*International Journal of*
*Mathematical Modelling & Computations*
Vol. 05, No. 03, Summer 2015, 259- 266

# Non-Standard Finite Difference Method for Numerical Solution of Second Order Linear Fredholm Integro-Differential Equations

P. K. Pandey *

*Department of Mathematics, Dyal Singh College ( University of Delhi), Lodhi Road, New Delhi- 110003, India.*

**Abstract.**In this article we have considered a non-standard finite difference method for the solution of second order Fredholm integro differential equation type initial value problems. The non-standard finite difference method and the composite trapezoidal quadrature method is used to transform the Fredholm integro-differential equation into a system of equations. We have also developed a numerical method for the numerical approximation of the derivative of the solution of the problems. The numerical results in experiment on some model problems show the simplicity and efficiency of the method. Numerical results showed that the proposed method is convergent and at least second order of accurate.

Received: 8 February 2015, Revised: 23 May 2015, Accepted: 2 August 2015.

**Keywords:** Composite Trapezoidal Method, Fredholm Integro-Differential Equations, Initial Value Problem, Non-Linear Equation, Non-Standard Finite Difference Method, Quadrature Formulas.

**Index to information contained in this paper**



## 1. Introduction

The occurrence of differential equations and integral equations are of common in many areas of sciences and engineering. However, the research work in these particular area resulted in a new specific topic, where both differential and integral operators appeared together in the same equation. This new type of equations, were termed as integro-differential equations. In a particular the conversion of a boundary value problems in the area of study of differential equation to integro-differential equations, with constant limits of integration, was termed as Fredholm

*Corresponding author. Email: pramod_10p@hotmail.com

integro-differential equations. This class of problems has gained importance in the literature for the variety of their applications. In most cases it is impossible to obtain solutions of these problems using analytical methods. There are different approaches and varieties of numerical and analytical methods are used to solve the Fredholm integro-differential equations namely difference method and compact finite difference method[2, 9, 21], an extrapolation method[1], Taylor series[20], method of regularization[13, 17], variational method[4], Adomian decomposition method[19], variational iterations method[15] and references therein reported in recent past.

In this article we consider a method for the numerical solution of the following linear Fredholm integro-differential equations of the form

$$y''(x) = f(x, y) + \int_a^b K(x, t)y(t)dt, \quad a \leqslant x \leqslant b, \tag{1}$$

subject to the initial conditions

$$y(a) = \alpha, \quad y'(a) = \beta$$

where $\alpha$, and $\beta$ are real constant. The functions $f(x, y)$ and the kernel $K(x, t)$ are known. The solution y(x) is to be determined.

This class of problems has gained importance in the literature for the variety of their applications. In most cases it is impossible to obtain solutions of these problems using analytical methods. In these cases we resort to approximate solution of the problems. The emphasis in this article will be on the development of an efficient numerical method to deal with approximate numerical solution of the integro differential equation than to proving theoretical concepts of convergence and existence. Thus the existence and uniqueness of the solution to problem (1) is assumed. We further assumed that problem (1) is well posed. The specific assumption on to ensure existence and uniqueness solution to problem (1) will not be considered. The theorems of uniqueness, existence, and convergence are important and can be found in the literature[2, 3, 5].

Last few decades have seen substantial progress in the development of approximate solutions of these problems by non conventional methods. One such method, a non standard finite difference methods have increasingly been recognized as efficient method for the numerical solution of initial value problems in ordinary differential equation [11, 14, 18]. The Non-standard finite difference method is a simple and generates impressive numerical result with high accuracy. Hence, the purpose of this article is to develop a non-standard finite difference method similar to [14] for numerical solution of the second-order initial value problems of Fredholm integro-differential equation (1).

To the best of our knowledge, no similar method for the numerical solution of problem (1) has been discussed in the literature so far. We hope that others may find the proposed method an improvement and appealing to those existing finite difference methods for the numerical solution of integro differential equations.

We have presented our work in this article as follows. In the next section we derived a non-standard finite difference method. In Section 3, we have discussed local

truncation error in propose method (5). The application of the proposed method to the problems in (1) has been presented and illustrative numerical results have been produced to show the efficiency of the new method in Section 4. Discussion and conclusion on the performance of the new method are presented in Section 5.

## 2. The Non-Standard Finite Difference Method

We define $N$ finite numbers of nodal points of the domain $[a,b]$, in which the solution of the problem (1) is desired, as $a \leqslant x_0 < x_1 < x_2 < ..... < x_N = b$ using uniform step length $h$ such that $x_i = a + ih, \quad i = 0, 1, 2, ....., N$. Suppose that we wish to determine the numerical approximation of the theoretical solution $y(x)$ of the problem (1) at the nodal point $x_i, \quad i = 1, 2, ....., N$. We denote the numerical approximation of $y(x)$ at node $x = x_i$ as $y_i$ . Let us denote $f_i$ as the approximation of the theoretical value of the source function $f(x, y(x))$ at node $x = x_i, \quad i = 0, 1, 2, ....., N$. Further we have assumed that $K(x, t)$ is separable kernel otherwise by using the Taylor series expansion for the kernel, reduce it to separable kernel. Thus the integro-differential equation (1) at node $x_i$ may be written as

$$y''(x_i) = f(x_i, y_i) + \int_a^b K(x_i, t)y(t)dt \tag{2}$$

We approximate the integral that appeared in equation (2) by the repeated / composite trapezoidal quadrature method[6] which will yield the following

$$\int_a^b K(x_i, t)y(t)dt = \sum_{j=0}^{n} [K(x_i, t_j)\lambda_j y(t_j) + E_{tj}] \tag{3}$$

where $a \leqslant t_0 < t_1 < t_2 < ..... < t_N = b, j = 0, 1, 2, ....., N$ using uniform step length $h$ such that $t_j = a + jh, \quad j = 0, 1, 2, ....., N$, $E_{tj}$ is the truncation error in $j^{th}$ interval and quadrature nodes $\lambda_j, j = 0, 1, 2, ....., N$ are numerical coefficients that

$$\lambda_j = \begin{cases} \frac{1}{2}h & \text{if } j = 0, N \\ h & \text{otherwise } j = 1, 2, ....., N-1 \end{cases}$$

do not depend on the function $y(t)$. The term $E_{tj}$ in (3) depend on $N$ and large $N$ reduces $E_{tj}$ considerably. Thus substituting the value of integral term in (2) from (3) after neglecting the error terms, we have

$$y_i'' = f_i + \sum_{j=0}^{N} K_{i,j}\lambda_j y_j \tag{4}$$

Let us assume a local assumption as in [8] that no previous truncation errors have been made i.e. $y(x_i) = y_i$ and following the ideas in [10, 12], we propose non-standard finite difference method for the approximation of the analytical solution $y(x_{i+1})$ and derivative of analytical solution $y'(x_{i+1})$ of the problem (1) at node

$x = x_{i+1}$ as,

$$y_{i+1} = y_i + hy'_i + \frac{3h^2F_i^2}{2(3F_i - hF'_i)}, \quad i = 0, 1, 2, ....., N-1, \tag{5}$$

$$y'_{i+1} = y'_i + \frac{2hF_i^2}{(2F_i - hF'_i)}, \quad i = 0, 1, 2, ....., N-1, \tag{6}$$

where $F_i = f_i + \sum_{j=0}^{N} K_{i,j}\lambda_j y_j$ and $F'_i = \frac{\partial F_i}{\partial x}$. Thus we will obtain the system of nonlinear equations at each nodal point $x_{i+1}, i = 0, 1, 2, ...., N-1$.

For the computational purpose in Section 4, we have used the following finite difference approximation in place of $hF'_i$ in (5) and (6).

$$hF'_i = F_{i+1} - F_i \tag{7}$$

Thus from (7) we can write (5) and (6) as,

$$y_{i+1} = y_i + hy'_i - \frac{3h^2F_i^2}{2(F_{i+1} - 4F_i)}, \quad i = 0, 1, 2, ....., N-1, \tag{8}$$

and

$$y'_{i+1} = y'_i - \frac{2hF_i^2}{F_{i+1} - 3F_i}, \quad i = 0, 1, 2, ....., N-1, \tag{9}$$

which is an implicit nonlinear system of equations. We have to solve a nonlinear system with a large number of equations. So there are some complexity in the system and so computation is difficult. However we have applied an iterative method to solve above system of nonlinear equations (8) and (9).

## 3. Local Truncation Error

The local truncation error at the node $x = x_{i+1}$ using the exact arithmetic, is given as :

$$T_{i+1} = y(x_i + h) - y_{i+1}$$

At the nodal point $x = x_{i+1}$, $i = 0, 1, 2, ...., N-1$, the truncation error $T_{i+1}$ in method (5) may be written as [6],

$$\begin{aligned} T_{i+1} &= y_{i+1} - y_i - hy'_i - \frac{3h^2}{2}\frac{F_i^2}{3F_i - hF'_i} \\ &= y_{i+1} - y_i - hy'_i - \frac{h^2}{2}y''_i(1 - \frac{hF'_i}{3F_i})^{-1} \end{aligned}$$

Writing the Taylor series expansion for $y$ at nodal point $x = x_i$ and binomial expansion using along with $y_i^{(3)} = F_i'$ and $y_i'' = F_i$, we have

$$T_{i+1} = \frac{h^4}{24}(3y_i^{(4)} - 4\frac{(y_i^{(3)})^2}{y_i''}). \tag{10}$$

Thus we have obtained a truncation error at each node of $O(h^4)$. Similarly we can estimate local truncation error in method (6) at each node $x_i$ as,

$$T'_{i+1} = \frac{h^3}{12}(2y_i^{(4)} - 3\frac{(y_i^{(3)})^2}{y_i''}). \tag{11}$$

## 4. Numerical Results

To illustrate our method and demonstrate its computational efficiency, we have considered two model problems. In each model problem, we took uniform step size $h$. In Table 1 - Table 3, we have shown $MAY$ and $MDY$, the maximum absolute error in the solution $y$ and derivative of the solution $y'$ of the problems (1) for different values of $N$. We have used the following formulas in computation of $MAY$ and $MDY$,

$$MAY = \max_{1 \leqslant i \leqslant N} |y(x_i) - y_i|.$$

$$MDY = \max_{1 \leqslant i \leqslant N} |y'(x_i) - y_i'|.$$

The order of the convergence $(O_N)$ of the method (11) is estimated by the formula

$$(O_N) = \log_m(\frac{MAY_N}{MAY_{mN}}),$$

where m can be estimated by considering the ratio of $N's$.

We have used Newton-Raphson iteration method to solve the system of nonlinear equations arised from equation (8). All computations were performed on a MS Window 2007 professional operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Duo Core 2.20 Ghz PC. The solutions are computed on $N$ nodes and iteration is continued until either the maximum difference between two successive iterates is less than $10^{(-10)}$ or the number of iteration reached $10^3$.

**Problem 1.** The model linear problem given by

$$y''(x) = 9y(x) + \frac{\exp(-3) - 1.0}{3} + \int_0^1 y(t)dt, \quad 0 \leqslant x \leqslant 1$$

subject to initial conditions

$$y(0) = 1, \quad and \quad y'(0) = -1.$$

The analytical solution of the problem is $y(x) = \exp(\frac{-1}{3}x)$. The $MAY$, $MDY$ computed by method (8) and (9) for different values of $N$ and no. of iterations $Iter.$ are presented in Table 1.

**Problem 2.** The model linear problem [16] given by

$$y''(x) = y(x) + \exp 1 \int_{-1}^{0} ty(t-1)dt, \quad -1 \leqslant x \leqslant 0$$

subject to initial conditions

$$y(0) = 1, \quad and \quad y'(0) = -1.$$

The analytical solution is $y(x) = \exp(-x)$. The $MAY$, $MDY$ computed by method (8) and (9) for different values of $N$ and no. of iterations $Iter.$ are presented in Table 2.

**Problem 3.** The model linear problem [7] given by

$$y''(x) = y(x) - x + \int_{0}^{1} xty(t)dt, \qquad 0 \leqslant x \leqslant 1$$

subject to initial conditions

$$y(0) = 1, \quad and \quad y'(0) = 1.$$

The analytical solution is $y(x) = \exp(x)$. The $MAY$, $MDY$ computed by method (8)and (9) for different values of $N$ and no. of iterations $Iter.$ are presented in Table 3.

We have described a numerical method for solving second order Fredholm type integro-differential equation initial value problem and three model problems considered to illustrate the preciseness and effectiveness of the proposed method. Numerical results for example 1 which is presented in table 1, for different values of $N$ show decreases with uniform step size maximum absolute errors in our method decrease. Similar observation can be found in result of example 2 and 3. Over all method (5-6) is convergent and convergence of the method does not depends on choice of step size $h$ .

## 5. Conclusion

A uniform step size non-standard finite difference method to find the numerical solution of Fredholm integro differential equation type initial value problems has been developed. This method has been used for transforming Fredholm integro differential equation to system of algebraic equations i.e. each nodal point $x = x_i, i = 1, 2...N$, we will obtain a system of algebraic equations given by (5) and (6). Thus we have obtained a nonlinear system of equations, which is always difficult to be solved, disadvantage of the proposed method. The propose method produces good approximate numerical value of the solution and derivative of the solution for variety of model problems with uniform step size. The numerical results for the model problems showed that the proposed method is computationally efficient. The rate of convergence of the present method is quadratic. It may be an

advantage of the proposed method that we get numerical value of the derivative of the solution which otherwise approximated by the difference method. The idea presented in this article leads to the possibility to develop non-standard uniform step size difference methods to solve higher order integro differential equations. Works in these directions are in progress.

## References


[1] S. H. Chang, On certain extrapolation methods for the numerical solution of integro-differential equations. J. Math. Comp, **39** (1982) 165 - 171 .

[2] L. M. Delves and J. L. Mohamed, Computational Methods for Integral Equations. Cambridge University Press, Cambridge, (1985).

[3] E. Hairer, S. P. Norsett and G. Wanner, Solving Ordinary Differential Equations I Nonstiff Problems (Second Revised Edition). Springer-Verlag New York, Inc. New York, NY, USA, (1993).

[4] J. H. He, Variational iteration method for autonomous ordinary differential systems. Appl. Math. Comput, **114 (2/3)** (2000) 115 - 123 .

[5] S. Hu, Z. Wan and M.Khavanin, On the existence and uniqueness for nonlinear integro-differential equations. Jour. Math Phy. Sci, **21 (2)** (1987) 93 - 103 .

[6] M. K. Jain, S. R. K. Iyenger and R. K. Jain, Numerical Methods for Scientific and Engineering Computation (2/e). Willey Eastern Limited, New Delhi, (1987).

[7] H. M. Jaradat, F. Awawdeh and O. Alsayyed, Series Solutions to the High-order Integro-differential Equations. Analele Universitatii Oradea Fasc. Matematica, Tom XVI, (2009) 247 - 257 .

[8] J. D. Lambert, Numerical Methods for Ordinary Differential Systems. John Wiley, England, ( 1991).

[9] E. Liz and Nieto, J. J. Nieto, Boundary value problems for second order integro-differential equations of Fredholm type. J. Comput. Appl. Math, **72** (1996) 215 - 225.

[10] P. K. Pandey and S. S. A. Jaboob, Explicit Method in Solving Ordinary Differential Equations of the Second Order. Int. J. of Pure and Applied Mathematics, **76 (2)** (2012) 233 - 239 .

[11] P. K. Pandey, Nonlinear Explicit Method for first Order Initial Value Problems. Acta Technica Jaurinensis, **6 (2)** (2013) 118 - 125.

[12] P. K. Pandey, Numerical Solution of Linear Fredholm Integro-Differential Equations by Method of Non-standard Finite Difference Method. To be appear in Applications & Applied Mathematics: An International Journal, (2015).

[13] D. L. Phillips, A technique for the numerical solution of certain integral equations of the first kind. J. Assoc. Comput. Mach, **9** (1962) 84 - 96 .

[14] H. Ramos, A non-standard explicit integration scheme for initial value problems. Applied Mathematics and Computation, **189 (1)** (2007) 710 -718 .

[15] R. Saadati, B. Raftari, H. Abibi, S. M. Vaezpour and S. Shakeri, A Comparison Between the Variational Iteration Method and Trapezoidal Rule for Solving Linear Integro-Differential Equations. World Applied Sciences Journal, **4** (2008) 321 - 325 .

[16] A. Saadatmandia and M. Dehghanb, Numerical solution of the higher-order linear Fredholm integro-differential-difference equation with variable coefficients. Computers & Mathematics with Applications, **59 (8)** (2010) 2996 - 3004 .

[17] A. N. Tikhonov, On the solution of incorrectly posed problem and the method of regularization. Soviet Math, **4** (1963) 1035 - 1038 .

[18] F. D. Van Niekerk, Rational one step method for initial value problem. Comput. Math. Applic, **16 (12)** (1988) 1035 - 1039 .

[19] A. M. Wazwaz, A reliable modification of the Adomian decomposition method. Appl. Math. Comput, **(102)** (1999) 77 - 86 .

[20] S. Yalcinbas, Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equations. Appl. Math. Comput, **127** (2002) 195 - 206 .

[21] J. Zhao and R. M. Corless, Compact finite difference method has been used for integro-differential equations. Appl. Math. Comput , **177** (2006) 271 - 288 .

Table 1. Maximum absolute error (Problem 1).

| | Maximum absolute error | | | | | |
|---|---|---|---|---|---|---|
| | $N = 16$ | $N = 32$ | $N = 64$ | $N = 128$ | $N = 256$ | $N = 512$ |
| $MAY$ | .71585178(-4) | .12278557(-4) | .24437904(-5) | .17881393(-6) | .23841858(-6) | .11920929(-6) |
| $MDY$ | .14553467(-2) | .36346415(-3) | .90907015(-4) | .22138158(-4) | .51955381(-5) | .14404455(-5) |
| *Iter.* | 6 | 4 | 4 | 3 | 2 | 2 |

Table 2. Maximum absolute error (Problem 2).

| | Maximum absolute error | | | | | |
|---|---|---|---|---|---|---|
| | $N = 16$ | $N = 32$ | $N = 64$ | $N = 128$ | $N = 256$ | $N = 512$ |
| $MAY$ | .12316704(-2) | .25701523(-3) | .56982040(-4) | .12636185(-4) | .28610229(-5) | .14305115(-5) |
| $MDY$ | .15672922(-1) | .39410591(-2) | .98633766(-3) | .24580956(-3) | .61511993(-4) | .16450882(-4) |
| *Iter.* | 6 | 8 | 10 | 5 | 4 | 2 |

Table 3. Maximum absolute error (Problem 3).

| | Maximum absolute error | | | | | |
|---|---|---|---|---|---|---|
| | $N = 16$ | $N = 32$ | $N = 64$ | $N = 128$ | $N = 256$ | $N = 512$ |
| $MAY$ | .14159679(-2) | .34070015(-3) | .83923340(-4) | .21219254(-4) | .45299530(-5) | .21457672(-5) |
| $MDY$ | .25238991(-1) | .62396526(-2) | .15521049(-2) | .38790703(-3) | .95367432(-4) | .25033951(-4) |
| *Iter.* | 5 | 4 | 3 | 3 | 2 | 2 |