International Journal of
Mathematical Modelling & Computations
Vol. 01, No. 04, 2011, 217- 226

The Scaling Law for the Discrete Kinetic Growth Percolation Model

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Received: 12 August 2011; Accepted: 18 November 2011.

Abstract. The critical exponent of the total number of finite clusters α is calculated directly without using scaling hypothesis both below and above the percolation threshold p_c based on a kinetic growth percolation model in two and three dimensions. Simultaneously, we can calculate other critical exponents β and γ, and show that the scaling law $\alpha + 2\beta + \gamma = 2$ has been held in the simulation result above the percolation threshold p_c.

Keywords: Percolation problem, Scaling law, Kinetic growth model, Leath algorithm.

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1. Introduction

The percolation problem is one of the most basic and important models for various phase transition phenomena in statistical physics. In a number of fields a variety of experimental discussions are reported, but Florys discussion [5] on aggregation in polymer science is the first one directly related to percolation. A mathematical definition of the percolation problem was given by Broatbent and Hammersley [2] in 1957. The exact values of threshold concentrations and critical exponents for the percolation problem are found in only several 2-D lattices [8] from 1965 to 1982. As for the method of the calculation, the series expansion or the Monte Carlo simulation has mainly been used [4]. The series expansion has been studied by Sykes et al. [7, 19, 20] powerfully. Using these series expansion some of the critical exponents were obtained [3, 7 ,21]. The Monte Carlo simulation has been studied by many researchers. Previous works of the critical exponents γ and β were

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reported by Stool and Domb [18], Leath and Reich [12], Nakanishi and Stanley [15, 16] and Hoshen et al. [9]. The critical exponent α was reported by Kirkpatrick [11].

Since Kasteleyn and Fortuin [10] provided the analogy between the percolation problem and other critical phenomena, a number of studies have been made to examine the scaling law in the percolation problem.

A kinetic growth percolation algorithm was proposed by Leath [13,14] in 1976, and Alexandrowicz [1] in 1980 calculated critical exponents from 2 to 8 dimensions. They showed that the threshold value p_c and the critical exponents were corresponding to usual percolation results [17]. However, all the critical exponents by them were calculated below the threshold $p < p_c$, because they thought that the function type of the distribution function also critical exponents α and γ do not change below and above the threshold. The critical exponent β shows the property of the fraction of sites belonging to the infinite cluster. Therefore we have to calculate the property above the threshold $p_c < p$. It should be decided by using the results above the threshold whether the scaling law $\alpha + 2\beta + \gamma = 2$ has been held. However, they concluded that the scaling law was held by using the value below the threshold. They assumed that the function type of the distribution function obeyed the scaling hypothesis. It is necessary to calculate the critical exponents independently above the threshold to prove the scaling law. Above the threshold, however, it is difficult to decide the exponent a because of how the infinite cluster appears in the usual percolation method. A lot of previous works described only the exponents β and γ, and the exponent α is derived from scaling assumption [9]. On the other hand, as the kinetic growth percolation model (KGM) is able to generate only the finite cluster, we can avoid this difficulty.

In the next section kinetic growth model (Leath algorithm) will be described and the ritical exponents α, β, and γ below and above the critical point by using KGM in two and three dimensions will be given in Sec.3 and Sec.4, respectively. The final section will present our conclusions and discussion.

2. Kinetic Growth Model (LEATH ALGORITHM)

In a site percolation problem, each site has a prescribed probability p (the same for each site) of occupying a site by the particle, and a probability of $1 - p$ of non-occupying a site by the particle. A bond between the two nearest neighbor sites is formed when the two nearest neighbor sites are occupied by particles. The connected sites form a cluster. We obtain the critical probability p_c, when the cluster becomes infinitely large. It is known that the same critical value is obtained by growing up from one site at the origin (KGM [1], Leath algorithm [13,14]) .

KGM (Leath algorithm) is performed on a discrete site percolation as follows:

1) A particle is placed on the origin.
2) One site is selected out of the four (or the six in the case of a $3d$-simple cubic lattice) nearest neighbor sites in the case of a $2d$-square lattice. The site is occupied or empty, with probability p or $1 - p$, respectively. The occupied nearest neighbor sites are linked and construct a cluster.
3) The above processes are repeated until the cluster grows infinitely or the growth stops.

We repeat above steps (1) to (3) again and again until many sample clusters are obtained.

Usually, when the probability p is greater than the critical probability p_c, the infinite cluster is obtained and when the probability p is less than p_c, only the finite size of cluster is obtained. However, in the finite lattice a finite cluster which

spreads from end to end is regarded as an infinite cluster. Also critical exponents ν for the correlation length of 1.3 (in $2d$) and 0.83 (in $3d$) are obtained, which are the same as the exponents obtained from the conventional simulation method [17].

This method of KGM (Leath algorithm) can generate a larger size cluster, which allows us to simulate a larger size cluster in the same lattice size, because we treat only a single cluster. Therefore this simulation can be done close to the percolation threshold p_c, when we increase p below p_c. However, when we decrease p from a larger probability than p_c, many clusters created by this simulation spread over the finite lattice range. This simulation has not been used above p_c.

3. The Critical Exponents and the Percolation Threshold in two Dimensions

In the percolation problem, various amounts indicate singularity at the threshold p_c. Then, critical exponents of those quantities are defined by the similarity with the critical phenomena of the phase transitions. Typical critical exponents α, β, and γ can be calculated from the kth moment of cluster size by using the cluster size distribution $n_s(p)$ as follows,

$$M_k = \sum_s S^k n_s(p). \tag{1}$$

The mean cluster size corresponds to M_2, the strength of the infinite cluster to M_1, and the total number of finite clusters to M_0. From these moments the critical exponents $-\gamma$, β and $2-\alpha$ can be calculated against $|p-p_c|$, respectively.

It is suggested that these critical exponents depend only on the dimension of the lattice, but not on the lattice structure itself. If we assume the scaling assumption of the distribution function $n_s(p)$, the number of the independent exponents is two. Then, using the $2d$-KGM model, we derive the critical exponents α, β, and γ independently above the critical point and examine whether the scaling law $\alpha + 2\beta + \gamma = 2$ has been held.

Below, we further explain the calculation procedures for the exponents α, β, and γ.

i) The critical exponent β:

The probability P of sites belonging to the infinite cluster shows the critical behavior near the percolation threshold p_c. It goes to zero by the simple power law,

$$P \sim (p - p_c)^{\beta}, \tag{2}$$

and right at the critical point $p = p_c$ we have $P = 0$. But even below pc we regard a finite cluster as an infinite one if it touches 2 edges at the top and the bottom, because of the finite lattice. Therefore, we estimate a value that is smaller than the true p_c to be the percolation threshold. We show the mean value of fraction $P(p, L)$ for 10000 samples against p in Figure 1 together with the curve obtained by using the critical parameters estimated below. The fraction $P(p, L)$ is calculated as a ratio for the cluster to exceed the finite square lattice ($L = 3000$). The number of samples is 10000. The fraction $P(p, L)$ becomes non-zero from about $p = 0.585$, though the true percolation threshold p_c is about 0.5927. When p approaches p_c, critical phenomena are shown, but when p closely approaches the vicinity of p_c, the fraction $P(p, L)$ is slightly over counted and deviates from Equation (2) because of

some finite size effects. When p becomes farther than p_c, critical phenomena are not shown. We show the fraction $\log(P(p,L))$ against $\log(ppc)$ in Figure 2, where p_c is tuned in order to obtain a straight line. The critical exponent β is calculated by using values (gray solid circles in Figure 1 or Figure 2) from $p = 0.593$ to $p = 0.615$. When the adjustable value of p_c was assumed to be 0.5926, to take into account some finite size effects [9], the critical exponent β becomes 0.139 ± 0.001 in Figure 2 (a). When the adjustable value of p_c was assumed to be 0.5927, the critical exponent β becomes 0.141 ± 0.001 in Figure 2 (b). These parameters allow for the good result even in the 3000×3000 square lattice. We believe this is the first time the exponent β has been obtained in KGM.

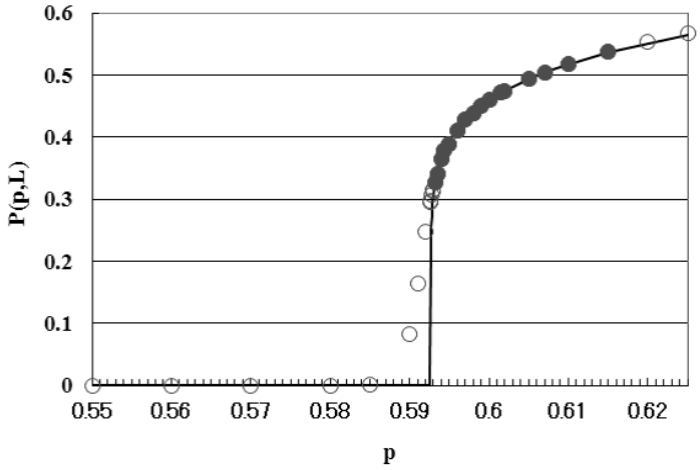
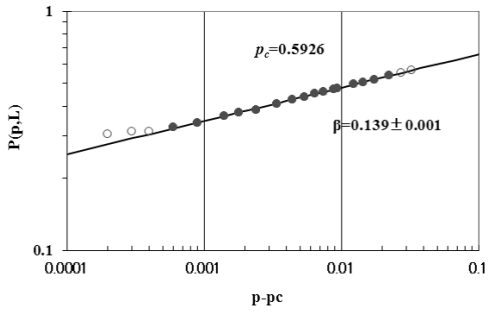
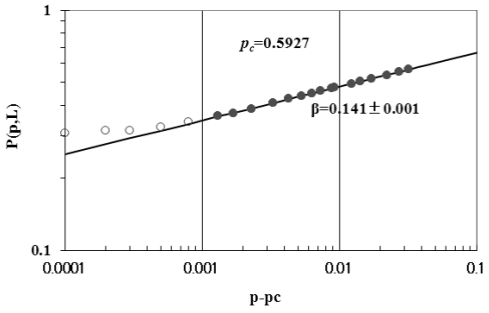


Figure 1. The mean value of percolation fraction $P(p,L)$x is shown by circles. The solid line is obtained by using the critical parameter of Table 1.



(a) The percolation fraction $log(P(p,L))$ is plotted against $log(pp_c)$. We try to find the best straight line by tuning p_c. The value of p_c obtained is 0.5926 and the slope is obtained $\beta = 0.139$.



(b) The percolation fraction $log(P(p,L))$ is plotted against $log(pp_c)$. When the adjustable value of p_c is assumed to be 0.5927, the critical exponent becomes 0.141.

Figure 2.

ii) The critical exponent α:

The total number of finite clusters M_0 varies as

$$M_0 \sim |p - p_c|^{2-\alpha} \tag{3}$$

By using KGM, the total number of finite clusters is counted as follows. We name clusters containing s sites s-clusters. We define n_s as the number of s-clusters per lattice site. The s-cluster consists of s occupied sites and t empty sites. If we make N sample clusters by using KGM, s-clusters will be counted at the ratio of $n_s s$. N is determined in the following manner. We sum of each $(s + t)$ sites of the cluster generated one by one. The simulation is repeated until the total summation of sites satisfies the Equation (4). ($L^2 = 3000 \times 3000 = 9000000$).

$$\sum_{i=1}^{N}(s_i + t_i) \cong 9000000. \tag{4}$$

The frequency distribution $f(s)$ of the s-cluster proportional to $n_s s$ against size s is obtained. Then we sum up $f(s)/s$ by over the size s. We obtain the total number of finite clusters M_0. This process corresponds to counting the number of finite clusters in the usual percolation problem. This total number decreases because of the appearances of larger clusters by increasing the correlation length, when p approaches below the threshold p_c. As the correlation length becomes infinite above p_c, infinite clusters appear. However, the finite clusters still appear above p_c. We show the total number of finite clusters M_0 against p in Figure 3 together with the curve obtained by using the critical parameters estimated as below. The total number of finite clusters is estimated slightly larger because only small clusters in the finite lattice were counted. The threshold p_c is estimated to a slightly different value from 0.5927 depending on the direction approached from above and from below p_c. Therefore, when p approaches the threshold from below, the threshold $p_c(L)$ is estimated a slightly larger than $p_c(L = \infty)$. When p approaches the threshold from above, the threshold $p_c(L)$ is estimated slightly smaller than $p_c(L = \infty)$. We show $\log(M_0)$ against $\log|p_c p|$ in Figure 4, where each p_c is tuned in order to obtain a straight line. As shown in Figure 3 and Figure 4, the power law is valid considerably in the large range below the threshold p_c. Therefore, the power law exponent α does not greatly depend on the threshold (gray solid circles in Figure 4). The critical exponent *alpha* is calculated by using values from $p = 0.450$ to $p = 0.587$ below the threshold p_c. When the adjustable value of p_c was assumed to be 0.5927, the critical exponent α becomes 0.66 ± 0.02 below the threshold p_c (gray solid circles in Figure 4 (a)). On the other hand, the critical exponent α becomes 0.31 ± 0.03 above the threshold p_c as shown in Figure 4 (a) with gray solid squares. As our simulation is done in the range of p near p_c from $p = 0.597$ to $p = 0.610$, the critical exponent α greatly depends on the value of p_c. When the range of the calculation is extended, the critical exponent α shows a tendency to increase. When the adjustable value of p_c was assumed to be 0.5918, to take into account some finite size effects, the critical exponent α becomes 0.66 ± 0.02 above the p_c in Figure 4 (b) with gray solid squares and 0.66 ± 0.02 below the threshold p_c in Figure 4 (b) with gray solid circles, respectively.

iii) The critical exponent γ:

When p approaches the threshold p_c the mean cluster size diverges as follows

$$S \sim |p - p_c|^{-\gamma} \tag{5}$$

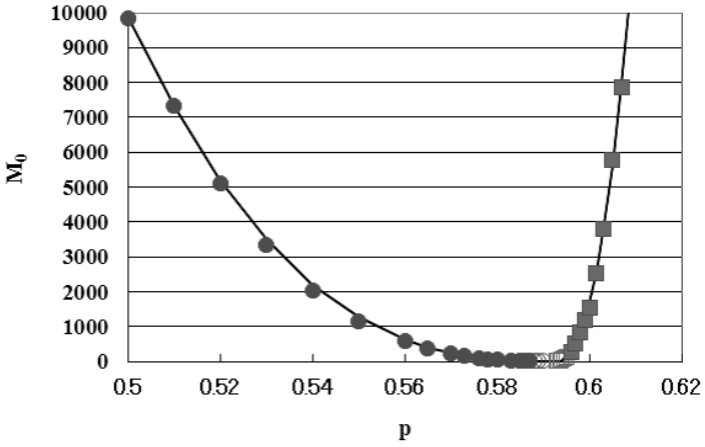
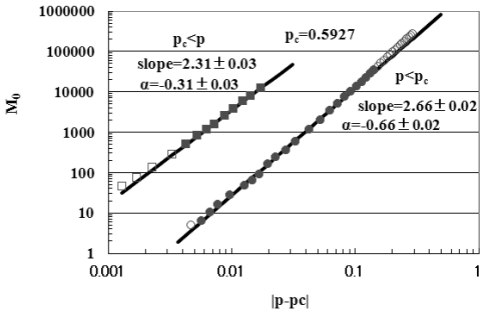
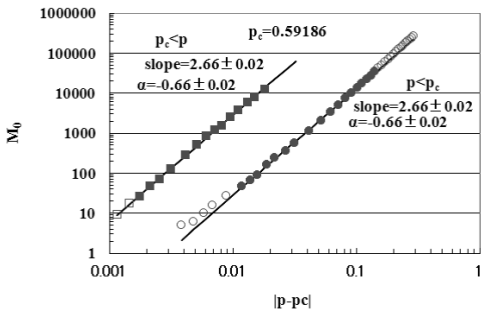


Figure 3. The total number of finite clusters M_0 is plotted against p by circles (below p_c) and by squares (above p_c). The solid line is obtained by using the critical parameters of Table 1.



(a) The total number of finite clusters $log(M_0)$ is plotted against $log(|pp_c|)$. When the adjustable p_c is assumed to be 0.5927, from the slope we have obtained $\alpha = 0.66$ (below p_c) and $\alpha = 0.31$ (above p_c).



(b) The total number of finite clusters $log(M_0)$ is plotted against $log(|pp_c|)$. We try to find the best straight line by tuning p_c, below and above p_c,, respectively. From the slope we have obtained $\alpha = 0.66$ both below and above p_c.

Figure 4.

The shift of the percolation threshold p_c to take into account some finite size effects is similar to the total number of finite clusters. We show the mean cluster size S for 10000 samples against p in Figure 5 together with the curve obtained by using the critical parameters estimated as below. The shape of this curve is like the well known susceptibility ξ in magnetization. As shown in Figure 6 (a), when p_c is assumed to be 0.5927, exponent γ becomes 2.30 ± 0.01 approaching p_c from $p < p_c$ (gray solid circles) and 2.03 ± 0.02 from above (gray solid squares), respectively. As shown in Figure 6 (b), both exponents γ become 2.38 if we assumed p_c to be 0.5934 when p approaches to p_c from below (gray solid circles) and to be 0.5918 when p approaches from above (gray solid squares), respectively [9]. These critical exponents in two dimensions are shown in Table 1.

Table 1. The critical exponents are given below and above p_c in two dimensions.

	p_c	$p < p_c$	$p_c < p$	previous work[17]
α	0.5927	-0.66 ± 0.02	-0.31 ± 0.03	$\frac{-2}{3}$
	0.5918	-0.66 ± 0.02	-0.66 ± 0.03	
γ	0.5927	2.30 ± 0.01	2.03 ± 0.02	$\frac{43}{18}$
	0.5934	2.38 ± 0.01	-	
	0.5918	-	2.38 ± 0.03	
β	0.5927	-	0.141 ± 0.001	$\frac{5}{36}$
	0.5926	-	1.39 ± 0.001	

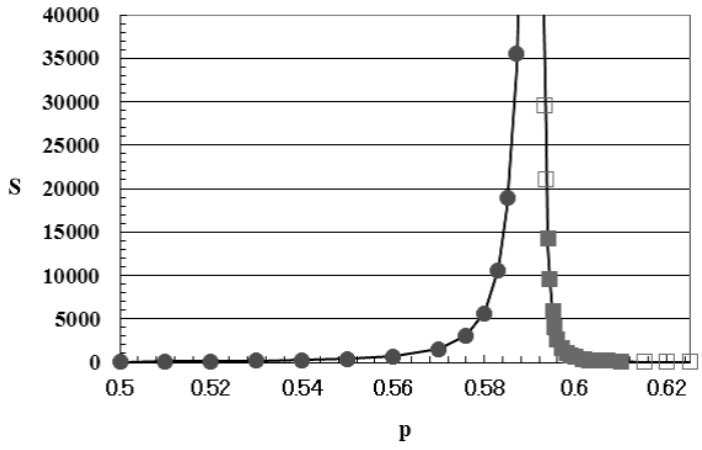
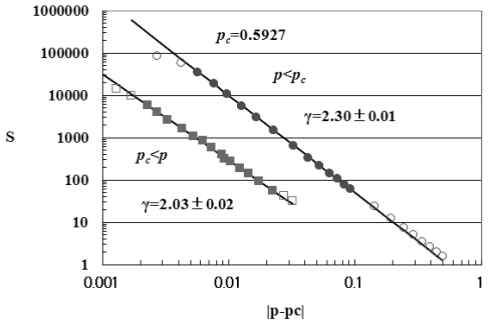
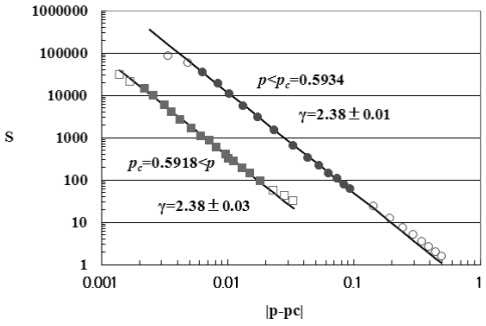


Figure 5. The mean cluster size S is plotted against p by circles (below p_c) and by squares (above p_c). The solid line is obtained by using the critical parameters of Table 1.



(a) The mean cluster size $log(S)$ is plotted against $log(|pp_c|)$. When the adjustable value of p_c is assumed to be 0.5927, from the slope we have obtained $\gamma = 2.30$ (below p_c) and $\gamma = 2.03$ (above p_c).



(b) The mean cluster size $log(S)$ is plotted against $log(|pp_c|)$. We try to find the best straight line by tuning p_c, below and above p_c, respectively. From the slope we have obtained $\gamma = 2.38$ both below and above p_c.

Figure 6.

4. The Critical Exponents and the Percolation Threshold in three Dimensions

Using the same calculation procedures in the previous section, we obtain the critical exponents α, β, and γ.

i) The critical exponent β:

The fraction $P(p, L)$ is calculated as a ratio for the cluster to exceed the finite cubic lattice ($L = 200$). The number of samples is 10000 for each value of p. We show the fraction $\log(P(p, L))$ against $\log(pp_c)$ in Figure 7, where p_c is tuned in order to obtain a straight line. The critical exponent β is calculated by using values (gray solid circles in Figure 7) from $p = 0.313$ to $p = 0.333$. When the adjustable value of p_c was assumed to be 0.3114, to take into account some finite size effects, the critical exponent β becomes 0.409 ± 0.004 in Figure 7. When the adjustable value of p_c was assumed to be 0.3116, the critical exponent β becomes 0.392 ± 0.004. Good results were obtained even in the $200 \times 200 \times 200$ cubic lattice.

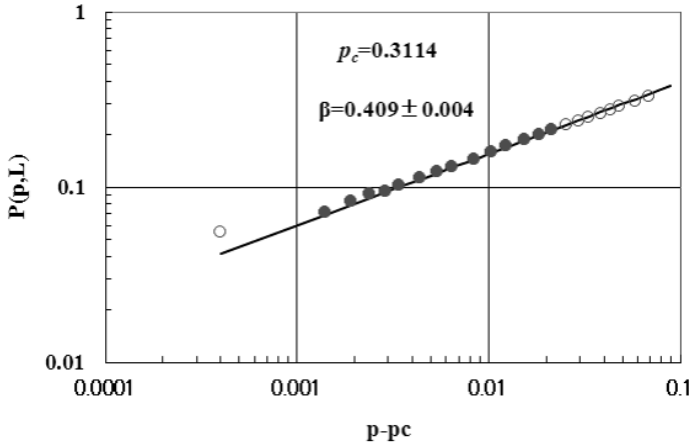


Figure 7. The percolation fraction $log(P(p, L))$ is plotted against $log(pp_c)$. We try to find the best straight line by tuning p_c. The value of pc obtained is 0.3114 and the slope is obtained $\beta = 0.409$.

ii) The critical exponent α:

The simulation is repeated until the total summation of sites satisfies the Equatin (5). ($L^3 = 200 \times 200 \times 200 = 8000000$).

$$\sum_{i=1}^{N}(s_i + t_i) \cong 8000000. \tag{6}$$

We show $\log(M_0)$ against $\log|p_c p|$ in Figure 8 , where p_c is tuned in order to obtain a straight line, below and above p_c, respectively. When the adjustable value of p_c was assumed to be 0.308, to take into account some finite size effects, the critical exponent α becomes 0.62 ± 0.04 above the p_c in Figure 8 with gray solid squares. When the adjustable value of p_c was assumed to be 0.318, the critical exponent α becomes 0.63 ± 0.03 below the threshold p_c in Figure 8 with gray solid circles.

iii) The critical exponent γ:

The mean cluster size S is calculated for 10000 samples for each value of p. We show $\log(S)$ against $\log|pp_c|$ in Figure 9, where p_c is tuned in order to obtain a straight line, below and above p_c, respectively. When p_c is assumed to be 0.3126, the exponent γ becomes 1.796 ± 0.005 approaching to p_c from $p < p_c$ (gray solid circles). When p_c is assumed to be 0.3012, the exponent γ becomes 1.79 ± 0.02 approaching to p_c from above (gray solid squares).
These critical exponents in three dimensions are shown in Table 2.

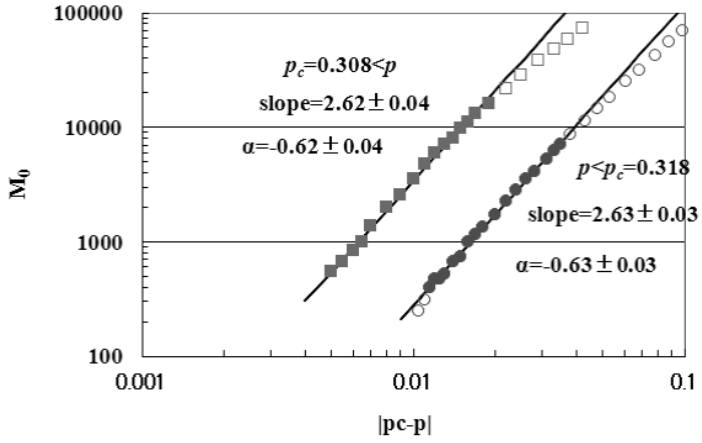


Figure 8. The total number of finite clusters $log(M_0)$ is plotted against $log(|pp_c|)$. We try to find the best straight line by tuning p_c, below and above p_c, respectively. From the slope we have obtained $\alpha = 0.63$ (below p_c) and $\alpha = 0.62$ (above p_c).

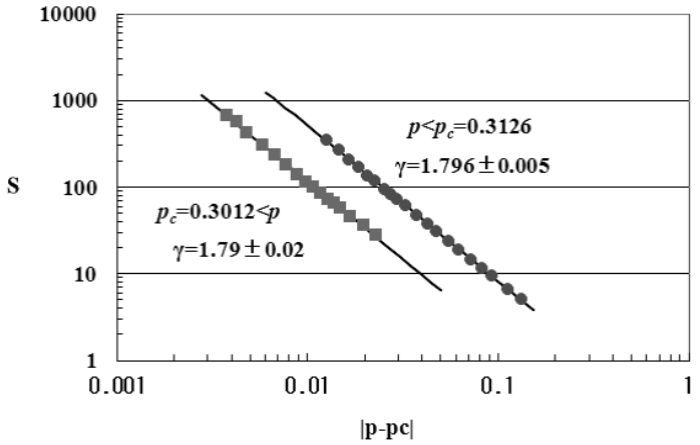


Figure 9. The mean cluster size $log(S)$ is plotted against $log(|pp_c|)$. We try to find the best straight line by tuning p_c, below and above p_c, respectively. From the slope we have obtained $\gamma = 1.79$ both below and above p_c.

Table 2. The critical exponents are given below and above p_c in three dimensions.

	p_c	$p < p_c$	$p_c < p$	previous work[17]
α	0.318 0.308	-0.63 ± 0.03 -	- -0.62 ± 0.04	-0.62
γ	0.3126 0.3102	1.796 ± 0.005 -	- 1.79 ± 0.02	1.80
β	0.3114	-	0.409 ± 0.004	0.41

5. Conclusion and Discussion

In two dimensions as shown in Table 1, critical exponents above the threshold p_c are sensitive to the adjustable value of the threshold p_c. When the adjustable value of the threshold p_c is assumed to be 0.5918, we obtain the same critical exponent $\alpha = 0.66 \pm 0.02$ below and above p_c. This exponent $\alpha = 0.66 \pm 0.02$ above p_c will be a value that we obtained for the first time in KGM. For the

critical exponent γ, we obtain the same value 2.38 ± 0.03 below and above p_c, when the adjustable value of the threshold p_c is assumed to be 0.5918 above p_c and 0.5934 below p_c. For the critical exponent β, when the adjustable value of the threshold p_c is assumed to be 0.5926, we obtained the value 0.139 ± 0.001. This exponent is a value obtained for the first time in KGM. For above p_c the scaling law $\alpha + 2\beta + \gamma = 0.66 + 2 \times 0.139 + 2.38 = 1.998$ which is almost 2 within the error range. When the adjustable value of the threshold p_c is assumed to be 0.5927 ($p_c(L = 8)$), we obtain $\alpha = 0.31 \pm 0.03$, $\gamma = 2.03 \pm 0.02$ and $\beta = 0.141 \pm 0.001$. The scaling law $\alpha + 2\beta + \gamma$ becomes 2.002 also.

In three dimensions as shown in Table 2, when the adjustable value of the threshold p_c is assumed to be 0.308 above p_c and 0.318 below p_c, the critical exponent $\alpha = 0.62 \pm 0.04$ above p_c and $\alpha = 0.63 \pm 0.03$ below p_c, respectively. We obtain the almost same value for the critical exponent γ both 1.796 ± 0.03 below and 1.79 ± 0.02 above p_c, when the adjustable value of the threshold p_c is assumed to be 0.3126 below pc and 0.3102 above p_c. For the critical exponent β, when the adjustable value of the threshold p_c is assumed to be 0.3114, we obtained a value of 0.409 ± 0.004. Above p_c the scaling law $\alpha + 2\beta + \gamma = 0.62 + 2 \times 0.409 + 1.79 = 1.988$ which is almost 2 within the error range.

Therefore, we may conclude that the scaling law is held in two and three dimensions.

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