547172	0.040556
$4\pi/5$	-0.808970	-0.709028	0.099941	-0.758680	0.050289	-0.794277	0.014692
$9\pi/10$	-0.951028	-0.819877	0.131151	-0.936128	0.014900	-0.946926	0.004102
π	-1.000000	-0.857487	0.142512	-1.107491	0.107491	-1.021123	0.021123

Also, The matrix for conditions are computed as

$$U_0 = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \quad \lambda_0 = 1$$
$$U_1 = [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \quad \lambda_1 = 0$$
$$U_2 = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \quad \lambda_2 = -1.$$

putting the above matrix in the matrix equation (18) and solving the system, the Taylor coefficients are obtained as

$$y^{(0)}(0) = 1, \quad y^{(1)}(0) = 0, \quad y^{(2)}(0) = -1, \quad y^{(3)}(0) = -0.010686,$$
$$y^{(4)}(0) = 1.137288, \quad y^{(5)}(0) = -0.13584, \quad y^{(6)}(0) = -0.8354504,$$
$$y^{(7)}(0) = -0.4526872,$$

By replacing into (3), we get

$$y(x) = 1 - \frac{x^2}{2} - 0.001781x^3 + 0.047387x^4 - 0.001132x^5 - 0.00116x^6 - 0.00009x^7,$$

The gained solutions for $N = 6, 7, 8$ are compared with the exact solution $y(x) = \cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$ in Table 1.

Example 4. Our last example is the nonlinear equation as

$$y^{(4)}(x-1)+xy^{(3)}(x-1)+xy''(x-1)+y'(x-2)+y(x-2) = f(x)+\int_0^2 \{ty(t-1)^3+e^{-3}y(t-2)^2\}\mathrm{d}t$$

with the conditions

$$y(0) = 1, \quad y'(0) = -1.$$

Table 2. Error analysis of Example 4 for the x value

		N=5		N=7	
x_i	Exact solution	Numerical solution	$E(x_i)$	Numerical solution	$E(x_i)$
0	1.000000	1.000000	0.000000	1.000000	0.000000
0.2	0.818730	0.819362	0.000632	0.818858	0.000128
0.4	0.670320	0.670355	0.000035	0.670347	0.000027
0.6	0.548811	0.544802	0.004009	0.549997	0.001186
0.8	0.449328	0.437368	0.011960	0.451493	0.002165
1.0	0.367879	0.345018	0.022861	0.371388	0.003509
1.2	0.301194	0.266474	0.034720	0.306491	0.005297
1.4	0.246596	0.201674	0.044922	0.254240	0.007644
1.6	0.201896	0.151223	0.050673	0.212581	0.010685
1.8	0.165298	0.115851	0.049447	0.179866	0.014568
2.0	0.135335	0.095872	0.039463	0.154744	0.019409

We approximate the solution $y(x)$ by the Taylor polynomial

$$y(x) = \sum_{n=0}^{5} \frac{y^{(n)}(0)}{n!}(x-c)^n,$$

whereas $a = 0$, $b = 2$, $c = 0$, $P_{00}(x) = 1$, $P_{10}(x) = 1$, $P_{20}(x) = x$, $P_{30}(x) = x$, $P_{40}(x) = 1$, $K_{20}(x,t) = e^{-3}$, $K_{30}(x,t) = t$, $\tau_{00} = \tau_{10} = 2$, $\tau_{20} = \tau_{30} = \tau_{40} = 2$, $f(x) = e^{-x+1} - \frac{1}{9}e^3 - \frac{1}{2}e + \frac{23}{18}e^{-3}$. To find a Taylor polynomial solution, we proceed as before. Then the Taylor coefficients are obtained as

$$y^{(0)}(0) = 1, \quad y^{(1)}(0) = -1, \quad y^{(2)}(0) = 1.071776, \quad y^{(3)}(0) = -1.674588,$$
$$y^{(4)}(0) = 2.457288, \quad y^{(5)}(0) = -1.69908.$$

By putting into (3), we get

$$y(x) = 1 - x + 0.535888x^2 - 0.279098x^3 + 0.102387x^4 + 0.014159x^5.$$

The gained solutions for $N = 5, 7$ are compared with the exact solution $y(x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$ in Table 2.

5. Conclusion

The method proposed in this paper is applied to find approximate and exact solutions of general nonlinear Fredholm integro-differential-difference equations. The method is based on computing the coefficients in the Taylor polynomial of solution of a nonlinear Fredholm integro-differential-difference equation. It is observed that the method has the best advantage when the known functions in an equation can be expanded to Taylor series which converge rapidly .

the method can be developed and applied to a integro-differential-difference equations

$$\sum_{k=0}^{m}\sum_{j=0}^{n}P_{kj}(x)y^{(k)}(x-\tau_{kj})=f(x)+\int_a^b\sum_{p=0}^{q}\sum_{l=0}^{s}K_{pl}(x,t)g(t-\tau_{pl},y(t-\tau_{pl}))\mathrm{d}t$$

but some modifications are required.

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