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Solving Linear Sixth-Order Boundary Value Problems by Using Hyperbolic Uniform Spline Method

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Abstract. In this paper, a numerical method is developed for solving a linear sixth order boundary value problem (6VBP) by using the hyperbolic uniform spline of order 3 (lower order). There is proved to be first-order convergent. Numerical results confirm the order of convergence predicted by the analysis.

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1. Introduction

When an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in when this instability is an ordinary convection, the ordinary differential equation is a sixth order ordinary differential equation. Much attention have been given to solve the sixth-order boundary value problems, which have application in various branches of applied sciences. These problems are generally arise in the mathematical modeling of viscoelastic flows [4]. A spline has been widely applied for the numerical solutions of some ordinary and partial differential equations in the numerical analysis. Many authors have used numerical and approximate methods to solve sixth-order BVPs. Some of the details about the numerical methods can be found in references [9–11]. In a series of paper [2, 3], Caglar et al. solved a two, three, five and six order BVPs by using third, fourth and sixth-degree splines. Lamnii et al. [6, 7] discussed a boundary-value

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problems based on spline interpolation and quasi-interpolation with second order convergence. The numerical analysis literature contains little on the solution of boundary value problems by using the hyperbolic B-splines, generally we find the splines used in the above mentioned papers are all non hyperbolic B-splines with higher degrees, which effect the computational efficiency in pratical application. This motivates us to use hyperbolic B-splines of order 3 (lower order) to solve these problems. In this paper we study a method based on the hyperbolic B-splines of order 3 for constructing numerical solutions to six-order boundary value problems (6BVPs) of the form:

$$y^{(6)}(\theta) + f(\theta)y(\theta) = g(\theta), \tag{1}$$

with boundary conditions:

$$y(a) = a_0,\ y'(a) = a_1,\ y''(a) = a_2,\ y(b) = b_0,\ y'(b) = b_1,\ y''(b) = b_2, \tag{2}$$

where $f(\theta)$ and $g(\theta)$ are given continuous functions defined in the bounded interval $[a, b]$, $a_i(i = 0, 1, 2)$, and $b_i(i = 0, 1, 2)$ are real constants.

The rest of paper is organized as follows. In Section 2, we give a explicit representation of B-splines of order 3, for more details see [1, 8]. The interpolation hyperbolic B-splines is developed in Section 3. Solution and the convergence analysis is presented in Section 4. Numerical examples are presented in Section 5.

2. Hyperbolic B-splines of order 3

In this section, we briefly give a explicit representation of Uniform Hyperbolic B-splines of order 3 (UH B-splines) and we give the interesting properties of UH B-splines of order 3, for more details see [1, 8]. To do this, we need the following notations. Suppose k be an intergr such that $k \geqslant 3$. Let $m_k = 2^k$ and $h_k = \frac{b-a}{m_k-2}$. Put

$$\begin{cases} \theta^k_{-2} = \theta^k_{-1} = \theta^k_0 = a, \\ \theta^k_i = a + ih, i = 1...m_k - 3, \\ \theta^k_{m_k-2} = \theta^k_{m_k-1} = \theta_{m_k} = b, \end{cases} \tag{3}$$

the set of knots that subdivide the interval $I = [a, b]$ uniformly. The hyperbolic tension splines space of order 3 is defined as follows

$$\mathcal{V}_k = \{s \in \mathcal{C}^1(I) :\ s_{|_{[\theta^k_i,\theta^k_{i+1}]}} \in \Gamma_3\} \text{ where } \Gamma_3 = span\{1, \cosh(\theta), \sinh(\theta)\}.$$

The dimension of $\mathcal{V}_k$ is m_k and the third-order hyperbolic B-splines are given by: for $i = 0, 1, ..., m_k - 5$,

$$\nu_{i,k}(\theta) = C_k \begin{cases} 2\left(\sinh(\frac{\theta-\theta^k_i}{2})\right)^2, & \theta^k_i \leqslant \theta < \theta^k_{i+1}; \\ 2\cosh(h_k) - \cosh(\theta^k_{i+2} - \theta) - \cosh(\theta - \theta^k_{i+1}), & \theta^k_{i+1} \leqslant \theta < \theta^k_{i+2}; \\ 2\left(\sinh(\frac{\theta^k_{i+3}-\theta}{2})\right)^2, & \theta^k_{i+2} \leqslant \theta < \theta^k_{i+3}; \\ 0, & \text{otherwise.} \end{cases}$$

with the respective left and right hand side boundary hyperbolic B-splines are

$$\nu_{-2,k}(\theta) = C_k \begin{cases} 4\left(\sinh(\frac{-\theta+a+h_k}{2})\right)^2, & \theta_0^k \leqslant \theta < \theta_1^k \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{-1,k}(\theta) = C_k \begin{cases} 1+2\cosh(h_k)-2\cosh(h_k-\theta+a)-\cosh(-a+\theta), & \theta_0^k \leqslant \theta < \theta_1^k \\ 2\left(\sinh(\frac{a+2h_k-\theta}{2})\right)^2, & \theta_1^k \leqslant \theta < \theta_2^k \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{m_k-4,k}(\theta) = C_k \begin{cases} 2\left(\sinh(\frac{\theta+2h_k-b}{2})\right)^2, & \theta_{m_k-4}^k \leqslant \theta < \theta_{m_k-3}^k \\ 1+2\cosh(h_k)-\cosh(b-\theta)-2\cosh(\theta+h_k-b), & \theta_{m_k-3}^k \leqslant \theta < \theta_{m_k-2}^k \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{m_k-3,k}(\theta) = C_k \begin{cases} 4\left(\sinh(\frac{\theta+h_k-b}{2})\right)^2, & \theta_{m_k-3}^k \leqslant \theta < \theta_{m_k-2}^k \\ 0, & \text{otherwise.} \end{cases}$$

where $C_k = \frac{1}{4\left(\sinh(\frac{h_k}{2})\right)^2}$.

The hyperbolic B-splines of order 3 possess all the desirable properties of classical polynomial B-splines, see [8]. In this paper, we limit ourselves to list some of them

- $\nu_{i,k}(\theta)$ is supported on the interval $[\theta_i^k, \theta_{i+1}^k]$;
- Positivity : $\nu_{i,k}(\theta) \geqslant 0, \quad \forall \theta \in [\theta_i^k, \theta_{i+1}^k]$;
- Partition of unity: $\displaystyle\sum_{i=-2}^{m_k-3} \nu_{i,k}(\theta) = 1$.

Table 1. The values of $\nu_{i,k}(\theta)$ and $\nu'_{i,k}(\theta)$ at the knots.

	θ_i^k	θ_{i+1}^k	θ_{i+2}^k	θ_{i+3}^k	*else*
$\nu_{i,k}(\theta)$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$\nu'_{i,k}(\theta)$	0	$\frac{1}{2\tanh(\frac{h_k}{2})}$	$\frac{-1}{2\tanh(\frac{h_k}{2})}$	0	0

In Figure 1, we present the graphs of the hyperbolic B-spline $\nu_{i,k}$ of order 3, for $i = -2, \cdots, 5$, with $k = 3$ and $[a, b] = [0, 1]$.

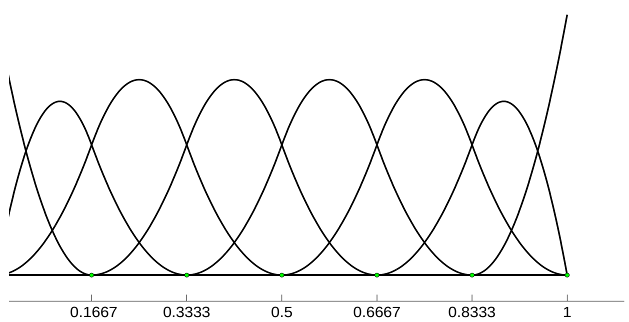

Figure 1. Hyperbolic B-spline $\nu_{i,k}$ of order 3, for $k = 3$ and $[a, b] = [0, 1]$.

3. Hyperbolic B-splines interpolation

In this section, we will construct an approximate of $y^{(6)}(\theta_j^k)$ by using Taylor series expansion.

According to Schoenberg-Whitney theorem (see [?]), for a given function $y(\theta)$ sufficiently smooth there exists a unique hyperbolic spline

$$s(\theta) = \sum_{i=-2}^{m_k-3} \mu_i \nu_{i,k}(\theta) \in \mathcal{V}_k$$

satisfying the interpolation conditions:

$$s(\theta_j^k) = y(\theta_j^k), \quad j = 0, 1, \cdots, m_k - 2; \tag{4}$$

$$s'(a) = y'(a), \quad s'(b) = y'(b). \tag{5}$$

$$s''(a) = y''(a), \quad s''(b) = y''(b). \tag{6}$$

For $j = 0, 1, \cdots, m_k - 3$, let $m_j = s'(\theta_j^k)$ and for $j = 1, 2, \cdots, m_k - 3$, let

$$M_j = \frac{s(\theta_j^k + h_k) - 2s(\theta_j^k) + s(\theta_j^k - h_k)}{h_k^2}.$$

By using the Taylor series expansion we have:

$$m_j = s^{'}(\theta_j^k) = y^{'}(\theta_j^k) - \frac{1}{180} h_k^4 y^{(5)}(\theta_j^k) + O(h_k^6); \tag{7}$$

$$M_j = y^{''}(\theta_j^k) + \frac{1}{12} h_k^2 y^{(4)}(\theta_j^k) + \frac{1}{360} h_k^4 y^{(6)}(\theta_j^k) + O(h_k^6); \tag{8}$$

Now we can apply M_j to construct $y^{(3)}(\theta_j^k)$ and $y^{(4)}(\theta_j^k)$ for $j = 2, 3, \cdots, m_k - 4$, $y^{(5)}(\theta_j^k)$ and $y^{(6)}(\theta_j^k)$ for $j = 3, 4, \cdots, m_k - 5$, as follows, where the errors obtained by Taylor series expansion.

$$\frac{M_{j+1} - M_{j-1}}{2h_k} = y^{(3)}(\theta_j^k) + \frac{1}{4} h_k^2 y^{(5)}(\theta_j^k) + O(h_k^4); \tag{9}$$

$$\frac{M_{j+1} - 2M_j + M_{j-1}}{h_k^2} = y^{(4)}(\theta_j^k) + \frac{1}{6} h_k^4 y^{(6)}(\theta_j^k) + O(h_k^6); \tag{10}$$

$$\frac{M_{j+2} - 2M_{j+1} + 2M_{j-1} - M_{j-2}}{2h_k^3} = y^{(5)}(\theta_j^k) + O(h_k^2); \tag{11}$$

$$\frac{M_{j+2} - 4M_{j+1} + 6M_j - 4M_{j-1} + M_{j-2}}{h_k^4} = y^{(6)}(\theta_j^k) + O(h_k^2); \tag{12}$$

By Table 1 and the equations (9), (10), (11) and (12), we get:

Table 2. Approximation values of $y(\theta_j^k)$,$y^{'}(\theta_j^k)$, and $y^{''}(\theta_j^k)$.

	Approximate value	Representation in μ_j	Order
$y(\theta_j^k)$	$s(\theta_j^k)$	$\frac{\mu_{j-2}+\mu_{j-1}}{2}$	$O(h_k^4)$
$y^{'}(\theta_j^k)$	m_j	$\frac{\mu_{j-1}-\mu_{j-2}}{2\tanh(\frac{h_k}{2})}$	$O(h_k^4)$
$y^{''}(\theta_j^k)$	$M_j=\frac{s_{j+1}-2s_j+s_{j-1}}{h_k^2}$	$\frac{\mu_{j-3}-\mu_{j-2}-\mu_{j-1}+\mu_j}{2h_k^2}$	$O(h_k^2)$

Table 3. Approximation values of $y^{(3)}(\theta_j^k)$, and $y^{(4)}(\theta_j^k)$.

	Approximate value	Representation in μ_j	Order
$y^{(3)}(\theta_j^k)$	$\frac{M_{j+1}-M_{j-1}}{2h_k}$	$\frac{-\mu_{j-4}+\mu_{j-3}+2\mu_{j-2}-2\mu_{j-1}-\mu_j+\mu_{j+1}}{4h_k^3}$	$O(h_k^2)$
$y^{(4)}(\theta_j^k)$	$\frac{M_{j+1}-2M_j+M_{j-1}}{h_k^2}$	$\frac{\mu_{j-4}-3\mu_{j-3}+2\mu_{j-2}+2\mu_{j-1}-3\mu_j+\mu_{j+1}}{2h_k^4}$	$O(h_k^4)$

Table 4. Approximation values of $y^{(5)}(\theta_j^k)$, and $y^{(6)}(\theta_j^k)$.

	Approximate value	Representation in μ_j	Order
$y^{(5)}(\theta_j^k)$	$\frac{M_{j+2}-2M_{j+1}+2M_{j-1}-M_{j-2}}{2h_k^3}$	$\frac{-\mu_{j-5}+3\mu_{j-4}-\mu_{j-3}-5\mu_{j-2}+5\mu_{j-1}+\mu_j-3\mu_{j+1}+\mu_{j+2}}{4h_k^5}$	$O(h_k^2)$
$y^{(6)}(\theta_j^k)$	$\frac{M_{j+2}-4M_{j+1}+6M_j-4M_{j-1}+M_{j-2}}{h_k^4}$	$\frac{\mu_{j-5}-5\mu_{j-4}+9\mu_{j-3}-5\mu_{j-2}-5\mu_{j-1}+9\mu_j-5\mu_{j+1}+\mu_{j+2}}{2h_k^6}$	$O(h_k^2)$

4. Hyperbolic B-splines solutions of 6 BVP

Let $s(\theta) = \sum_{i=-2}^{m_k-3} \mu_i \nu_{i,k}(\theta)$ be the approximate solution of (1) and $\widetilde{s}(\theta) = \sum_{i=-2}^{m_k-3} \widetilde{\mu}_i \nu_{i,k}(\theta)$ be the approximate spline of $s(\theta)$. Discretize (1) at θ_j^k for $j = 3, \cdots, m_k-5$, we get :

$$y^{(6)}(\theta_j^k) + f(\theta_j^k)y(\theta_j^k) = g(\theta_j^k), \quad j = 3, 4, \cdots, m_k - 5. \tag{13}$$

Now, by using Table 2 and 3, the equation (13) becomes

$$\frac{\mu_{j-5} - 5\mu_{j-4} + 9\mu_{j-3} - 5\mu_{j-2} - 5\mu_{j-1} + 9\mu_j - 5\mu_{j+1} + \mu_{j+2}}{2h_k^6} + f_j \frac{\mu_{j-2} + \mu_{j-1}}{2} = g_j + O(h_k^2) \tag{14}$$

where $f_j = f(\theta_j^k)$ and $g_j = g(\theta_j^k)$. Consequently,

$$(\mu_{j-5}-5\mu_{j-4}+9\mu_{j-3}-5\mu_{j-2}-5\mu_{j-1}+9\mu_j-5\mu_{j+1}+\mu_{j+2})+f_j h_k^6(\mu_{j-2}+\mu_{j-1}) = 2g_j h_k^6+O(h_k^8) \tag{15}$$

By dropping $O(h_k^8)$ from (15), we yield a linear system with m_k-7 linear equations in m_k unknowns $\mu_j,\ j = -2, -1, \cdots, m_k-3$. So seven more equations are needed. On the other hand, by using the sixth boundary conditions (2), we get

$$\begin{cases} \mu_{-2} = a_0; \\ \mu_{-1} - \mu_{-2} = 2a_1 \tanh(\frac{h_k}{2}). \end{cases}$$

Thus,

$$\begin{cases} \mu_{-2} = a_0; \\ \mu_{-1} = a_0 + 2a_1 \tanh(\frac{h_k}{2}). \end{cases} \quad (16)$$

By using a similar technique, we get:

$$\begin{cases} \mu_{m_k-3} = b_0; \\ \mu_{m_k-3} - \mu_{m_k-4} = 2b_1 \tanh(\frac{h_k}{2}). \end{cases}$$

Thus,

$$\begin{cases} \mu_{m_k-3} = b_0 \\ \mu_{m_k-4} = b_0 - 2b_1 \tanh(\frac{h_k}{2}) \end{cases} \quad (17)$$

To build the three other equations we propose the following formulas

$$y'_{j-1} + 4y'_j + y'_{j+1} = \frac{3}{h_k}(y_{j+1} - y_{j-1}) + O(h_k^4) \quad (18)$$

and

$$y''_{j-1} + 10y''_j + y''_{j+1} = \frac{12}{h_k^2}(y_{j+1} - 2y_j + y_{j-1}) + O(h_k^4) \quad (19)$$

which can be easily demonstrated using a Taylor series expansion. By formulas (18) and (19), Table 2 and 3, we can construct an approximate formulae for $y^{(6)}(a)$, as follows

$$y^{(6)}(a) = \frac{10M_5 - 61M_4 + 180M_3 - 286M_2 + 250M_1 - 45M_0 - 12\Omega_{h_k}}{h_k^4} + O(h_k^2) \quad (20)$$

with $\Omega_{h_k} = \frac{3}{h_k}(m_2 - m_0)$. By using $M_0 = a_2, m_0 = a_1$ and $tanh(\frac{h_k}{2}) \simeq \frac{h_k}{2}$ for smaller h_k, and by turning (20), the coefficients are determined as follows

$$288\mu_0+223\mu_1-395\mu_2+231\mu_3-71\mu_4+10\mu_5 = 2h_k^6 g_0-2h_k^6 f_0 a_0-250\mu_{-2}+536\mu_{-1}+90h_k^2 a_2-72ha_1+O(h_k^8) \quad (21)$$

The two other equations we need are an approximate of $y^{(6)}(\theta_2^k)$ and $y^{(6)}(\theta_{m_k-4}^k)$, by using $M_0 = a_2, M_{m_k-2} = b_2$ and Table 4, we yield :

$$y^{(6)}(\theta_2^k) = \frac{M_4 - 4M_3 + 6M_2 - 4M_1 + a_2}{h_k^4} + O(h_k^2) \quad (22)$$

and

$$y^{(6)}(\theta_{m_k-4}^k) = \frac{b_2 - 4M_{m_k-3} + 6M_{m_k-4} - 4M_{m_k-5} + M_{m_k-6}}{h_k^4} + O(h_k^2) \quad (23)$$

By turning (22), (23) and (13), the coefficients are determined as follows

$$-6\mu_0 - 5\mu_1 + 9\mu_2 - 5\mu_3 + \mu_4 + f_2 h_k^6(\mu_1 + \mu_0) = 2h_k^6 g_2 + 4\mu_{-2} - 10\mu_{-1} - 2h_k^2 a_2 + O(h_k^8) \quad (24)$$

and

$$\mu_{m_k-9}-5\mu_{m_k-8}+9\mu_{m_k-7}-5\mu_{m_k-6}-6\mu_{m_k-5}+h_k^6 f_{m_k-4}(\mu_{m_k-5}+\mu_{m_k-6})= \\ 2h_k^6 g_{m_k-4}+4\mu_{m_k-3}-10\mu_{m_k-4}-2h_k^2 b_2+O(h_k^8) \qquad (25)$$

Take (21),(24),(15) and (25), we get m_k-4 linear equations with $\mu_i,\ i=0,1,\cdots,m_k-5$, as unknowns since μ_{-2}, μ_{-1}, μ_{m_k-4} and μ_{m_k-3} have been yielded from (16)and (17).
Let $C=[\mu_0,\mu_1,\cdots,\mu_{m_k-5}]^T$, $\widetilde{C}=[\widetilde{\mu}_0,\widetilde{\mu}_1,\cdots,\widetilde{\mu}_{m_k-5}]^T$, $D=[d_0,d_1,\cdots,d_{m_k-5}]^T$, $E=[e_0,e_1,\cdots,e_{m_k-5}]^T$ and using equations (15), (16), (17) and (21), we get

$$(A+h_k^6 FB)C=D+E; \qquad (26)$$

$$(A+h_k^6 FB)\widetilde{C}=D, \qquad (27)$$

where A and B are the following $(m_k-4)\times(m_k-4)$ matrix:

$$A=\begin{pmatrix} 288 & 223 & -395 & 231 & -71 & 10 & & & & & \\ -6 & -5 & 9 & -5 & 1 & & & & & & \\ 9 & -5 & -5 & 9 & -5 & 1 & & & & & \\ -5 & 9 & -5 & -5 & 9 & -5 & 1 & & & & \\ 1 & -5 & 9 & -5 & -5 & 9 & -5 & 1 & & & \\ & \cdots & \cdots & \cdots & \cdots & & & & & & \\ & & \cdots & \cdots & \cdots & \cdots & & & & & \\ & & 1 & -5 & 9 & -5 & -5 & 9 & -5 & 1 & \\ & & & 1 & -5 & 9 & -5 & -5 & 9 & -5 & \\ & & & & 1 & -5 & 9 & -5 & -5 & 9 & \\ & & & & & 1 & -5 & 9 & -5 & -6 & \end{pmatrix},\ B=\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & \cdots & & & \\ & & & \cdots & & \\ & & & 1 & 1 & \\ & & & & 1 & 1 \end{pmatrix}$$

and where D, F and E are the following matrix

$$D=\begin{pmatrix} 2h_k^6 g_0-2h_k^6 f_0 a_0-250\mu_{-2}+536\mu_{-1}+90h_k^2 a_2-72ha_1 \\ 2h_k^6 g_2+4\mu_{-2}-10\mu_{-1}-2h_k^2 a_2 \\ 2h_k^6 g_3-\mu_{-2}+5\mu_{-1} \\ 2h_k^6 g_4-\mu_{-1} \\ 2h_k^6 g_5 \\ \vdots \\ 2h_k^6 g_{m_k-7} \\ 2h_k^6 g_{m_k-6}-\mu_{m_k-4} \\ 2h_k^6 g_{m_k-5}-\mu_{m_k-3}+5\mu_{m_k-4} \\ 2h_k^6 g_{m_k-4}+4\mu_{m_k-3}-10\mu_{m_k-4}-2h_k^2 b_2 \end{pmatrix},$$

$F=\mathrm{diag}(0,f_2,f_3,\cdots,f_{m_k-4})$ and $E=[O(h_k^8),O(h_k^8),\cdots,O(h_k^8)]^T$.

After solving the linear system (27), $\widetilde{\mu}_i,\ i=0,1,\cdots,m_k-5$, $\widetilde{\mu}_{-2}=\mu_{-2}$, $\widetilde{\mu}_{-1}=\mu_{-1}$, $\widetilde{\mu}_{m_k-4}=\mu_{m_k-4}$, and $\widetilde{\mu}_{m_k-3}=\mu_{m_k-3}$ will be used together to get the approximation spline solution $\widetilde{s}(\theta)=\sum_{i=-2}^{m_k-3}\widetilde{\mu}_i\nu_{i,k}(\theta)$.

In order to determine the bound of $\|C-\widetilde{C}\|_\infty$, we need the following lemma.

LEMMA 4.1 *The matrix A is inversible.*

Proof

It suffices to prove that for all $D = [d_0, d_1, \cdots, d_{m_k-5}]^T \in \mathbb{R}^{m_k-4}$ such that $A\ D = 0$, we have $D = 0$. Indeed, If we put

$$z(\theta) = \sum_{j=-2}^{m_k-7} d_{j+2}\nu_{j,k}(\theta) + \sum_{j=m_k-6}^{m_k-3} 0\ \nu_{j,k}(\theta),$$

then $z^{(6)}(\theta_i^k) = 0,$ for all $i = 5, 6, \cdots, m_k - 7.$
On the other hand, using the fact that z is hyperbolic spline function of $\mathcal{C}^1$, we deduce that $z(\theta) = \alpha + \beta\cosh(\theta) + \gamma\sinh(\theta)$ in $[\theta_5^k, \theta_6^k]$. From $z^{(6)}(\theta_5^k) = 0$ and $z^{(6)}(\theta_6^k) = 0$, we have,

$$\begin{cases} \beta\cosh(\theta_5^k) + \gamma\sinh(\theta_5^k) = 0; \\ \beta\cosh(\theta_6^k) + \gamma\sinh(\theta_6^k) = 0; \end{cases}$$

we deduce $\beta = 0$ and $\gamma = 0$. Consequently, $z^{(6)}(\theta) = 0$ and $z'(\theta) = 0$ in all the interval $[\theta_5^k, \theta_6^k]$. In a same way , we have in all the other subintervals of $[\theta_5^k, \theta_{m_k-7}^k]$, $z^{(6)}(\theta) = 0$ and $z'(\theta) = 0$.
Consequently, we have

$$\begin{cases} z'(\theta_5^k) = 0 \\ z'(\theta_6^k) = 0 \\ \quad\vdots \\ z'(\theta_{m_k-8}^k) = 0 \\ z'(\theta_{m_k-7}^k) = 0 \end{cases}$$

thus,

$$\begin{cases} \frac{d_6-d_5}{2\tanh(\frac{h_k}{2})} = 0 \\ \frac{d_7-d_6}{2\tanh(\frac{h_k}{2})} = 0 \\ \quad\vdots \\ \frac{d_{m_k-7}-d_{m_k-8}}{2\tanh(\frac{h_k}{2})} = 0 \\ \frac{d_{m_k-6}-d_{m_k-7}}{2\tanh(\frac{h_k}{2})} = 0 \end{cases}$$

so $d_5 = d_6 = d_7 = \cdots = d_{m_k-7} = d_{m_k-6}$, we have also

$$\begin{cases} 288d_0 + 223d_1 - 395d_2 + 231d_3 - 71d_4 + 10d_5 = 0 \\ -6d_0 - 5d_1 + 9d_2 - 5d_3 + d_4 = 0 \\ 9d_0 - 5d_1 - 5d_2 + 9d_3 - 5d_4 + d_5 = 0 \\ -5d_0 + 9d_1 - 5d_2 - 5d_3 + 9d_4 - 5d_5 + d_6 = 0 \\ \quad\vdots \\ d_{m_k-10} - 5d_{m_k-9} + 9d_{m_k-8} - 5d_{m_k-7} - 5d_{m_k-6} + 9d_{m_k-5} = 0 \\ d_{m_k-9} - 5d_{m_k-8} + 9d_{m_k-7} - 5d_{m_k-6} + d_{m_k-5} = 0 \end{cases} \tag{28}$$

finally we have $d_0 = d_1 = d_2 = \ldots = d_{m_k-5} = 0$ which in turn gives $D = 0$. ■

Proposition 4.2 *If we assume that* $h_k^6\|A^{-1}\|_\infty\|B\|_\infty\|F\|_\infty \leqslant \frac{1}{2}$, *then there exists* *a constant K which depends only of the functions f and g such that*

$$\|C-\widetilde{C}\|_\infty \leqslant Kh_k.$$

Proof

From (26), (27) and Lemma 1, we have
$C-\widetilde{C}=(I+h_k^6A^{-1}BF)^{-1}A^{-1}E$. Since $E=O(h_k^8)$, there exists a constant K_1 such that $\|E\|_\infty \leqslant K_1h_k^8$. Consequently

$$\|C-\widetilde{C}\|_\infty \leqslant K_1h_k^8\|A^{-1}\|_\infty\|(I+h_k^6A^{-1}BF)^{-1}\|_\infty.$$

Using the inequality $h_k^6\|A^{-1}\|_\infty\|B\|_\infty\|F\|_\infty \leqslant \frac{1}{2}$, and $\|B\|_\infty=2$, we deduce that

$$\|C-\widetilde{C}\|_\infty \leqslant K_1\frac{h_k^7\|A^{-1}\|_\infty}{1-h_k^6\|A^{-1}\|_\infty\|B\|_\infty\|F\|_\infty}h_k \leqslant \frac{K_1(b-a)}{\|2F\|_\infty}h_k.$$

Hence, $|s(\theta)-\widetilde{s(\theta)}| \leqslant \|C-\widetilde{C}\|_\infty \sum_{i=-2}^{m_k-3}\nu_{i,k}(\theta)=\|C-\widetilde{C}\|_\infty \simeq O(h_k)$.

Therefore, we get

$$|y(\theta)-\widetilde{s(\theta)}| \leqslant |y(\theta)-s(\theta)|+|s(\theta)-\widetilde{s(\theta)}| \leqslant O(h_k^4)+O(h_k)\simeq O(h_k).$$

■

5. Numerical results

To test our method, we considered three examples of sixth-order boundary value problems (6BVPs) of the form ((2),(1)).

Example 5.1

We consider the following boundary-value problem

$$\begin{cases} y^{(6)}+xy=-(24+11x+x^3)e^x, & x\in[0,1]; \\ y(0)=0,\ y^{'}(0)=1,\ y^{''}(0)=0, \\ y(1)=0,\ y^{'}(1)=-e,\ y^{''}(1)=-4e, \end{cases} \tag{29}$$

The exact solution is $y(x)=x(1-x)e^x$.

As an example, we give in Figure 2, the graph of the exact solution and the graph of hyperbolic-spline solution with $k=5$. In Table 5, we give the maximum absolute errors computed at various points of the interval $[0,1]$, for problem (29), and the convergence order.

Table 5. Maximum absolute error and order of convergence for Problem (29).

k	4	5	6	7	8	9
Error	6.501e-002	3.093e-002	1.509e-002	7.461e-003	3.709e-003	1.844e-003
Order	1.0717	1.0346	1.0168	1.0083	1.0079	

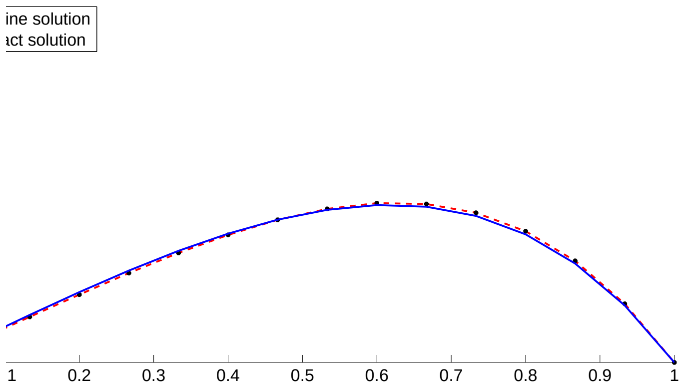

Figure 2. Exact solution and hyperbolic-spline solution for $k = 5$.

Example 5.2

Consider the following boundary-value problem

$$\begin{cases} y^{(6)} - y = 6(1-2x)\sinh(x) - 21\cosh(x), & x \in [0,1]; \\ y(0) = 0,\ y^{'}(0) = 1,\ y^{''}(0) = -2, & \\ y(1) = 0,\ y'(1) = -\cosh(1),\ y^{''}(1) = -2(cosh(1) + sinh(1)), & \end{cases} \tag{30}$$

The exact solution is $y(x) = x(1-x)\cosh(x)$.
Result has been shown for different values of k in Table 6.

Table 6. Maximum absolute error and order of convergence for Problem (30).

k	4	5	6	7	8	9
Error	4.433e-002	2.095e-002	1.010e-002	4.881e-003	2.314e-003	1.047e-003
Order	1.0813	1.0525	1.0491	1.0767	1.1441	

Example 5.3

Consider the following boundary-value problem

$$\begin{cases} y^{(6)} - y = -6\cosh(x), & x \in [0,1]; \\ y(0) = 0,\ y^{'}(0) = 1,\ y^{''}(0) = -2, & . \\ y(1) = 0,\ y'(1) = -\sinh(1),\ y^{''}(1) = -2\cosh(1), & \end{cases} \tag{31}$$

The exact solution is $y(x) = (1-x)\sinh(x)$.
Result has been shown for different values of k in Table 7.

Table 7. Maximum absolute error and order of convergence for Problem (31).

k	4	5	6	7	8	9
Error	3.887e-002	1.828e-002	8.878e-003	4.376e-003	2.172e-003	1.088e-003
Order	1.0883	1.0419	1.0206	1.0105	0.9973	

6. Conclusion

Numerical results confirm the order of convergence predicted by the analysis. Experimental results demonstrate that our method is more effective for the problems where the exact solution is hyperbolic (see Tables 6 and 7). The construction of this type of approximants requires the solution of linear systems of lower order compared with the methods given in the literature, see for example [6, 7]. So, its extension to general linear and nonlinear boundary value problems using the hyperbolic (tension) B-splines of order more than 4, 5, $\cdots$ is under progress.

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