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On Quasilinear Elliptic Systems Involving Multiple Critical Exponents

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Abstract. In this paper, we consider the existence of a non-trivial weak solution to a quasilinear elliptic system involving critical Hardy exponents. The main issue of the paper is to understand the behavior of these Palais-Smale sequences. Indeed, the principal difficulty here is that there is an asymptotic competition between the energy functional carried by the critical nonlinearities. Then by the variational method, we obtain the existence non-trivial week solution for the system.

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Index to information contained in this paper

1. Introduction

In this paper, we consider the existence of a non-trivial week solution to the following quasilinear elliptic system with multiple critical exponents

$$\begin{cases} -\mathrm{div}\left(\frac{|Du|^{p-2}Du}{|x|^{ap}}\right) - \mu\frac{|u|^{p-2}u}{|x|^{(a+1)p}} = \frac{\alpha}{|x|^{bp^*}}|u|^{\alpha-2}|v|^{\beta}u + \frac{|u|^{p^*(c)-2}}{|x|^{cp^*}}, & \text{in } R^N, \\ -\mathrm{div}\left(\frac{|Dv|^{p-2}Dv}{|x|^{ap}}\right) - \mu\frac{|v|^{p-2}v}{|x|^{(a+1)p}} = \frac{\beta}{|x|^{bp^*}}|u|^{\alpha}|v|^{\beta-2}v + \frac{|v|^{p^*(c)-2}}{|x|^{cp^*}}, & \text{in } R^N, \end{cases} \tag{1}$$

where $1 < p < N$, $0 < a < \frac{N-p}{p}$, $a \leqslant b$, $c < a+1$, $p^*(b) = \frac{Np}{N-(a+1-b)p}$, $p^*(c) = \frac{Np}{N-(a+1-c)p}$,

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($p^*(b)$, $p^*(c)$ are critical Hardy exponents). $0 \leqslant \mu \leqslant \overline{\mu} := (\frac{N-(a+1-c)p}{p})^p$, $\alpha > 1, \beta > 1$, $\alpha + \beta = p^*(b)$, and $\overline{\mu}$ is the best constant of the Hardy inequality:

$$\int_{R^N} \frac{|u|^p}{|x|^{(a+1)p}} dx \leqslant \frac{1}{\overline{\mu}} \int_{R^N} \frac{|Du|^p}{|x|^{ap}} dx, \tag{2}$$

for any $u \in C_0^\infty(R^N)$.

Let $D_a^{1,p}(R^N)$ be the completion of $C_0^\infty(R^N)$ under the norm

$$\|u\|_{D_a^{1,p}(R^N)} = |\|Du\||_{L^p(R^N,|x|^{-ap})},$$

where for all $\alpha \geqslant 0$, $q \geqslant 1$, we define

$$\|u\|_{L^q(R^N,|x|^{-\alpha})} := \Big(\int_{R^N} \frac{|u|^q}{|x|^\alpha} dx\Big)^{\frac{1}{q}}.$$

For $0 \leqslant \mu \leqslant \overline{\mu}$, it follow from the Hardy inequality (2) that

$$\|u\|_\mu := \Big(\int_{R^N} \frac{|Du|^p}{|x|^{ap}} dx - \mu \int_{R^N} \frac{|u|^p}{|x|^{(a+1)p}} dx\Big)^{\frac{1}{p}},$$

is well-define on $D_a^{1,p}(R^N)$ and that $\|.\|_\mu$ is comparable to the norm $\|.\|_{D_a^{1,p}(R^N)}$.

From the Caffarrelli-Khon-Nirenberg inequality [4], there is a positive constant $C_{a,s} > 0$ such that,

$$\Big(\int_{R^N} |x|^{-p^*(s)s} |u|^{p^*(s)} dx\Big)^{\frac{p}{p^*(s)}} \leqslant C_{a,s} \int_{R^N} |x|^{-ap} |Du|^p dx, \tag{3}$$

for all $u \in D_a^{1,p}(R^N)$, where $a \leqslant s \leqslant a+1$, $p^*(s) = \frac{Np}{N-(a+1-s)p}$. Inequality (3) implies that the imbedding $D_a^{1,p}(R^N) \hookrightarrow L^{p^*(s)}(R^N, |x|^{-p^*(s)s})$ is continuous. Now, we define the product space

$$D := D_a^{1,p}(R^N) \times D_a^{1,p}(R^N),$$

with the norm

$$\|(u,v)\|^p = \|u\|_\mu^p + \|v\|_\mu^p.$$

From (2) and (3), it follows that the energy functional of the problem (1),

$$
\begin{aligned}
J(u,v) &= \frac{1}{p}\int_{R^N}\Big(\frac{|Du|^p}{|x|^{ap}}+\frac{|Dv|^p}{|x|^{ap}}-\mu\frac{|u|^p+|v|^p}{|x|^{(a+1)p}}\Big)dx \\
&\quad -\frac{1}{p^*(b)}\int_{R^N}\frac{|u|^\alpha|v|^\beta}{|x|^{bp^*(b)}}dx-\frac{1}{p^*(c)}\int_{R^N}\frac{|u|^{p^*(c)}}{|x|^{cp^*(c)}}dx-\frac{1}{p^*(c)}\int_{R^N}\frac{|v|^{p^*(c)}}{|x|^{cp^*(c)}}dx \\
&= \frac{1}{p}\|(u,v)\|^p-\frac{1}{p^*(b)}\int_{R^N}\frac{|u|^\alpha|v|^\beta}{|x|^{bp^*(b)}}dx \\
&\quad -\frac{1}{p^*(c)}\int_{R^N}\frac{|u|^{p^*(c)}}{|x|^{cp^*(c)}}dx-\frac{1}{p^*(c)}\int_{R^N}\frac{|v|^{p^*(c)}}{|x|^{cp^*(c)}}dx,
\end{aligned}
$$

is well-defined in D, and $J \in C^1(D)$. Furthermore, the critical points of J are week solutions to Eq. (1). Note that both $\|.\|_{D_a^{1,p}(R^N)}$ and $\|.\|_{L^{p^*(s)}(R^N,|x|^{-p^*(s)s})}$ are invariant under the rotation transformation and the conformal one parameter transformation group

$$\Upsilon_r : D_a^{1,p}(R^N) \to D_a^{1,p}(R^N); \quad u(x) \mapsto \Upsilon_r[u](x) = r^{\frac{N-(a+1)p}{p}}u(rx),$$

the imbedding $D_a^{1,p}(R^N) \hookrightarrow L^{p^*(s)}(R^N, |x|^{-p^*(s)s})$ is not compact. The loss of compactness here will make the direct method of variation invalid. In this sense, the exponents $p^*(b)$ and $p^*(c)$ in the right hand side of (1) are critical. The solvability of quasilinear (or semilinear) elliptic system with singular weights and one critical exponent was recently studied by several authors, e.g.,[5, 6, 8, 11–13, 17, 18] and the references therein studied the Dirichlet problem in a smooth bounded domain; while [1, 7, 19] and the references therein studied the whole space case, they also get some asymptotic properties of the week solution, which are essential to study the bounded domain case.

The large references cite above dealt with one critical exponent and some other subcritical exponents. The key step is to estimate the concentration of the solution to the "limit" equation in the whole space, then to bound the minimax energy level of Mountain Pass type. The main issue of the paper is to understand the behavior of these Palais Smale sequences. Indeed, the principal difficulty here is that there is an asymptotic competition between the energy functional carried by the critical nonlinearities.

In this paper, following the ideas in [9], we will study the problem (1). The rest of this paper is organized as follows. In section 2, we get the existence of local Palais-Smale sequences by verifying the geometric conditions of the Mountain Pass Lemma due to Ambrosetti and Rabinowitz ([3] ,see also [15, 16]). In section 3, we study the concentration properties of the Palais-Smale sequence of a zero weak limit. In section 4, we first deduce by contradiction to eliminate the possibility of a zero weak limit case. Then, applying the ideas modified from [10, 14] and a monotonic inequality, we shall prove that the nontrivial weak limit of the Palais-Smale sequence is indeed a weak solution to (1).

2. Existence of local Palais-Smale sequence

In this section, we will prove that the energy functional J satisfies the geometric conditions of the Mountain Pass Lemma and that the minimax energy level of the Mountain Pass type is bounded.

In this paper, we suppose that $B_\rho = \Big\{(u,v) \in D : \|(u,v)\| \leqslant \rho\Big\}$.

Lemma 2.1 *For $\mu \in [0,\overline{\mu})$, J satisfies*
(i) $J(0,0) = 0$ and there exist $\rho, \tau > 0$, such that $J|_{\partial B_\rho} \geqslant \tau > 0$;
(ii) For any $(u,v) \in D\backslash\{(0,0)\}$, there exist $t_1 > 0$, such that $J(t_1u, t_1v) \leqslant 0$, and $\|(t_1u,t_1v)\| > \rho$.

Proof It is obvious that $J(0,0) = 0$. From the young inequality, (2) and (3), we have

$$\begin{aligned}
\int_{R^N} \frac{|u|^\alpha |v|^\beta}{|x|^{bp^*(b)}}dx &\leqslant \frac{\alpha}{\alpha+\beta}\int_{R^N}\frac{|u|^{\alpha+\beta}}{|x|^{bp^*(b)}}dx + \frac{\beta}{\alpha+\beta}\int_{R^N}\frac{|v|^{\alpha+\beta}}{|x|^{bp^*(b)}}dx \\
&= \frac{\alpha}{\alpha+\beta}\int_{R^N}\frac{|u|^{p^*(b)}}{|x|^{bp^*(b)}}dx + \frac{\beta}{\alpha+\beta}\int_{R^N}\frac{|v|^{p^*(b)}}{|x|^{bp^*(b)}}dx \\
&\leqslant \frac{\alpha+\beta}{\alpha+\beta}\Big[\int_{R^N}\frac{|u|^{p^*(b)}}{|x|^{bp^*(b)}}dx + \int_{R^N}\frac{|v|^{p^*(b)}}{|x|^{bp^*(b)}}dx\Big] \\
&\leqslant \frac{C_{a,b}}{p^*(b)}\Big[\|u\|^{p^*(b)} + \|v\|^{p^*(b)}\Big] \\
&= \frac{C_{a,b}}{p^*(b)}\|(u,v)\|^{p^*(b)}. \qquad (4)
\end{aligned}$$

Also, by (3) and (4), one can get

$$\begin{aligned}
J(u,v) &\geqslant (\frac{1}{p} - \frac{\mu}{p\overline{\mu}})\|(u,v)\|^p - \frac{C_{a,b}}{p^*(b)}\|(u,v)\|^{p^*(b)} - \frac{C_{a,b}}{p^*(c)}\|(u,v)\|^{p^*(c)} \\
&= \Big(\frac{1}{p} - \frac{\mu}{p\overline{\mu}} - \frac{C_{a,b}}{p^*(b)}\|(u,v)\|^{p^*(b)-p} - \frac{C_{a,b}}{p^*(c)}\|(u,v)\|^{p^*(c)-p}\Big)\|(u,v)\|^p.
\end{aligned}$$

Then, since $\mu \in [0,\overline{\mu})$ and $p^*(b)$, $p^*(c) > b$, there exist $\rho > 0$ small enough such that

$$J(u,v)|_{\partial B_\rho} \geqslant \tau > 0.$$

(ii) For any $(u,v) \in D\backslash\{(0,0)\}$, $t > 0$, we have

$$\begin{aligned}
J(tu,tv) &= \frac{t^p}{p}\|(u,v)\|^p - \frac{t^{p^*(b)}}{p^*(b)}\int_{R^N}\frac{|u|^\alpha|v|^\beta}{|x|^{bp^*(t)}}dt \\
&\quad -\frac{t^{p^*(c)}}{p^*(c)}\int_{R^N}\frac{|u|^{p^*(c)}}{|x|^{cp^*(c)}}dx - \frac{t^{p^*(c)}}{p^*(c)}\int_{R^N}\frac{|v|^{p^*(c)}}{|x|^{cp^*(c)}}dx \\
&\to -\infty \quad as \quad t \to +\infty. \qquad (5)
\end{aligned}$$

Therefore, there exist $t_1 > 0$ large enough, such that $J(t_1u,t_1v) \leqslant 0$ and $\|(t_1u,t_1v)\| > \rho$. ■

For ρ, τ, t_1 given as in the Lemma 1, let Γ be the set of all passes which connect $(0,0)$ and (t_1u,t_1v), i.e.,

$$\Gamma := \Big\{\gamma \in C([0,1],D) : \gamma(0) = (0,0), \gamma(1) = (t_1u,t_1v)\Big\}.$$

Now, we define

$$M_{u,v} = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)).$$

From the first statement of Lemma 1, $M_{u,v}$ is bounded from below by $\tau > 0$. In order to bound $M_{u,v}$ from above, let's recall some results about the extremal functions and best constants to the following inequality involving the Hardy potential and critical Sobolev exponent (cf.[19]):

$$K_{a,s} = \inf \frac{\int_{R^N} \left(\frac{|Du|^p}{|x|^{ap}} - \mu \frac{|u|^p}{|x|^{(a+1)p}}\right) dx}{\left(\int_{R^N} \frac{|u|^{p^*(s)}}{|x|^{p^*(s)s}} dx\right)^{\frac{p}{p^*(s)}}}, \tag{6}$$

for $a \leqslant s < a+1$. Also, the authors in [19] had shown that the best constant $K_{a,s}$ in (6) is achievable, i.e., there exists $U_{a,s} \in D_a^{1,p}(R^N)\backslash\{0\}$, such that $\|U_{a,s}\|_{L^{p^*(s)}(R^N,|x|^{-p^*(s)s})} = 1$, and

$$K_{a,s} = \int_{R^N} \Big(\frac{|DU_{a,s}|^p}{|x|^{ap}} - \mu \frac{|U_{a,s}|^p}{|x|^{(a+1)p}}\Big) dx.$$

Also, we define

$$K_{\alpha,\beta,a,s} = \inf \frac{\int_{R^N} \left(\frac{|Du|^p}{|x|^{ap}} + \frac{|Dv|^p}{|x|^{ap}} - \mu \frac{|u|^p + |v|^p}{|x|^{(a+1)p}}\right) dx}{\left(\int_{R^N} \frac{|u|^\alpha |v|^\beta}{|x|^{p^*(s)s}} dx\right)^{\frac{p}{p^*(s)}}}. \tag{7}$$

Then we have (it's proof is the same as that of Theorem 5 in [2])

$$K_{\alpha,\beta,a,s} = \Big(\big(\frac{\alpha}{\beta}\big)^{\frac{\beta}{\alpha+\beta}} + \big(\frac{\beta}{\alpha}\big)^{\frac{\alpha}{\alpha+\beta}}\Big) K_{a,s}.$$

Lemma 2.2 *For $\mu \in [0, \overline{\mu})$, there exists $(u,v) \in D\backslash\{(0,0)\}$ such that*

$$0 < M_{u,v} < M_* := min\Big\{\frac{a+1-b}{N} K_{a,b}^{\frac{N}{(a+1-b)p}}, \frac{a+1-c}{N} K_{a,c}^{\frac{N}{(a+1-c)p}}\Big\}.$$

Proof Without loss of generality, we assume that

$$\frac{a+1-b}{N} K_{a,b}^{\frac{N}{(a+1-b)p}} < \frac{a+1-c}{N} K_{a,c}^{\frac{N}{(a+1-c)p}}.$$

Let's define functions f, u_1 and v_1 as

$$u_1 = \alpha^{\frac{1}{p}} U_{a,b}, \quad v_1 = \beta^{\frac{1}{p}} U_{a,b}, \quad f(t) = \frac{t^p}{p}(\alpha+\beta)\|U_{a,b}\|_\mu^p - \frac{1}{p^*(b)} \alpha^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}} t^{p^*(b)},$$

then

$$f'(t) = t^{p-1}(\alpha+\beta)\|U_{a,b}\|_\mu^p - \alpha^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}} t^{p^*(b)-1}.$$

If $f'(t)=0$ then $t=t_{max}=\frac{(\alpha+\beta)^{\frac{1}{p^*(b)-p}}\|U_{a,b}\|_\mu^{\frac{p}{p^*(b)-p}}}{(\alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{p}})^{\frac{1}{p^*(b)-p}}}$. Since $\frac{p^*(b)-p}{p}<1$, so,

$$\begin{aligned}
\sup_{t\geqslant 0} f(t) &= \Big(\frac{1}{p}-\frac{1}{p^*(b)}\Big)\Big(\frac{(\alpha+\beta)\int_{R^N}(\frac{|DU_{a,b}|}{|x|^{ap}}-\mu\frac{|U_{a,b}|}{|x|^{(a+1)p}})dx}{(\alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{p}})^{\frac{p}{p^*(b)}}}\Big)^{\frac{p^*(b)}{p^*(b)-p}}\\
&\leqslant \Big(\frac{1}{p}-\frac{1}{p^*(b)}\Big)\Big[\Big(\frac{\alpha\int_{R^N}(\frac{|DU_{a,b}|}{|x|^{ap}}-\mu\frac{|U_{a,b}|}{|x|^{(a+1)p}})dx}{(\alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{p}})^{\frac{p}{p^*(b)}}}\Big)^{\frac{p^*(b)}{p^*(b)-p}}\\
&+\Big(\frac{\beta\int_{R^N}(\frac{|DU_{a,b}|}{|x|^{ap}}-\mu\frac{|U_{a,b}|}{|x|^{(a+1)p}})dx}{(\alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{p}})^{\frac{p}{p^*(b)}}}\Big)^{\frac{p^*(b)}{p^*(b)-p}}\Big]\\
&\leqslant \Big(\frac{1}{p}-\frac{1}{p^*(b)}\Big)\Big[((\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}}+(\frac{\beta}{\alpha})^{\frac{\alpha}{\alpha+\beta}})K_{a,b}\Big]^{\frac{p^*(b)}{p^*(b)-p}}=\frac{a+1-b}{N}K_{\alpha,\beta,a,b}.
\end{aligned}$$

Therefore, from the definitions of J, f, u_1 and v_1, we have

$$M_{u,v}\leqslant \sup_{t\geqslant 0} J(tu_1,tv_1)\leqslant \sup_{t\geqslant 0} f(t)\leqslant \frac{a+1-b}{N}K_{\alpha,\beta,a,b}.$$

Assume that the equalities hold in the above inequality, let t'_1 and t'_2 be the maximum points of $J(tu_1,tv_1)$ and respectively, then

$$J(t'_1u_1,t'_2v_1)=f(t'_2).$$

From the definitions of J, f, u_1 and v_1, it follows that

$$f(t'_1)-\frac{t^{p^*(c)}}{p^*(c)}\int_{R^N}\frac{|u|^{p^*(c)}}{|x|^{p^*(c)c}}dx-\frac{t^{p^*(c)}}{p^*(c)}\int_{R^N}\frac{|v|^{p^*(c)}}{|x|^{p^*(c)c}}dx=f(t'_2).$$

Then we have that $f(t'_1)>f(t'_2)$, which contradicts to the fact that t'_2 is the maximum point of $f(t)$. At last, we have that

$$M_{U_{a,b}}\leqslant \frac{a+1-b}{N}K_{\alpha,\beta,a,b}^{\frac{N}{(a+1-b)p}}.$$

Therefore, the proof is completed. ■

PROPOSITION 2.3 *For $\mu\in[0,\overline{\mu})$, there exists a local Palais-Smale sequence $\{(u_k,v_k)\}\subset D$ at energy level M, that is,*

$$\lim_{k\to+\infty} J'(u_k,v_k)=0,\ and\ \lim_{k\to+\infty} J(u_k,v_k)=M,\ \ forsome\ \ M\in(0,M_*].$$

Proof From the Lemmas 1 and 2 the energy functional J satisfies the geometric conditions of the Mountain Pass Lemma and that the minimax energy level of the Mountain Pass lemma, and for (u,v) defined as in Lemma 2, the minimax level of Mountain Pass type $M_{u,v}$ is finite, thus the existence of local Palais-Smale sequence is a direct consequence of the Mountain Pass lemma. ■

3. Palais-Smale sequence of zero weak limit

In this section, we shall study the concentration properties of the Palais-Smale sequence of a zero weak limit. First, we prove the following lemmas.

LEMMA 3.1 *Assume that a local Palais-Smale sequence* $\{(u_k, v_k)\} \subset D$ *is of a zero weak limit, i.e.,* $u_k \rightharpoonup 0$, $v_k \rightharpoonup 0$ *in* $D_a^{1,p}(R^N)$ *as* $k \to \infty$. *Then for any compact* $\omega \subset\subset R^N \backslash \{(0,0)\}$, *there exists a subsequence of* $\{(u_k, v_k)\}$, *such that*

$$\begin{aligned}\lim_{k\to+\infty}\int_\omega \frac{|u_k|^\alpha |v_k|^\beta}{|x|^{bp^*(b)}}dx &= \lim_{k\to+\infty}\int_\omega \frac{|u_k|^{p^*(c)}}{|x|^{cp^*(c)}}dx \\ &= \lim_{k\to+\infty}\int_\omega \frac{|v_k|^{p^*(c)}}{|x|^{cp^*(c)}}dx \\ &= \lim_{k\to+\infty}\int_\omega \frac{|u_k|^p + |v_k|^p}{|x|^{(a+1)p}}dx = 0, \end{aligned} \tag{8}$$

and

$$\lim_{k\to+\infty}\int_\omega \frac{|Du_k|^p + |Dv_k|^p}{|x|^{ap}}dx = 0. \tag{9}$$

Proof We split the proof of the following steps:

Step 1. For $\omega \subset\subset R^N \backslash \{(0,0)\}$, the following compact imbedding holds

$$D_a^{1,p}(\omega) \hookrightarrow\hookrightarrow L^q(\omega), \quad for\ 1 \leqslant q < p^*.$$

On the other hand, since $C_0^\infty(\omega) \subset C_0^\infty(R^N)$, and $D_a^{1,p}(R^N) \subset D_a^{1,p}(\omega)$, the convergence

$$u_k \rightharpoonup 0 \ \ and \ \ v_k \rightharpoonup 0 \ \ in\ D_a^{1,p}(R^N) \quad as\ k \to +\infty.$$

Thus, we have

$$u_k \to 0 \ \ and \ \ v_k \to 0 \ \ in\ L^q(\omega), \ \ 1 \leqslant q < p^*.$$

Note that $|x|^{-1}$ is bounded in $\omega \subset\subset R^N \backslash \{(0,0)\}$, there exists a positive constant C, such that

$$\begin{aligned}\int_\omega \frac{|u_k|^{p^*(b)}}{|x|^{bp^*(b)}}dx &\leqslant C\int_\omega |u_k|^{p^*(b)}, \\ \int_\omega \frac{|u_k|^{p^*(c)}}{|x|^{cp^*(c)}}dx &\leqslant C\int_\omega |u_k|^{p^*(c)}, \\ \int_\omega \frac{|v_k|^{p^*(b)}}{|x|^{bp^*(b)}}dx &\leqslant C\int_\omega |v_k|^{p^*(b)}, \\ \int_\omega \frac{|v_k|^{p^*(c)}}{|x|^{cp^*(c)}}dx &\leqslant C\int_\omega |v_k|^{p^*(c)},\end{aligned}$$

and

$$\int_\omega \frac{|u_k|^\alpha |v_k|^\beta}{|x|^{bp^*(b)}}dx \leqslant C\int_\omega |u_k|^\alpha |v_k|^\beta dx.$$

Since $1 < p$ and $p^*(b), p^*(c) < p^*$, we have that (8) holds.

Step 2. Let $\eta \in C_0^\infty(R^N \backslash \{(0,0)\})$ be a cut-off function, satisfying $0 \leqslant \eta \leqslant 1$, $\eta_\omega \equiv 1$.
From $J'(u_k, v_k) \to 0$ in the dual space $(D_a^{1,p}(R^N))'$ of $D_a^{1,p}(R^N)$ as $k \to +\infty$, we have

$$\begin{aligned} o(1)\|(\eta u_k, \eta v_k)\| &= < J'(u_k, v_k), (\eta u_k, \eta v_k) > \\ &= \int_{R^N} \frac{|Du_k|^{p-2}Du_k.D(\eta u_k) + |Dv_k|^{p-2}Dv_k.D(\eta v_k)}{|x|^{ap}}dx \\ &\quad -\mu\int_{R^N} \frac{|u_k|^p + |v_k|^p}{|x|^{(a+1)p}}dx - \int_{R^N} \frac{\eta|u_k|^\alpha|v_k|^\beta}{|x|^{bp^*(b)}}dx \\ &\quad -\int_{R^N} \frac{\eta\Big(|u_k|^{p^*(c)} + |v_k|^{p^*(c)}\Big)}{|x|^{cp^*(c)}}dx \end{aligned} \tag{10}$$

$$\begin{aligned} &= \int_{supp(\eta)} \frac{\eta\Big(|Du_k|^p + |Dv_k|^p\Big)}{|x|^{ap}}dx - \mu\int_{supp(\eta)} \frac{\eta\Big(|u_k|^p + |v_k|^p\Big)}{|x|^{(a+1)p}}dx \\ &\quad -\int_{supp(\eta)} \frac{\eta|u_k|^\alpha|v_k|^\beta}{|x|^{bp^*(b)}}dx - \int_{supp(\eta)} \frac{\eta\Big(|u_k|^{p^*(c)} + |v_k|^{p^*(c)}\Big)}{|x|^{cp^*(c)}}dx \\ &\quad -\int_{R^N} \frac{\Big(u_k|Du_k|^{p-2}Du_k + v_k|Dv_k|^{p-2}Dv_k\Big).D\eta}{|x|^{ap}}dx, \end{aligned} \tag{11}$$

where $supp(\eta)$ is the support of η. The weak convergence of $\{(u_k, v_k)\}$ implies that $\{\|(u_k, v_k)\|\}$ is bounded. From the Hölder inequality, there exists a positive constant C such that

$$\begin{aligned}
&\Big|\int_{R^N}\frac{\Big(u_k|Du_k|^{p-2}Du_k+v_k|Dv_k|^{p-2}Dv_k\Big).D\eta}{|x|^{ap}}dx\Big|\\
&\qquad\leqslant\int_{R^N}\frac{\Big(u_k|Du_k|^{p-1}+v_k|Dv_k|^{p-1}\Big).|D\eta|}{|x|^{ap}}dx\\
&\qquad\leqslant C\int_{supp|D\eta|}\Big(u_k|Du_k|^{p-1}+v_k|Dv_k|^{p-1}\Big)dx\\
&\qquad\leqslant C\Big[|||Du_k|||^{p-1}_{L^p(supp|D\eta|)}\|u_k\|_{L^p(supp|D\eta|)}\\
&\qquad\qquad+|||Dv_k|||^{p-1}_{L^p(supp|D\eta|)}\|v_k\|_{L^p(supp|D\eta|)}\Big]\\
&\qquad\to 0,\quad as\ k\to+\infty.
\end{aligned}$$

Thus, we have

$$\int_{R^N}\frac{\eta\Big(|Du_k|^p+|Dv_k|^p\Big)}{|x|^{ap}}dx=<J'(u_k,v_k),(\eta u_k,\eta v_k)>+o(1)=o(1).$$

Then, is follows that

$$\int_{R^N}\frac{\Big(|Du_k|^p+|Dv_k|^p\Big)}{|x|^{ap}}dx\leqslant\lim_{k\to+\infty}\int_{R^N}\frac{\eta\Big(|Du_k|^p+|Dv_k|^p\Big)}{|x|^{ap}}dx=0,$$

which completes the proof of the lemma. ■

For any $\delta>0$, let,

$$\begin{aligned}
\alpha&:=\lim_{k\to+\infty}\sup\int_{B_\delta}\Big(\frac{|Du_k|^p}{|x|^{ap}}+\frac{|Dv_k|^p}{|x|^{ap}}-\mu\frac{|u_k|^p+|v_k|^p}{|x|^{(a+1)p}}\Big)dx,\\
\beta_1&:=\lim_{k\to+\infty}\sup\int_{B_\delta}\Big(\frac{|u_k|^\alpha|v_k|^\beta}{|x|^{bp^*(b)}}\Big)dx,\\
\gamma&:=\lim_{k\to+\infty}\sup\int_{B_\delta}\Big(\frac{(|u_k|^{p^*(c)}+|v_k|^{p^*(c)})}{|x|^{cp^*(c)}}\Big)dx.
\end{aligned}$$

Then, from Lemma 3, it follows that α,β and γ are independent of δ. Moreover, the following lemma shows us a relationship among α,β and γ.

LEMMA 3.2 *Assume that a local Palais-Smale sequence $\{(u_k,v_k)\}\subset D$ is of a zero weak limit. Then there holds*
(i) $K_{\alpha,\beta,a,b}\beta_1^{\frac{p}{p^*(b)}}\leqslant\alpha,\ \frac{K_{a,c}}{2^{\frac{p}{p^*(c)}}}\gamma^{\frac{p}{p^*(c)}}\leqslant\alpha;$
(ii) $\alpha\leqslant\beta_1+\gamma.$

Proof (i) Without loss of generality, we only prove the second inequality in (1).
Let $\eta\in C_0^\infty(R^N)$ be a cut-off function, satisfying $0\leqslant\eta\leqslant1$, $\eta_{B_\delta(0)}\equiv1$.

We have

$$\Big(\|u\|^p\Big)^{\frac{p^*(c)}{p}} + \Big(\|v\|^p\Big)^{\frac{p^*(c)}{p}} \leqslant \Big(\|u\|^p + \|v\|^p\Big)^{\frac{p^*(c)}{p}} + \Big(\|u\|^p + \|v\|^p\Big)^{\frac{p^*(c)}{p}}$$
$$= \|(u,v)\|^{p^*(c)} + \|(u,v)\|^{p^*(c)} = 2\|(u,v)\|^{p^*(c)}.$$

From inequality (7) and the definition of $K_{a,c}$ it follows that

$$K_{a,c}^{\frac{p^*(c)}{p}}\Big(\int_{R^N} \frac{|u|^{p^*(c)}}{|x|^{cp^*(c)}}dx\Big) \leqslant \Big(\int_{R^N}\Big(\frac{|Du|^p}{|x|^{cp}} - \mu\frac{|u|^p}{|x|^{(a+1)p}}\Big)dx\Big)^{\frac{p^*(c)}{p}}.$$

Then, we have

$$K_{a,c}^{\frac{p^*(c)}{p}}\Big(\int_{R^N} \frac{|u|^{p^*(c)}+|v|^{p^*(c)}}{|x|^{cp^*(c)}}dx\Big) \leqslant \Big(\int_{R^N}\Big(\frac{|Du|^p}{|x|^{cp}} - \mu\frac{|u|^p}{|x|^{(a+1)p}}\Big)dx\Big)^{\frac{p^*(c)}{p}}$$
$$+\Big(\int_{R^N}\Big(\frac{|Dv|^p}{|x|^{cp}} - \mu\frac{|v|^p}{|x|^{(a+1)p}}\Big)dx\Big)^{\frac{p^*(c)}{p}}$$
$$\leqslant 2\Big(\int_{R^N}\Big(\frac{|Du|^p}{|x|^{cp}} + \frac{|Dv|^p}{|x|^{cp}} - \mu\frac{|u|^p+|v|^p}{|x|^{(a+1)p}}\Big)dx\Big)^{\frac{p^*(c)}{p}}.$$

Then

$$\frac{K_{a,c}}{2^{\frac{p}{p^*(c)}}}\Big(\int_{R^N} \frac{|u|^{p^*(c)}+|v|^{p^*(c)}}{|x|^{cp^*(c)}}dx\Big)^{\frac{p}{p^*(c)}} \leqslant \int_{R^N}\Big(\frac{|Du|^p}{|x|^{cp}} + \frac{|Dv|^p}{|x|^{cp}} - \mu\frac{|u|^p+|v|^p}{|x|^{(a+1)p}}\Big)dx.$$

Also, we have

$$\frac{K_{a,c}}{2^{\frac{p}{p^*(c)}}}\Big(\int_{R^N} \frac{|\eta u_k|^{p^*(c)}+|\eta v_k|^{p^*(c)}}{|x|^{cp^*(c)}}dx\Big)^{\frac{p}{p^*(c)}}$$
$$\leqslant \int_{R^N}\Big(\frac{|D(\eta u_k)|^p+|D(\eta v_k)|^p}{|x|^{ap}} - \mu\frac{|\eta u_k|^p+|\eta v_k|^p}{|x|^{(a+1)p}}\Big)dx$$
$$= \int_{B_\delta}\Big(\frac{|D(\eta u_k)|^p+|D(\eta v_k)|^p}{|x|^{ap}} - \mu\frac{|\eta u_k|^p+|\eta v_k|^p}{|x|^{(a+1)p}}\Big)dx$$
$$+\int_{supp(\eta)\setminus B_\delta}\Big(\frac{|D(\eta u_k)|^p+|D(\eta v_k)|^p}{|x|^{ap}} - \mu\frac{|\eta u_k|^p+|\eta v_k|^p}{|x|^{(a+1)p}}\Big)dx.$$

Taking the supper limits at both sides and noting Lemma 3, we have

$$\frac{K_{a,c}}{2^{\frac{p}{p^*(c)}}}\gamma^{\frac{p}{p^*(c)}} \leqslant \alpha.$$

(ii) From $J'(u_k, v_k) \to 0$ in W' as $k \to +\infty$, we have

$$\begin{aligned}
o(1)\|(\eta u_k, \eta v_k)\| &= \langle J'(u_k, v_k), (\eta u_k, \eta v_k)\rangle \\
&= \int_{R^N} \frac{|Du_k|^{p-2}Du_k.D(\eta u_k) + |Dv_k|^{p-2}Dv_k.D(\eta v_k)}{|x|^{ap}}dx \\
&\quad -\mu \int_{R^N} \frac{\eta\Big(|u_k|^p + |v_k|^p\Big)}{|x|^{(a+1)p}}dx - \int_{R^N} \frac{\eta|u_k|^\alpha|v_k|^\beta}{|x|^{bp^*(b)}}dx \\
&\quad - \int_{R^N} \frac{\eta\Big(|u_k|^{p^*(c)} + |v_k|^{p^*(c)}\Big)}{|x|^{cp^*(c)}}dx \\
&\geqslant \int_{B_\delta} \frac{\Big(|Du_k|^p + |Dv_k|^p\Big)}{|x|^{ap}}dx - \mu \int_{B_\delta} \frac{\Big(|u_k|^p + |v_k|^p\Big)}{|x|^{(a+1)p}}dx \\
&\quad - \int_{B_\delta} \frac{|u_k|^\alpha|v_k|^\beta}{|x|^{bp^*(b)}}dx - \int_{B_\delta} \frac{\Big(|u_k|^{p^*(c)} + |v_k|^{p^*(c)}\Big)}{|x|^{cp^*(c)}}dx \\
&\quad - \int_{supp(\eta)\backslash B_\delta} \frac{\Big(|Du_k|^p + |Dv_k|^p\Big)}{|x|^{ap}}dx \\
&\quad +\mu \int_{supp(\eta)\backslash B_\delta} \frac{\Big(|u_k|^p + |v_k|^p\Big)}{|x|^{(a+1)p}}dx - \int_{supp(\eta)\backslash B_\delta} \frac{\eta|u_k|^\alpha|v_k|^\beta}{|x|^{bp^*(b)}} \\
&\quad - \int_{supp(\eta)\backslash B_\delta} \frac{\eta\Big(|u_k|^{p^*(c)} + |v_k|^{p^*(c)}\Big)}{|x|^{cp^*(c)}}dx \\
&\quad - \int_{supp|D\eta|\backslash B_\delta} \frac{|Du_k|^{p-1}u_k.|D\eta| + |Dv_k|^{p-1}v_k.|D\eta|}{|x|^{ap}}dx.
\end{aligned}$$

Noting that $supp(\eta)\backslash B_\delta$, $supp|D\eta|\backslash B_\delta \subset\subset R^N\backslash\{(0,0)\}$, and taking the upper limits at both sides. We have $\alpha \leqslant \beta_1 + \gamma$. ∎

Proposition 3.3 *Assume that $\{(u, v)\} \subset D$ is a local Palais-Smale sequence of J at some energy level $M \in (0, M_*)$ and $(u_k, v_k) \rightharpoonup 0$ in D as $k \to +\infty$. Then there exist a positive constant $\varepsilon_0 > 0$, such that one of the following two relation holds*
(1) $\beta_1 = \gamma = 0$;
(2) $\beta_1, \gamma \geqslant \varepsilon_0$,
for all $\delta > 0$.

Proof From the definition of energy functional J and the weak convergence $(u_k, v_k) \rightharpoonup 0$ as $k \to +\infty$ one can get

$$\begin{aligned}
M + o(1) = J(u_k, v_k) &= \frac{1}{p}\|(u_k, v_k)\|^p - \frac{1}{p^*(b)} \int_{R^N} \frac{|u_k|^\alpha|v_k|^\beta}{|x|^{bp^*(b)}}dx \\
&\quad -\frac{1}{p^*(c)}\|(u_k, v_k)\|^{p^*(c)}_{L^{p^*(c)}(R^N,|x|^{-cp^*(c)})},
\end{aligned} \tag{12}$$

and

$$o(1) = \langle J'(u_k, v_k), (u_k, v_k)\rangle$$
$$= \|(u_k, v_k)\|^p - \int_{R^N} \frac{|u_k|^\alpha |v_k|^\beta}{|x|^{bp^*(b)}} dx - \|(u_k, v_k)\|^{p^*(c)}_{L^{p^*(c)}(R^N, |x|^{-cp^*(c)})}. \tag{13}$$

Thus, we get

$$M + o(1) = J(u_k, v_k) - \frac{1}{p}\langle J'(u_k, v_k), (u_k, v_k)\rangle = \left(\frac{1}{p} - \frac{1}{p^*(b)}\right) \int_{R^N} \frac{|u_k|^\alpha |v_k|^\beta}{|x|^{bp^*(b)}} dx$$
$$- \left(\frac{1}{p} - \frac{1}{p^*(c)}\right) \|(u_k, v_k)\|^{p^*(c)}_{L^{p^*(c)}(R^N, |x|^{-cp^*(c)})}. \tag{14}$$

It follows that

$$\int_{R^N} \frac{|u_k|^\alpha |v_k|^\beta}{|x|^{bp^*(b)}} dx \leqslant \frac{NM}{a+1-b}, \quad \|(u_k, v_k)\|^{p^*(c)}_{L^{p^*(c)}(R^N, |x|^{-cp^*(c)})} \leqslant \frac{NM}{a+1-c},$$

and then

$$\beta_1 \leqslant \frac{NM}{a+1-b}, \ \ \gamma \leqslant \frac{NM}{a+1-c}. \tag{15}$$

On the other hand, from Lemma 4, one can get

$$K_{\alpha,\beta,a,b} \beta_1^{\frac{p}{p^*(b)}} \leqslant \alpha \leqslant \beta_1 + \gamma.$$

Therefore

$$K_{\alpha,\beta,a,b} \beta_1^{\frac{p}{p^*(b)}} \left(1 - K^{-1}_{\alpha,\beta,a,b} \beta_1^{\frac{p^*(b)-p}{p^*(b)}}\right) \leqslant \gamma.$$

From (15), and the domain of $M \in (0, M_*)$, on can get

$$1 - K^{-1}_{\alpha,\beta,a,b} \beta_1^{\frac{p^*(b)-p}{p^*(b)}} \geqslant 1 - K^{-1}_{\alpha,\beta,a,b} (\frac{NM}{a+1-b})^{\frac{p^*(b)-p}{p^*(b)}} > 0.$$

Then there exists $\delta_1 > 0$, such that $\beta_1^{\frac{p}{p^*(b)}} \leqslant \delta_1 \gamma$. Similarly, there exists $\delta_2 > 0$, such that $\gamma^{\frac{p}{p^*(c)}} \leqslant \delta_2 \beta_1$. Thus, there exists $\varepsilon_0 > 0$, such that either $\beta_1 = \gamma = 0$ or $\beta_1, \gamma \geqslant \varepsilon_0$. ■

4. Existence of nontrivial weak solution

In this section, by the ideas modified from [10, 14] and a monotonic inequality, we shall prove that the nontrivial weak limit of the Palais-Smale sequence of J is indeed a week solution to problem (1).

Lemma 4.1 *Assume that $\{(u_k, v_k)\} \subset D$ is the local Palais-Smale sequence as in Proposition 1 then*

$$\min\Big\{\lim\sup_{k\to+\infty}\int_{R^N}\frac{|u_k|^\alpha|v_k|^\beta}{|x|^{bp^*(b)}}dx,\ \lim\sup_{k\to+\infty}\int_{R^N}\Big(\frac{(|u_k|^{p^*(c)}+|v_k|^{p^*(c)})}{|x|^{cp^*(c)}}\Big)dx\Big\}>0.$$

Proof We deduce by contradiction. Without loss of generality, suppose that

$$\lim\sup_{k\to+\infty}\int_{R^N}\frac{|u_k|^\alpha|v_k|^\beta}{|x|^{bp^*(b)}}dx=0.$$

From (13) and (14), up to a subsequence

$$\|(u_k,v_k)\|^p=\|(u_k,v_k)\|^{p^*(c)}_{L^{p^*(c)}(R^N,|x|^{-cp^*(c)})}+o(1), \tag{16}$$

and

$$\frac{NM}{(a+1-c)p}+o(1)=\|(u_k,v_k)\|^{p^*(c)}_{L^{p^*(c)}(R^N,|x|^{-cp^*(c)})}. \tag{17}$$

Combining (16) and (7) with $s=c$, we have

$$K_{a,c}\|(u_k,v_k)\|^p_{L^{p^*(c)}(R^N,|x|^{-cp^*(c)})}\leqslant\|(u_k,v_k)\|^{p^*(c)}_{L^{p^*(c)}(R^N,|x|^{-cp^*(c)})}+o(1),$$

that is,

$$\|(u_k,v_k)\|^p_{L^{p^*(c)}(R^N,|x|^{-cp^*(c)})}\Big(K_{a,c}-\|(u_k,v_k)\|^{p^*(c)-p}_{L^{p^*(c)}(R^N,|x|^{-cp^*(c)})}\Big)\leqslant o(1). \tag{18}$$

From (15) and the domain of $M\in(0,M_*)$, we get

$$K_{a,c}-\|(u_k,v_k)\|^{p^*(c)-p}_{L^{p^*(c)}(R^N,|x|^{-cp^*(c)})}\geqslant K_{a,c}-\Big(\frac{NM}{(a+1-c)}\Big)^{\frac{p^*(c)-p}{p^*(c)}}>0.$$

Therefore, (18) contradicts (17), which implies that the conclusion of lemma is true. ■

For any sequence of positive numbers $\{r_k\}$, let $\tilde u_k(x):=r_k^{\frac{N-(a+1)p}{p}}u_k(r_kx)$ and $\tilde v_k(x):=r_k^{\frac{N-(a+1)p}{p}}v_k(r_kx)$.

If the sequence $\{(u_k,v_k)\}\subset D$ is a Palais-Smale sequence of J at energy level M, then $\{(u_k,v_k)\}$ is also Palais-Smale sequence of J at the same energy level. since $\|.\|$ and $\|.\|_{L^{p^*(c)}(R^N,|x|^{-cp^*(c)})}$
are invariant under the conformal transformation, it suffices to show that

$$J'\Big(\tilde u_k(x),\tilde v_k(x)\Big)\to0\ in\ D'\ as\quad k\to+\infty.$$

In fact, let $y=r_kx$, $U_k(y):=u_k(r_kx)$, $V_k(y):=v_k(r_kx)$, $\tilde\phi(y):=r_k^{\frac{-(N-(a+1)p)}{p}}\phi(\frac{y}{r_k})$ and $\tilde\psi(y):=r_k^{\frac{-(N-(a+1)p)}{p}}\psi(\frac{y}{r_k})$ for $(\phi,\psi)\in D$.

A direct calculation yields that $\|(\phi,\psi)\|_D = \|(\tilde{\phi},\tilde{\psi})\|_D$, and

$$\begin{aligned} D_{x_i}\tilde{u}_k(x) &= r_k^{\frac{N-(a+1)p}{p}} D_{x_i}u_k(r_k x) = r_k^{\frac{N-ap}{p}} D_{y_i}u_k(y), \\ D_{x_i}\tilde{v}_k(x) &= r_k^{\frac{N-(a+1)p}{p}} D_{x_i}v_k(r_k x) = r_k^{\frac{N-ap}{p}} D_{y_i}v_k(y), \\ D_{x_i}\phi(x) &= r_k^{\frac{N-(a+1)p}{p}} D_{x_i}\tilde{\phi}(y) = r_k^{\frac{N-ap}{p}} D_{y_i}\tilde{\phi}(y), \\ D_{x_i}\psi(x) &= r_k^{\frac{N-(a+1)p}{p}} D_{x_i}\tilde{\psi}(y) = r_k^{\frac{N-ap}{p}} D_{y_i}\tilde{\psi}(y). \end{aligned}$$

Thus

$$< J'(\tilde{u}_k,\tilde{v}_k),(\phi,\psi) >=< J'(u_k,v_k),(\tilde{\phi},\tilde{\psi}) >= \|J'(u_k,v_k)\|_{D'}\|(\tilde{\phi},\tilde{\psi})\|_D,$$

that is,

$$\|J'(\tilde{u}_k,\tilde{v}_k)\|_{D'} = \|J'(u_k,v_k)\|_{D'}.$$

Then from $J'(u_k,v_k)\to 0$ in D' as $k\to+\infty$, we have $J'(\tilde{u}_k,\tilde{v}_k)\to 0$ in D' as $k\to+\infty$.

LEMMA 4.2 *Assume that* $\{(u_k,v_k)\}\subset D$ *is the local Palais-Smale sequence as in Proposition 1 then there exists a positive* $\varepsilon_1\in(0,\frac{\varepsilon_0}{2}]$ *(*ε_0 *is given in Proposition 1), such that for any* $\varepsilon\in(0,\varepsilon_1)$ *there exists a subsequence, still denoted by* $\{(u_k,v_k)\}$*, and a sequence for positive number* $\{r_k\}$*, such that the scaled sequence* $\{(\tilde{u}_k,\tilde{v}_k)\}$ *satisfies*

$$\int_{B_1}\frac{|\tilde{u}_k|^\alpha|\tilde{v}_k|^\beta}{|x|^{bp^*(b)}}dx = \varepsilon, \quad \forall k\in\mathbb{N}. \tag{19}$$

Proof We only prove (19). Set $\rho = \limsup_{k\to+\infty}\int_{R^N}\frac{|u_k|^\alpha|v_k|^\beta}{|x|^{bp^*(b)}}dx$. From the Lemma 5, we know that $\rho>0$. Let $\varepsilon_1=\min\{\frac{\varepsilon_0}{2},\rho\}$, $\varepsilon\in(0,\varepsilon_1)$. From the absolute continuity of integration, there exists a
subsequence by $\{(u_k,v_k)\}$, and a sequence of positive number $\{r_k\}$, such that

$$\int_{B_{r_k}}\frac{|u_k|^\alpha|v_k|^\beta}{|x|^{bp^*(b)}}dx = \varepsilon, \quad \forall k\in\mathbb{N}.$$

By scaling, we have (19). ■

THEOREM 4.3 *For* $\mu\in[0,\overline{\mu})$*. there exists a nontrivial week solution to problem (1).*

Proof Suppose that $\{(u_k,v_k)\}\subset D$ is the local Palais-Smale sequence obtained in Proposition 1.

Step 1. $\{(u_k,v_k)\}$ is bounded in D.

Without loss of generality, assume that $b<c$, then $p^*(b)>p^*(c)$. From the Hölder inequality, (12) and (13), we have

$$\begin{aligned}
&M + o(1) + o(1)\|(u_k, v_k)\}\| \\
&\quad = J(u_k, v_k) - \frac{1}{p^*(c)}\langle J'(u_k, v_k), (u_k, v_k)\rangle \\
&\quad = \Big(\frac{1}{p} - \frac{1}{p^*(c)}\Big)\|\{(u_k, v_k)\}\|^p - \Big(\frac{1}{p^*(b)} - \frac{1}{p^*(c)}\Big)\int_{R^N} \frac{|u_k|^\alpha |v_k|^\beta}{|x|^{bp^*(b)}}dx \\
&\quad \geqslant \Big(\frac{1}{p} - \frac{1}{p^*(c)}\Big)\|\{(u_k, v_k)\}\| - \Big(\frac{1}{p^*(b)} - \frac{1}{p^*(c)}\Big)\frac{C_{a,b}}{p^*(b)}\|(u_k, v_k)\|^{p^*(b)}_{L^{p^*(b)}(R^N,|x|^{-bp^*(b)})}.
\end{aligned}$$

Since $p < p^*(c)$, it follows that $\{(u_k, v_k)\}$ is bounded in D.

Step 2. Various of convergence.

From the boundedness of $\{(u_k, v_k)\}$, there exists $\{(u_0, v_0)\} \subset D$, such that up to subsequence, there holds

$$u_k \rightharpoonup u_0,\ \ v_k \rightharpoonup v_0 \quad \textit{weakly in } D^{1,p}_a(R^N).$$

For any $\omega \subset\subset R^N\backslash\{0\}$, the compact imbedding $D^{1,p}_a(\omega) \hookrightarrow\hookrightarrow L^q(\omega)$, $1 \leqslant q < p^*$ implies that

$$u_k \to u_0,\ \ v_k \to v_0 \quad \textit{strongly a.e. in } R^N.$$

Furthermore, the boundedness of $\{(u_k, v_k)\}$ in D implies that

$$\Big\{|Du_k|^{p-2}Du_k\Big\},\ \Big\{|Dv_k|^{p-2}Dv_k\Big\},\ \Big\{|u_k|^{p-2}u_k\Big\},\ \Big\{|v_k|^{p-2}v_k\Big\},$$
$$\Big\{|u_k|^{\alpha-2}u_k|v_k|^\beta\Big\},\ \Big\{|u_k|^\alpha|v_k|^{\beta-2}v_k\Big\},\ \Big\{|u_k|^{p^*(c)-2}u_k\Big\} \text{ and } \Big\{|v_k|^{p^*(c)-2}v_k\Big\},$$

are bounded in $\Big(L^{p'}(R^N, |x|^{-ap})\Big)^N$, $L^{p'}\Big(R^N, |x|^{-(a+1)p}\Big)$, $L^{p^{*'}(b)}\Big(R^N, |x|^{-bp^*(b)}\Big)$ and $L^{p^{*'}(c)}\Big(R^N, |x|^{-cp^*(c)}\Big)$, respectively. Thus, for some $m_1, m_2 \in \Big(L^{p'}(R^N, |x|^{-ap})\Big)^N$ we have the following week convergence

$$\Big\{|Du_k|^{p-2}Du_k\Big\} \rightharpoonup m_1,\ \Big\{|Dv_k|^{p-2}Dv_k\Big\} \rightharpoonup m_2,\ in\ \Big(L^{p'}(R^N, |x|^{-ap})\Big)^N,$$

and

$$\Big\{|u_k|^{p-2}u_k\Big\} \rightharpoonup \Big\{|u_0|^{p-2}u_0\Big\},\ \ \Big\{|v_k|^{p-2}v_k\Big\} \rightharpoonup \Big\{|v_0|^{p-2}v_0\Big\}$$

in $L^{p'}(R^N, |x|^{-(a+1)p})$,

$$\Big\{|u_k|^{\alpha-2}u_k|v_k|^\beta\Big\} \rightharpoonup \Big\{|u_0|^{\alpha-2}u_0|v_0|^\beta\Big\},\ \ \Big\{|u_k|^\alpha v_k|v_k|^{\beta-2}\Big\} \rightharpoonup \Big\{|u_0|^\alpha v_0|v_0|^{\beta-2}\Big\}$$

in $L^{p^{*'}(b)}\Big(R^N, |x|^{-bp^*(b)}\Big)$ and

$$\left\{|u_k|^{p^*(c)-2}u_k\right\} \rightharpoonup \left\{|u_0|^{p^*(c)-2}u_0\right\}, \quad \left\{|v_k|^{p^*(c)-2}v_k\right\} \rightharpoonup \left\{|v_0|^{p^*(c)-2}v_0\right\}$$

in $L^{p^{*'}(c)}\left(R^N, |x|^{-cp^*(c)}\right)$.

Step 3. $(u_0, v_0) \neq 0$.

We deduce by contradiction, assuming that $(u_0, v_0) = (0, 0)$, then $\{(u_k, v_k)\}$ satisfies all the assumptions in Lemma 6. Thus, there exists a sequence $\{r_k\}$, such that up to a subsequence, $\{(\tilde{u}_k, \tilde{v}_k)\}$ still is a local Palais-Smale sequence of zero weak limit. Hence (19) holds. Note that (19) contradict the conclusions in Proposition 2. Thus $(u_0, v_0) \neq 0$.

Step 4. (u_0, v_0) is a week solution to (1) .

From the convergence $J'(u_k, v_k) \rightharpoonup 0$ in D^{-1} as $k \to +\infty$, one can get

$$\begin{aligned} o(1) &= < J'(u_k, v_k), (\phi, \psi) > \\ &= \int_{R^N} \frac{|Du_k|^{p-2}Du_k.D\phi + |Dv_k|^{p-2}Dv_k.D\psi}{|x|^{ap}}dx - \mu \int_{R^N} \frac{|u_k|^{p-2}u_k\phi + |v_k|^{p-2}v_k\psi}{|x|^{(a+1)p}}dx \\ &\quad - \int_{R^N} \frac{\alpha|u_k|^{\alpha-2}u_k|v_k|^{\beta}\phi + \beta|u_k|^{\alpha}|v_k|^{\beta-2}v_k\psi}{|x|^{bp^*(b)}}dx \\ &\quad - \int_{R^N} \Big(\frac{|u_k|^{p^*(c)-2}u_k\phi + |v_k|^{p^*(c)-2}v_k\psi}{|x|^{cp^*(c)}}\Big)dx. \end{aligned} \tag{20}$$

For any $(\phi, \psi) \in D$. From the medium convergence obtained in Step 2, we can easily pass the limit in the last three terms in (20). Then, it suffices to show that

$$\int_{R^N} \frac{|Du_k|^{p-2}Du_k.D\phi}{|x|^{ap}}dx \to \int_{R^N} \frac{|Du_0|^{p-2}Du_0.D\phi}{|x|^{ap}}dx, \quad as\ k \to +\infty, \tag{21}$$

$$\int_{R^N} \frac{|Dv_k|^{p-2}Dv_k.D\psi}{|x|^{ap}}dx \to \int_{R^N} \frac{|Dv_0|^{p-2}Dv_0.D\psi}{|x|^{ap}}dx, \quad as\ k \to +\infty. \tag{22}$$

To prove, it suffices to show that $Du_k \to Du_0$ and $Dv_k \to Dv_0$ a.e. in R^N. We shall modify the ideas in [10, 14] to be appropriate to our case. First, note that for any $X \neq Y \in R^N$, there holds

$$\Big(|X|^{p-2}X - |Y|^{p-2}Y, X - Y\Big) > 0.$$

We have

$$\Big(|Du_k|^{p-2}Du_k - |Du_0|^{p-2}Du_0\Big).D(u_k - u_0) \geqslant 0, \tag{23}$$

$$\Big(|Dv_k|^{p-2}Dv_k - |Dv_0|^{p-2}Dv_0\Big).D(v_k - v_0) \geqslant 0, \tag{24}$$

and the equality holds if and only if $Du_k = Du_0$ and $Dv_k = Dv_0$. For a given

$\varepsilon > 0$, $\sigma \in R$, we define a truncation function ϕ_ε and ψ_ε by

$$\phi_\varepsilon(\sigma) = \psi_\varepsilon(\sigma) = \begin{cases} \sigma, & \text{if } |\sigma| < \varepsilon, \\ \frac{\varepsilon\sigma}{|\sigma|}, & \text{if } |\sigma| \geqslant \varepsilon, \end{cases}$$

and $\sigma^N := \phi_N(\sigma) = \psi_N(\sigma)$ for $N \geqslant 1$. For an increasing sequence $\{\mathcal{R}_\mathrm{j}\}$ with $\mathcal{R}_\mathrm{j} \to +\infty$ as $\mathrm{j} \to +\infty$, $B_{\mathcal{R}_\mathrm{j}}$ is an open ball with radius $\mathcal{R}_\mathrm{j}$, and $\{B_{\mathcal{R}_\mathrm{j}}\}$ is an exhaustion of R^N. Let $\eta \in C_0^\infty(R^N)$ be a cut-off function, satisfying $0 \leqslant \eta \leqslant 1$, $\eta|_{B_{\mathcal{R}_\mathrm{j}}} \equiv 1$, $supp(\eta) \subset B_{\mathcal{R}_{\mathrm{j}+1}}$. We shall prove that for any n large enough, there holds

$$\limsup_{k\to+\infty} \Big[\int_{R^N} \frac{\Big(|Du_k|^{p-2}Du_k - |Du_0|^{p-2}Du_0\Big).D\Big(\eta\phi_\varepsilon o(u_k-u_0)^n\Big)}{|x|^{ap}}dx + \int_{R^N} \frac{\Big(|Dv_k|^{p-2}Dv_k - |Dv_0|^{p-2}Dv_0\Big).D\Big(\eta\psi_\varepsilon o(v_k-v_0)^n\Big)}{|x|^{ap}}dx\Big] \leqslant o(1), \quad (25)$$

as $\varepsilon \to 0$. In fact, since $u_k \to u_0$ and $v_k \to v_0$ a.e. $B_{\mathcal{R}_{\mathrm{j}+1}}$, for any $\delta > 0$, it follows from Egoroff's Theorem that there exists $E_\varepsilon \subset B_{\mathcal{R}_{\mathrm{j}+1}}$ such that $|B_{\mathcal{R}_{\mathrm{j}+1}} \backslash E_\delta|$, and (u_k, v_k) converges uniformly to (u_0, v_0) in E_δ. Then for a sufficiently large k, we have $|u_k(x) - u_0(x)| \leqslant \varepsilon$ and $|v_k(x) - v_0(x)| \leqslant \varepsilon$ for all $x \in E_\delta$, and thus $\phi_\varepsilon o((u_k(x) - u_0(x))^n) = (u_k(x) - u_0(x))^n$ and $\psi_\varepsilon o((v_k(x) - v_0(x))^n) = (v_k(x) - v_0(x))^n$ for all $x \in E_\delta$. We now define

$$A := \int_{R^N} \frac{(|Du_k|^{p-2}Du_k - |Du_0|^{p-2}Du_0).D(\eta\phi_\varepsilon o(u_k-u_0)^n)}{|x|^{ap}}dx,$$
$$B := \int_{R^N} \frac{(|Dv_k|^{p-2}Dv_k - |Dv_0|^{p-2}Dv_0).D(\eta\psi_\varepsilon o(v_k-v_0)^n)}{|x|^{ap}}dx,$$

and since $\phi_\varepsilon o((u_k-u_0)^n)$ and $\psi_\varepsilon o((v_k-v_0)^n)$ is bounded in $D_a^{1,p}(R^N)$, we can get

$$\begin{aligned} & A + B \\ = {} & \mu\int_{R^N} \frac{|u_k|^{p-2}u_k\eta\phi_\varepsilon o((u_k(x)-u_0(x))^n)}{|x|^{(a+1)p}}dx + \mu\int_{R^N} \frac{|v_k|^{p-2}v_k\eta\psi_\varepsilon o((v_k(x)-v_0(x))^n)}{|x|^{(a+1)p}}dx \\ & +\alpha\int_{R^N} \frac{|u_k|^{\alpha-2}u_k|v_k|^\beta\eta\phi_\varepsilon o((u_k(x)-u_0(x))^n)}{|x|^{bp^*(b)}}dx \\ & +\beta\int_{R^N} \frac{|u_k|^\alpha v_k|v_k|^{\beta-2}\eta\psi_\varepsilon o((v_k(x)-v_0(x))^n)}{|x|^{bp^*(b)}}dx \\ & + \int_{R^N} (\frac{|u_k|^{p^*(c)-2}|u_k|\eta\phi_\varepsilon o((u_k(x)-u_0(x))^n) + |v_k|^{p^*(c)-2}|v_k|\eta\psi_\varepsilon o((v_k(x)-v_0(x))^n)}{|x|^{cp^*(c)}})dx + o(1) \\ := {} & L_1 + L_2 + L_3 + o(1), \end{aligned}$$

where

$$L_1 := \mu \int_{R^N} \frac{|u_k|^{p-2}u_k\eta\phi_\varepsilon o((u_k(x)-u_0(x))^n)}{|x|^{(a+1)p}}dx + \mu \int_{R^N} \frac{|v_k|^{p-2}v_k\eta\psi_\varepsilon o((v_k(x)-v_0(x))^n)}{|x|^{(a+1)p}}dx,$$

$$L_2 := \alpha \int_{R^N} \frac{|u_k|^{\alpha-2}u_k|v_k|^{\beta}\eta\phi_\varepsilon o((u_k(x)-u_0(x))^n)}{|x|^{bp^*(b)}}dx + \beta \int_{R^N} \frac{|u_k|^{\alpha}v_k|v_k|^{\beta-2}\eta\psi_\varepsilon o((v_k(x)-v_0(x))^n)}{|x|^{bp^*(b)}}dx,$$

$$L_3 := \int_{R^N} \Big(\frac{|u_k|^{p^*(c)-2}|u_k|\eta\phi_\varepsilon o((u_k(x)-u_0(x))^n) + |v_k|^{p^*(c)-2}|v_k|\eta\psi_\varepsilon o((v_k(x)-v_0(x))^n)}{|x|^{cp^*(c)}}\Big)dx.$$

Here, one can get

$$\begin{aligned}|L_1| &\leqslant \mu \int_{R^N} \Big(\frac{|u_k|^{p-1}\eta\Big|\phi_\varepsilon o((u_k(x)-u_0(x))^n)\Big| + |v_k|^{p-1}\eta\Big|\psi_\varepsilon o((v_k(x)-v_0(x))^n)\Big|}{|x|^{(a+1)p}}\Big)dx \\ &\leqslant \mu \int_{E_\delta} \Big(\frac{|u_k|^{p-1}\eta|u_k(x)-u_0(x)| + |v_k|^{p-1}\eta|v_k(x)-v_0(x)|}{|x|^{(a+1)p}}\Big)dx \\ &\quad + \mu\varepsilon \int_{B_{\mathcal{R}_{j+1}}\setminus E_\delta} \Big(\frac{|u_k|^{p-1}+|v_k|^{p-1}}{|x|^{(a+1)p}}\Big)dx,\end{aligned}$$

on E_δ, (u_k, v_k) converges uniformly to (u_0, v_0); while in $B_{\mathcal{R}_{j+1}}\setminus E_\delta$, from the absolute continuity of integration, we have that for a large enough k and a small enough ε that L_1 can be as small as desired. Similarly, L_2 and L_3 also can be as small as desired. That is (25) is true.

On the other hand, we have

$$\begin{aligned}&\int_{E_\delta} \frac{\Big(|Du_k|^{p-2}Du_k - |Du_0|^{p-2}Du_0\Big).D(u_k-u_0)^n}{|x|^{ap}}dx \\ &\qquad + \int_{E_\delta} \frac{\Big(|Dv_k|^{p-2}Dv_k - |Dv_0|^{p-2}Dv_0\Big).D(v_k-v_0)^n}{|x|^{ap}}dx \\ &\leqslant \int_{R^N} \frac{\Big(|Du_k|^{p-2}Du_k - |Du_0|^{p-2}Du_0\Big).D(\eta\phi_\varepsilon o((u_k-u_0)^n)}{|x|^{ap}}dx \\ &\qquad + \int_{R^N} \frac{\Big(|Dv_k|^{p-2}Dv_k - |Dv_0|^{p-2}Dv_0\Big).D(\eta\psi_\varepsilon o((v_k-v_0)^n)}{|x|^{ap}}dx.\end{aligned}$$

It follow that

$$\Big(|Du_k|^{p-2}Du_k - |Du_0|^{p-2}Du_0\Big).D(u_k-u_0) \to 0, \quad in\ L^1(E_\delta, |x|^{-ap}),$$

$$\Big(|Dv_k|^{p-2}Dv_k - |Dv_0|^{p-2}Dv_0\Big).D(v_k-v_0) \to 0, \quad in\ L^1(E_\delta, |x|^{-ap}),$$

and thus, up to a subsequence, also a.e. in E_δ. Hence, we have, $Du_k \to Du_0$ and $Dv_k \to Dv_0$ a.e. in E_δ. Since δ is arbitrary, this implies that $Du_k \to Du_0$

and $Dv_k \to Dv_0$ a.e. in $B_{\mathcal{R}_{j+1}}$. By the diagonal process of Cantor, we have that $Du_k \to Du_0$ and $Dv_k \to Dv_0$ a.e. in R^N. Thus, (21) and (22) are hold, which completes the proof of Step 4 and that of Theorem 4.3. ■

5. Conclusions

The existence of a non-trivial weak solution to quasilinear elliptic system involving critical Hardy exponents by variational methods is an important issue in the area of partial differential equation. In this manuscript by using variational method, we have proved the existence of a non-trivial weak solution to a quasilinear elliptic system involving critical Hardy exponents.

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