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A Note on "A Sixth Order Method for Solving Nonlinear Equations"

P. Assari[a,*] and T. Lotfi[a]

[a]*Department of Mathematics, Isalmic Azad University, Hamedan Branch, PO. Code: 15743-65181, Iran.*

Abstract. In this study, we modify an iterative non-optimal without memory method in [4] [Mirzaee. F and Hamzeh. A, A sixth order method for solving nonlinear equations, International Journal of Mathematical Modelling & Computations. **4** (2014) 55- 60.], in such a way that is optimal. Therefore, we obtain convergence order eight with the same functional evaluations. To justify our proposed method, some numerical examples are given.

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1. Introduction

Based on Kung and Traub's conjecture [2] every multi-point iterative without memory method may obtain optimal convergence order 2^n using at most $n+1$ functional evaluations. For recent works see [1, 3]. Recently, Mirzaee and Hamzeh [4] introduced a three steps without memory method, in which they have asserted it has convergence order six.
Hence, this method is not optimal, since it uses four functional evaluations and has convergence order six. They should have obtained convergence order eight. Motivated by this fact, we modify the third step of their work in such away that our modified method gets the convergence order eight with the same number of functional evaluations. To show efficiency and accuracy of the modified method, we will test some numerical examples.

*Corresponding author. Email: lotfi@iauh.ac.ir

2. New Eighth order Method

Consider the following iterative method [4]

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \dfrac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)f(x_n)(2f(y_n) - f(x_n))}{f'(x_n)[4f(y_n)f(x_n) - 2f(y_n)^2 - f(x_n)^2]}, \end{cases} \tag{1}$$

with the following error equation

$$e_{n+1} = c_2c_3(-c_2^2 + c_3)e_n^6 + O(e_n^7). \tag{2}$$

Since this method uses four function evaluations per iteration and has of convergence order six, hence it is not optimal. To optimize this method, it suffices to modify it, so that we have convergence order eight. To this end, we use weight functions to approximate $f'(z_n)$ in the denominator of the third step:

$$f'(z_n) \simeq \frac{f'(x_n)}{h(v_n) \times g(s_n, t_n)}. \tag{3}$$

Therefore, we have a new eighth order method as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \dfrac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}, \\ x_{n+1} = z_n - h(v_n)g(s_n, t_n)\dfrac{f(z_n)}{f'(x_n)}. \end{cases} \tag{4}$$

In the following theorem, we provide some necessary conditions that method (4) is optimal. Let $f : D \to \mathbb{R}$ be sufficiently differentiable function with a simple root $\alpha \in D$, $D \subset \mathbb{R}$ be an open set, x_0 be close enough to α, then the method (4) is at least of eighth-order, and satisfies the error equation

$$e_{n+1} = c_2\left(c_2^2 - c_3\right)(32c_2^4 - 6c_2^2c_3 + c_3^2 + c_2c_4)e_n^8 + O(e_n^9), \tag{5}$$

where $v_n = \frac{f(y_n)}{f(x_n)}$, $s_n = \frac{f(z_n)}{f(y_n)}$, $t_n = \frac{f(z_n)}{f(x_n)}$, and $h(v_n), g(s_n, t_n)$ are two real valued weight functions that satisfy the conditions

$$h(0) = 1, h'(0) = 2, h''(0) = 10, h'''(0) = 72, \tag{6}$$

$$g(0,0) = 1, g_s(0,0) = \frac{\partial g(s,t)}{\partial s} = 1, g_t(0,0) = \frac{\partial g(s,t)}{\partial t} = 2, \tag{7}$$

and $e_n = x_n - \alpha$ and $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}, j = 1, 2, \cdots$.

Proof We use the Mathematica to establish the desired conditions.

$\mathrm{f[e\_]} = \mathrm{f1a} * \left(\mathrm{e} + \sum_{\mathrm{k}=2}^{5} \mathrm{c_k} * \mathrm{e^k}\right);$

$\mathrm{ey} = \mathrm{e} - \mathrm{Series[f[e]/f'[e], \{e, 0, 8\}]};\quad (*\mathrm{ey} = \mathrm{y} - \alpha*)$

$\mathrm{ez = e - \frac{f[ey]-f[e]}{2f[ey]-f[e]} * \frac{f[e]}{f'[e]};}$ $\quad$ $(*\mathrm{ez = z} - \alpha *)$
$\mathrm{v := \frac{f[ey]}{f[e]}; s := \frac{f[ez]}{f[ey]}; t := \frac{f[ez]}{f[e]};}$
$\mathrm{h[v] := h0 + h1v + \frac{1}{2}h2v^2 + \frac{1}{6}h3v^3;}$ $\quad$ $\mathrm{(*h0 = h(0), h1 = h'(0), h2 = h''(0), h3 = h'''(0)*)}$
$\mathrm{g[s,t] := g0 + g1s + g2t;}$ $\quad$ $\mathrm{(*g0 = g(0,0), g1 = g_s(0,0), g2 = g_t(0,0)*)}$
$\mathrm{\hat{e} = ez - (h[v] * g[s,t])\frac{f[ez]}{f'[e]};}$
$\mathrm{Coefficient[\hat{e}, e^4]//FullSimplify}$
$\mathrm{Out[a] : -(-1 + g0h0)c_2(c_2^2 - c_3).}$
$\mathrm{g0 = 1; h0 = 1;}$
$\mathrm{Coefficient[\hat{e}, e^5]//FullSimplify}$
$\mathrm{Out[b] : -(-2 + h1)c_2^2(c_2^2).}$
$\mathrm{h1 = 2;}$
$\mathrm{Coefficient[\hat{e}, e^6]//FullSimplify}$
$\mathrm{Out[c] : -c_2(c_2^2 - c_3)\big((-6 + g1 + h2)c_2^2 - (-1 + g1)c_3\big).}$
$\mathrm{g1 = 1; h2 = 10;}$
$\mathrm{Coefficient[\hat{e}, e^7]//FullSimplify}$
$\mathrm{Out[d] : -c_2^2(c_2^2 - c_3)\big((-14 + g2 + h3)c_2^2 - (-2 + g2)c_3\big).}$
$\mathrm{g2 = 2; h3 = 72;}$
$\mathrm{Coefficient[\hat{e}, e^8]//FullSimplify}$
$\mathrm{Out[e] : c_2\left(c_2^2 - c_3\right)(32c_2^4 - 6c_2^2c_3 + c_3^2 + c_2c_4).}$

Therefore, we have

$$e_{n+1} = c_2\left(c_2^2 - c_3\right)(32c_2^4 - 6c_2^2c_3 + c_3^2 + c_2c_4)e_n^8 + O(e_n^9).$$

■

To construct new methods with suitable weight functions, we consider the following weight functions

$$h_1(v) = 1 + 2v + 5v^2 + 12v^3,$$
$$h_2(v) = \frac{1}{1 - 2v - v^2},$$
$$g_1(s,t) = 1 + s + 2t,$$
$$g_2(s,t) = \frac{1+s}{1-2t}.$$

3. Numerical Results

Now we show the convergence behavior of the developed method in action. For this purpose, some test problems are chosen along with their initial approximations and the exact zeros in Tables 1-4. The errors $|x_n - \alpha|$ denote approximations to the sought zeros, and $a(-b)$ stands for $a \times 10^{-b}$. Moreover, COC indicates the computational order of convergence [5] and is computed by

$$COC = \frac{\log|f(x_n)/f(x_{n-1})|}{\log|f(x_{n-1})/f(x_{n-2})|}. \tag{8}$$

In what follows, we have reported the numerical results for ten mentioned test problems. As it can be seen, our new method (4) supports their theoretical aspects. As it can be observed in Tables 1-4, our modified method (4) works in action.

$$
\begin{aligned}
&f_1(x) = e^{x^3-x} - \cos(x^2-1) + x^3 + 1, && x_0 = -1.65, && \alpha = -1,\\
&f_2(x) = (x-2)(x^6 + x^3 + 1)e^{-x^2}, && x_0 = 1.8, && \alpha = 2,\\
&f_3(x) = e^x \sin x + \log(x^4 - 3x + 1), && x_0 = 0.3, && \alpha = 0,\\
&f_4(x) = e^{x^2-3x} \sin x + \log(x^2 + 1), && x_0 = 0.35, && \alpha = 0,\\
&f_5(x) = \tfrac{1}{2}(e^{x-2} - 1), && x_0 = 2.5, && \alpha = 2,\\
&f_6(x) = (x-2)(x^{10} + x + 2)e^{-5x}, && x_0 = 2.2, && \alpha = 2.
\end{aligned}
$$

Table 1. Numerical results with g_1, h_1

	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	COC
$f_1(x)$	3.8427(−5)	5.2146(−36)	3.9925(−283)	8.0000
$f_2(x)$	1.3575(−5)	2.3039(−37)	1.5865(−291)	8.0000
$f_3(x)$	2.8856(−1)	5.0778(−3)	2.5111(−16)	8.2405
$f_4(x)$	3.8654(−4)	5.7019(−25)	1.2735(−191)	8.0002
$f_5(x)$	6.2271(−5)	1.6941(−35)	5.0842(−280)	8.0000
$f_6(x)$	5.4001(−6)	2.3101(−44)	2.5931(−351)	8.0000

Table 2. Numerical results with g_2, h_1

	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	COC
$f_1(x)$	3.7609(−5)	4.3899(−36)	1.5119(−283)	8.0000
$f_2(x)$	1.2481(−5)	1.1766(−37)	7.3400(−294)	8.0000
$f_3(x)$	2.6742(−1)	3.6032(−3)	1.6158(−17)	8.2652
$f_4(x)$	4.7740(−4)	3.0895(−24)	9.4607(−186)	8.0003
$f_5(x)$	6.1902(−5)	1.6154(−35)	3.4743(−280)	8.0000
$f_6(x)$	5.7265(−6)	3.6941(−44)	1.1861(−349)	8.0000

Table 3. Numerical results with g_1, h_2

	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	COC
$f_1(x)$	3.8024(−5)	3.37206(−36)	1.2894(−284)	8.0000
$f_2(x)$	1.0910(−5)	6.61109(−39)	1.2017(−304)	8.0000
$f_3(x)$	2.9784(−1)	2.4786(−4)	5.0336(−28)	8.0788
$f_4(x)$	3.8229(−4)	4.4570(−26)	1.5179(−201)	8.0002
$f_5(x)$	9.9658(−7)	5.6304(−52)	5.8452(−414)	8.0000
$f_6(x)$	2.0020(−6)	8.2463(−48)	6.8356(−379)	8.0000

4. Conclusion

We have obtained an improvement of a sixth-order iterative method. Theorem (2) shows that the order of convergence of the present method is eight. Per iteration the present method requires four evaluations of the function and there is no need for its derivatives and therefore has the efficiency index equal to 1.682. Numerical tests demonstrate that the present method is preferable to the classical Ostrowskis method in high precision computations.

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Table 4. Numerical results with g_2, h_2

	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	COC
$f_1(x)$	$3.7205(-5)$	$2.8334(-36)$	$3.2048(-285)$	8.0000
$f_2(x)$	$9.8143(-6)$	$2.8346(-39)$	$1.3728(-307)$	8.0000
$f_3(x)$	$3.3039(-1)$	$3.5427(-4)$	$8.7725(-27)$	8.0347
$f_4(x)$	$4.7323(-4)$	$2.4580(-25)$	$1.2988(-195)$	8.0002
$f_5(x)$	$6.1656(-7)$	$1.2086(-53)$	$2.6345(-427)$	8.0000
$f_6(x)$	$2.3295(-6)$	$2.7716(-47)$	$1.1131(-374)$	8.0000

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