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Variational Homotopy Perturbation Method for Solving the Nonlinear Gas Dynamics Equation

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Abstract. Noor A. et al. [7] analyze a technique by combining the variational iteration method and the homotopy perturbation method which is called the variational homotopy perturbation method (VHPM) for solving higher dimensional initial boundary value problems. In this paper, we consider the VHPM to obtain exact solution to Gas Dynamics equation.

Keywords: Variational Homotopy Perturbation Method, Nonlinear Gas Dynamics Equation.

Index to information contained in this paper

1. Introduction

In this paper, we consider the Gas Dynamics equation

$$u_t + \frac{1}{2}(u^2)_x = u(1-u) + f(x,t) \qquad 0 \leqslant x \leqslant 1, \quad t \geqslant 0 \qquad (1)$$

Where $u_t = \frac{\partial u}{\partial t}, (u^2)_x = 2uu_x$.

Recently various iterative methods, are employed for the numerical and analytic solution of partial differential equation. In this paper, the variational homotopy perturbation method is applied to solve partial differential equations. This method is suggested by combining the variational iteration technique and the homotopy perturbation method. In this algorithm, the correct functional is developed and the Lagrange multipliers are calculated optimally via variational theory [2-4, 6]. Finally, the homotopy perturbation is implemented on the correct functional and the comparison of like powers of ρ gives solutions of various orders. The developed

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algorithm takes full advantage of variational iteration and the homotopy perturbation methods. In this letter we solve Gas Dynamics equation via VHPM.

2. Variational Homotopy Perturbation Method

To convey the basic idea of the Variational homotopy perturbation method, we consider the following general differential equation :

$$Lu + Nu = g(x), \tag{2}$$

where L is a linear operator, N is a nonlinear operator, and g(x) is the forcing term. According to variational iteration method , we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau))d\tau, \tag{3}$$

where λ is a Lagrange multiplier [4], which can be identified optimally via a variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. That is, $\delta\tilde{u}_n = 0$. Now, we apply the homotopy perturbation method,

$$\sum_{n=0}^{\infty} \rho^{(n)} u_n = u_0(x) + \rho\int_0^x \lambda(\tau)(\sum_{n=0}^{\infty} \rho^{(n)} L(u_n(t)) + N(\sum_{n=0}^{\infty} \rho^{(n)} \tilde{u}_n(\tau))d\tau - \int_0^x \lambda(\tau)g(\tau)d\tau, \tag{4}$$

Which is the variational homotopy perturbation method and is formulated by the coupling of variational iteration method and Adomian's polynomials. A comparison of like powers of ρ gives solutions of various orders.

3. Implementation of the VHPM

Consider the Gas Dynamics equation:

$$u_t + \frac{1}{2}(u^2)_x = u(1-u) + f(x,t) \qquad 0 \leqslant x \leqslant 1, \quad t \geqslant 0$$

with initial condition

$$u(x,0) = g(x)$$

At first, we consider the homogeneous Gas Dynamics equation and then take into account a general form.

$$u_t + \frac{1}{2}(u^2)_x = u(1-u) \tag{5}$$

subject to the initial condition

$$u(x,0) = e^{-x},$$

The correct functional is given by

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \lambda(\tau)(\frac{\partial u_k}{\partial \tau} + \frac{1}{2}\frac{\partial \tilde{u}_k^2}{\partial x} - \tilde{u}_k + \tilde{u}_k^2)d\tau \tag{6}$$

Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda(\tau) = -1$, which yields the following iteration formula:

$$u_{k+1}(x,t) = e^{-x} - \int_0^t (\frac{\partial u_k}{\partial \tau} + \frac{1}{2}\frac{\partial u_k^2}{\partial x} - u_k + u_k^2)d\tau \tag{7}$$

Applying the variational homotopy perturbation method, we have

$$\begin{aligned} u_0 + \rho u_1 + \rho^2 u_2 + \cdots = e^{-x} &- \rho \int_0^t (\frac{\partial u_0}{\partial t} + \rho\frac{\partial u_1}{\partial t} + \ldots) \\ &- \rho \int_0^t (u_0 + \rho u_1 + \ldots)(\frac{\partial u_0}{\partial x} + \rho\frac{\partial u_1}{\partial x} + \ldots)d\tau \\ &+ \rho \int_0^t (u_0 + \rho u_1 + \ldots)d\tau \\ &- \rho \int_0^t (u_0 + \rho u_1 + \ldots)^2 d\tau \end{aligned} \tag{8}$$

Comparing the coefficient of like powers of ρ, we have

$$\begin{aligned} \rho^{(0)} : u_0(x,t) &= e^{-x}, \\ \rho^{(1)} : u_1(x,t) &= e^{-x}t, \\ \rho^{(2)} : u_2(x,t) &= e^{-x}\frac{t^2}{2}, \\ &\vdots \end{aligned} \tag{9}$$

and so on.

$$u(x,t) = e^{-x} + e^{-x}t + e^{-x}\frac{t^2}{2} + \cdots = e^{-x}(1 + t + \frac{t^2}{2} + \ldots) = e^{(t-x)}. \tag{10}$$

There for we obtain the exact solution [5]

Now, we consider the nonhomogeneous Gas Dynamics equation

$$u_t + \frac{1}{2}(u^2)_x = u(1-u) + f(x,t) \tag{11}$$

where

$$f(x,t) = -e^{(t-x)}, \quad u(x,0) = 1 - e^{-x}$$

The correct functional is given by

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \lambda(\tau)(\frac{\partial u_k}{\partial \tau} + \frac{1}{2}\frac{\partial \tilde{u}_k^2}{\partial x} - \tilde{u}_k + \tilde{u}_k^2 - f(x,\tau))d\tau \tag{12}$$

Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda(\tau) = -1$, which yields the following iteration formula:

$$u_{k+1}(x,t) = e^{-x} - \int_0^t (\frac{\partial u_k}{\partial \tau} + \frac{1}{2}\frac{\partial u_k^2}{\partial x} - u_k + u_k^2 - f(x,\tau))d\tau \tag{13}$$

Applying the variational homotopy perturbation method, we have

$$\begin{aligned} u_0 + \rho u_1 + \rho^2 u_2 + \cdots = 1 - e^{-x} - \rho \int_0^t (\frac{\partial u_0}{\partial t} + \rho \frac{\partial u_1}{\partial t} + \ldots) \\ - \rho \int_0^t (u_0 + \rho u_1 + \ldots)(\frac{\partial u_0}{\partial x} + \rho \frac{\partial u_1}{\partial x} + \ldots)d\tau \\ + \rho \int_0^t (u_0 + \rho u_1 + \cdots)d\tau \\ - \rho \int_0^t (u_0 + \rho u_1 + \cdots)^2 d\tau \\ - \int_0^t e^{\tau - x} d\tau \end{aligned} \tag{14}$$

Comparing the coefficient of like powers of ρ, we have

$$\begin{aligned} \rho^{(0)} &: u_0(x,t) = 1 - e^{t-x}, \\ \rho^{(1)} &: u_1(x,t) = 0, \\ \rho^{(2)} &: u_2(x,t) = 0, \\ &\vdots \end{aligned} \tag{15}$$

and so on.

$$u(x,t) = 1 - e^{t-x} \tag{16}$$

There for we obtain the exact solution [1]

4. Conclusion

In this paper, The VHPM is successfully applied on the nonlinear Gas Dynamics equation and the approximations obtained by VHPM converge to its exact solution. It is worth mentioning that the VHPM is applied without any discretization, restrictive assumption, or transformation and is free from round-off errors. In our work we use the MATLAB software to calculate the series obtained from the VHPM.

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