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Application of Partial Differential Equations in Snow Mechanics

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Abstract. In the present work, failure of a snow slab is analyzed by accounting Normal mode criteria. The analysis has been extended to include residual stress into the model (in addition to body forces). Intensity of crack energy release rate, and displacement components have been derived and their values have been estimated. The obtained results have been compared with the existing snow slab failure models [4-7]. The criteria of snow slab failure have been critically examined.The maximum stress in the snow pack and critical intensity factor of crack development has also been studied in the model. The results of the present model can be used for the avalanche release information.

Keywords: Equilibrium Equation, Maximum Stress, Principal Normal Stress.

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1. Introduction

One of the most fascinating material on earth is snow which covers entire Himalayan regions during the peak winter period. Though it looks very fascinating and very attractive to look and play for but, it has its own limitations and dangers associated with it. Avalanches are one such danger's associated in Hilly bound snow areas. Avalanches are very destructive in nature. It destroys buildings, bridges, which comes on its way. Many casualties occur during the winter as a result of an avalanche. It is always a concern for the people living around this region. The exact cause an avalanche is very difficult to predict however one can identify the causes associated with it. So far, it has been identified that there are three major causes for an avalanche to release;

1. Pressure metamorphism

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2. Temperature metamorphism
3. Due to wind drifting

1. Pressure metamorphism: Avalanches of this type are basically due to over pressure caused as a result of snow precipitation and unable to bear the load either due to slope saturation or due to extra loading.
2. Temperature metamorphism: When snow is allowed to stand over long time [without any further precipitation], the temperature within snow pack gets into metamorphic state due to temperature gradient within the snow pack. As a result, the instability causes due to growth of depth hoar crystals, the crack initiates at some point within the snow pack.
3. Due to wind drifting: Wind induced activities are predominant in the hilly bound snow areas. After the snow fall and a gap of 8 - 10 hours, wind activities disrupt snow lying on windward areas. As a result, leeward areas get filled with unwanted snow which is of potential threat for the avalanches.
In view of the importance of an avalanche, failure analysis of a slab has been undertaken in the present work. The model has been analysed for the earlier criteria of the failure at the beginning and the improvements over the same in the present investigations.

2. Analysis

1. Snow failure: laboratory experiments have shown that the mechanical behavior of snow depends on the rate of loading at low rates, snow shows predominately nonlinear viscous behavior. Considerable strain energy can be dissipated. At high rates the elastic properties dominate and samples break after very limited deformation (brittle failure). Schweizer (1999) found the transition between the ductile and the brittle state of failure was at strain rate of about $10^{-3}s^{-1}$, for the snow type tested (small rounded particles, size<0.5). For larger grains (size$=1mm$)the transition is shifted toward lower strain rates at about $10^{-4}s^{-1}$.

3. Equilibrium Equation

Crack initiation and development has been modeled from the following fundamental Equations (1,2)

$$\sigma_{ij,j}=0\rightarrow\begin{cases}\partial\sigma_{xx}/\partial x+\partial\sigma_{yx}/\partial y+F_1=0\\ \partial\sigma_{yy}/\partial y+\partial\sigma_{xy}/\partial x+F_2=0\end{cases}\tag{1}$$

Where σ_{ij} and F_i are respectively stress tensor and body force. Prinicipal normal stress (σ_1 and σ_2) which is one of the important parameter for the establishment of peak stress or maximum stress (τu) in the snow pack (slab) is obtained by the following constitutive equations by Coulomb-Mohr criterion (C-M criterion) see Figure 1.

$$|\sigma_1-\sigma_2|+m|\sigma_1+\sigma_2|=2\tau_u\tag{2}$$

$$\begin{pmatrix}\sigma_{xx}-\sigma & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}-\sigma\end{pmatrix}\tag{3}$$

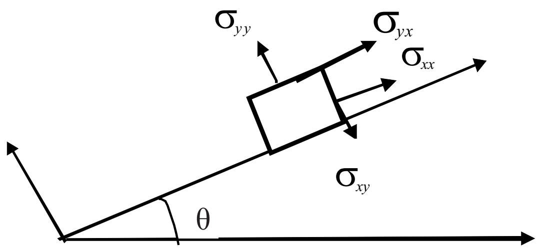


Figure 1. C-M criterion

$$m = \frac{\mu}{\sqrt{1+\mu^2}} \text{ (where)} \mu = -\tan\theta \tag{4}$$

Let $\sigma_{xx} = 0$ by Equation (1) we get, $F_1 = \rho g \sin(\theta)$, $\sigma_{xy} = -\rho g y \sin\theta$ and $\sigma_{yy} = -\rho g y \cos\theta$,consequently principal normal stress:

$$\sigma_{12} = (\sigma_{xx} + \sigma_{yy})/2 + ([(\sigma_{xx} - \sigma_{yy})/2]^2 + \sigma_{xy}^2)^{1/2} \tag{5}$$

so:

$$\sigma_1 - \sigma_2 = 2\rho g y \sqrt{\left(\frac{\cos^2\theta}{2} + \sin^2\theta\right)}, \quad \sigma_1 + \sigma_2 = -\rho g y \cos\theta \tag{6}$$

Substituting Equation (6) into (2) we get the expression for ultimate stress or peak stress

$$\tau_u = \rho g y[(1 + 3\sin^2\theta)^{1/2} - \sin 2\theta/2] \tag{7}$$

4. Residual Stress

Residual stress that remain after the cause of the stresses (external forces, heat gradient) has been removed has also been computed by using the following relation from the concept of beam theory

$$\sigma_r = -\sigma_0/2[1 - (\sigma_0/\varepsilon E)^2] \tag{8}$$

Where σ_0, ε, and E are respectively yield stress, strain tensor and elastic modules. Equation (1) for the shear model [5] (Figure 2) simplifies to

$$\partial\sigma_{xy}/\partial y - (1/H)(\sigma_r - \sigma_g) = 0 \quad \text{or} \quad \sigma_{xx} = (L/H)(\sigma_g - \sigma_r) \tag{9}$$

Where L is critical length and $\sigma_g = \rho g H \sin\theta$ so

$$\sigma_t = \rho g H \sin\theta \tag{10}$$

Where σ_t is tensile stress. The values of $\sigma_t/\rho g$ are taken in the ranges of Table 1 $0.5 \leqslant \sigma_t/\rho g \leqslant 3$.

5. Energy Release Rate and Stress Intensity Factors

Energy Release rate and Stress Intensity Factors which are one of the fundamental parameters in the avalanche release where one can estimate the damage potential have been computed by following relations [3-6]:
$G = (1/w)(\partial U/\partial L)$ where U is complementary strain energy and w is thickness of snow slab. so:

$$U = 1/2 \int \sigma_{xx}\varepsilon_x dV = 1/2([1-\upsilon^2]/E)\int \sigma_{xx}^2 dV = (L^3 w/HE)[1-\upsilon^2](\sigma_g - \sigma_r)^2 \quad (11)$$

Hence:

$$G = (3L^2/HE)(1-\upsilon^2)(\sigma_g - \sigma_r)^2 \quad (12)$$

$$K = (GE)^{1/2} = L(\sigma_g - \sigma_r)\{3(1-\upsilon^2)/H\}^{1/2} \quad (13)$$

Where K is the intensity Factor. Computations of G and K has been carried out for various parameters [L, υ, H, and E (only for G)]. The results are shown in Figure 2

6. Estimation of Critical Length

Critical length of the avalanche crack is estimated in terms of residual stress, gravitational tensor and ultimate (peak) stress from the concepts of Hooke's Law [7-8]:

$$\varepsilon_x = 1/E[\sigma_{xx} - \upsilon(\sigma_{yy} + \sigma_{zz})], \quad \varepsilon_y = 1/E[\sigma_{yy} - \upsilon(\sigma_{xx} + \sigma_{zz})] \quad (14)$$

suppose $\varepsilon_y = \sigma_z = 0$ therefore

$$\sigma_{yy} = \upsilon\sigma_{xx} \quad (15)$$

Substituting Equations from (15) into (14) we get,

$$\sigma_{yy} = \upsilon\sigma_{xx} \quad \sigma_{xx} = E\varepsilon_x/(1-\upsilon^2) \quad (16)$$

According to definition of energy per unit volume:

$$E = \int \sigma_{xx} d\varepsilon_x = [(1-\upsilon^2)/E]\sigma_{xx}^2 \quad (17)$$

We knowD

$$E = \text{body force} \times \text{distance} = [\sigma_p - \sigma_r]\delta/H \quad (18)$$

Combining (17), (18) we get:

$$\{(\sigma_p - \sigma_r)/H\}.\delta = [(1-\upsilon^2)/E]\{(L/H)(\sigma_g - \sigma_r)\}$$

Therefore

$$L = [H/(\sigma_g - \sigma_r)]\{4G^*(\sigma_g - \sigma_r)\delta/(H[1-\upsilon])\}^{1/2} \quad (19)$$

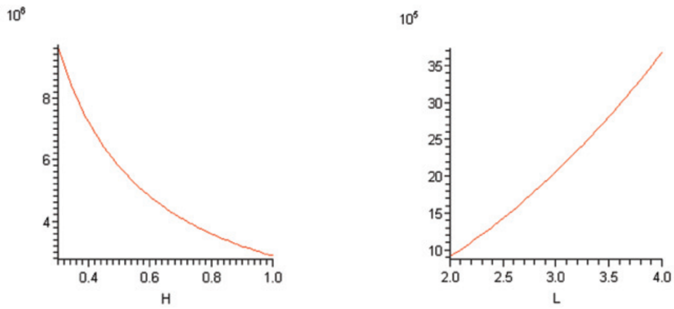
Where L is critical length., G^* is a shear modulus and given by $G^* = E/2(1+v)$. The graph of L are shown in Figure 3

Table 1. Data of flow parameters

Parameters	typical value	range
Density:ρ	200kg/m^3	100-400
Poissons ratio: v	0.2	0.1-0.4
Slope angle: θ	38 degree	30-45
Slab depth: $H(=R)$	0.5 m	0.3-1
Elastic modulus: E	1 Mpa	0.5-10
Shear modulus: G^*	0.5 Mpa	0.1-5
Peak strength/residual strength:σ_p/σ_r	1.5	1.25-2
Weak layer thickness: $d(=100\delta)$	10mm	1-15mm
Slope angle at run out zone: θR	10	0-11
Eddy viscosity: η	1000m/s^2	400-1000
Viscosity: μ	0.2	0.1-0.3

7. Results

Snow slab failure has been modeled in the present investigations. Existing models have been reviewed and analyzed for the stability of the snow pack. Snow stability condition has been evolved and analyzed for the existing data [7].

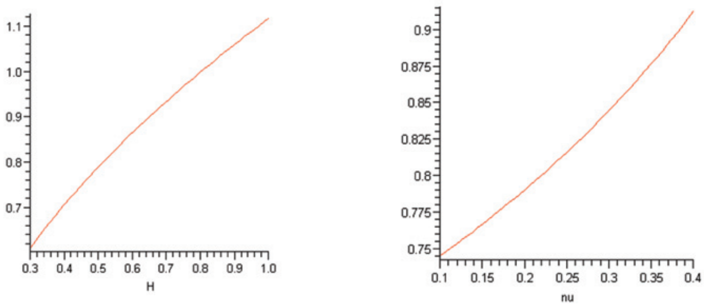


(a) Energy release rate in term of slab depth

(b) Energy release rate in term of slab length

Figure 2. Energy release rates

The results are found to be qualitatively in good agreement with the physics of flow and deformation. The results can be used for the avalanche control, mitigation and forecasting aspect point of view.



(a) Critical length in term of slab depth

(b) Critical length in term of poisson ratio

Figure 3. Critical lengths

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