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Fusion Frames in Hilbert Spaces

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Abstract. Fusion frames are an extension to frames that provide a framework for applications and providing efficient and robust information processing algorithms. In this article we study the erasure of subspaces of a fusion frame.

Keywords: Frame, Fusion Frame, Exact fusion frame, Bessel fusion sequence, Orthonormal fusion basis.

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1. Introduction

The concept of frame in Hilbert spaces has been introduced by Duffin and Schaeffer[11] in 1952 to study some deep problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann, and Meyer[9], frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames. Traditionally, frames have been used in signal processing, image processing, data compression and sampling theory. A frame is a countable family of vectors in a separable Hilbert space which allows a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame elements. The fusion frames which were introduced by Casazza and Kutyniok in [3] and Fornasier in [13] are a natural generalization of frame theory and related to the construction of global frame from local frames in Hilbert spaces. In addition a similar idea was used by Sun [18]. We extend some of known results of frames to fusion frames.

The paper is organized as follows: In §2, we first briefly recall the definitions and basic properties, then we get several characterizations of fusion frames. In §3, we study the erasure of subspaces of a fusion frame.

Let $\mathcal{H}$ be a separable Hilbert space and let I, J, J_i be countable (or finite) index sets. If W is a closed subspace of $\mathcal{H}$, we denote the orthogonal projection of $\mathcal{H}$

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onto W by π_W. Let $B(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$, then we denote the range and the null space of $T \in B(\mathcal{H},\mathcal{K})$ by $\mathcal{R}_T$ and $\mathcal{N}_T$, respectively.

A family of vectors $\mathcal{F} = \{f_i\}_{i\in I}$ is called a frame for $\mathcal{H}$ if there exist constants $0 < A \leqslant B < \infty$ such that,

$$A\|f\|^2 \leqslant \sum_{i\in I} |< f\ ,\ f_i >|^2 \leqslant B\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{1}$$

The constants A and B are called frame bounds. If we only have the right-hand inequality of (1), we call $\mathcal{F}$ a Bessel sequence. The representation space associated with a frame is $\ell^2(I)$. If $\mathcal{F} = \{f_i\}_{i\in I}$ is a Bessel sequence, the *synthesis operator* for $\mathcal{F}$ is the bounded linear operator $T_\mathcal{F} : \ell^2(I) \to \mathcal{H}$, given by $T_\mathcal{F}(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i$. The *analysis operator* for $\mathcal{F}$ is $T^*_\mathcal{F} : \mathcal{H} \to \ell^2(I)$ and satisfies: $T^*_\mathcal{F} f = \{< f, f_i >\}_{i\in I}$. By composing $T_\mathcal{F}$ and $T^*_\mathcal{F}$ we obtain the frame operator

$$S_\mathcal{F} : \mathcal{H} \to \mathcal{H}, \qquad S_\mathcal{F} f = T_\mathcal{F} T^*_\mathcal{F} f = \sum_{i\in I} < f, f_i > f_i,$$

which is a positive, self-adjoint and invertible operator and the following reconstruction formula holds for all $f \in \mathcal{H}$:

$$f = \sum_{i\in I} < f, \widetilde{f}_i > f_i = \sum_{i\in I} < f, f_i > \widetilde{f}_i.$$

where $\widetilde{f}_i = S^{-1}_\mathcal{F} f_i$ $(i \in I)$. Also $\widetilde{\mathcal{F}} = \{\widetilde{f}_i\}_{i\in I}$ is a frame for $\mathcal{H}$ and called the *canonical dual* frame of $\mathcal{F} = \{f_i\}_{i\in I}$. In general, the Bessel sequence $\mathcal{G} = \{g_i\}_{i\in I}$ is called a *dual* of the frame $\mathcal{F} = \{f_i\}_{i\in I}$ if the following formula holds

$$f = T_\mathcal{F} T^*_\mathcal{G}(f) = \sum_{i\in I} < f, g_i > f_i \qquad \forall f \in \mathcal{H}.$$

Moreover a Riesz basis for $\mathcal{H}$ is a family of the form $\{U(e_j)\}_{j\in J}$, where $\{e_j\}_{j\in J}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator. For more details about the theory and applications of frames and Riesz bases we refer the reader to [7, 8, 12, 15, 16, 19].

2. Review of Fusion Frames

In this section we briefly recall the basic definitions and results which we will need later. We also present some useful new results about fusion frames. For more information we refer the reader to [1, 5, 6, 10, 14, 17].

Let $\mathcal{W} = \{W_i\}_{i\in I}$ be a sequence of closed subspaces in $\mathcal{H}$, and let $\mathcal{A} = \{\alpha_i\}_{i\in I}$ be a family of weights, i.e., $\alpha_i > 0$ for all $i \in I$. We say that $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$, if there exist constants $0 < C \leqslant D < \infty$ such that,

$$C\|f\|^2 \leqslant \sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(f)\|^2 \leqslant D\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{2}$$

The numbers C, D are called the fusion frame bounds. The family $\mathcal{W}_\alpha$ is called a C-tight fusion frame if $C = D$, it is a Parseval fusion frame if $C = D = 1$, and a v-uniform if $\alpha = \alpha_i = \alpha_j$ for all $i, j \in I$. If the right-hand inequality of (2) holds, then

we say that $\mathcal{W}_\alpha$ is a Bessel fusion sequence with Bessel fusion bound D. Moreover $\mathcal{W} = \{W_i\}_{i\in I}$ is called an orthonormal fusion basis for $\mathcal{H}$ if $\mathcal{H} = \bigoplus_{i\in I} W_i$.

In order to analyze a signal $f \in \mathcal{H}$. we denote the representation space associated with a fusion frame by

$$\ell^2(\mathcal{H}, I) = \Big\{\{f_k\}_{k\in I} | f_k \in \mathcal{H} \text{ and } \sum_{k\in I} \|f_k\|^2 < \infty\Big\}$$

The *synthesis* operator $T_{\mathcal{W}_\alpha} : \ell^2(\mathcal{H}, I) \longrightarrow \mathcal{H}$ is defined by

$$T_{\mathcal{W}_\alpha}(\{f_i\}_{i\in I}) = \sum_{i\in I} \alpha_i \pi_{W_i}(f_i) \qquad \forall \{f_i\}_{i\in I} \in \ell^2(\mathcal{H}, I)$$

and the associated adjoint operator $T^*_{\mathcal{W}_\alpha} : H \longrightarrow \ell^2(H, I)$ given by

$$T^*_{\mathcal{W}_\alpha}(f) = \{\alpha_i \pi_{W_i}(f)\}_{i\in I} \qquad \forall f \in \mathcal{H}$$

is called the *analysis* operator. Let $\{e_j\}_{j\in J}$ be an orthonormal basis for $\mathcal{H}$. Define $e_{ij} = \{\delta_{ik} e_j\}_{k\in I}$ for all $i \in I, j \in J$ where δ_{ik} is the Kronecker delta, then $\{e_{ij}\}_{i\in I, j\in J}$ is an orthonormal basis for $\ell^2(\mathcal{H}, I)$. The sequence $\{e_{ij}\}_{i\in I, j\in J}$ is called the associated orthonormal basis to $\{e_j\}_{j\in J}$ in $\ell^2(\mathcal{H}, I)$. By composing $T_{\mathcal{W}_\alpha}$ and $T^*_{\mathcal{W}_\alpha}$ we obtain the frame operator

$$S_{\mathcal{W}_\alpha} : \mathcal{H} \longrightarrow \mathcal{H} \quad , \quad S_{\mathcal{W}_\alpha}(f) = T_{\mathcal{W}_\alpha} T^*_{\mathcal{W}_\alpha}(f) = \sum_{i\in I} \alpha_i^2 \pi_{W_i}(f), \tag{3}$$

which is a bounded, invertible, and positive operator. This provides the reconstruction formula

$$f = \sum_{i\in I} \alpha_i^2 S^{-1}_{\mathcal{W}_\alpha} \pi_{W_i}(f) = \sum_{i\in I} \alpha_i^2 \pi_{W_i} S^{-1}_{\mathcal{W}_\alpha}(f) \qquad \forall f \in \mathcal{H}. \tag{4}$$

Indeed, it can be proven that, a sequence $\mathcal{W}_\alpha$ is a Bessel fusion sequence with Bessel fusion bound D for $\mathcal{H}$ if and only if the synthesis operator $T_{\mathcal{W}_\alpha}$ is a well-defined bounded operator from $\ell^2(\mathcal{H}, I)$ into $\mathcal{H}$ and $\|T_{\mathcal{W}_\alpha}\| \leqslant \sqrt{D}$.

The next theorem is analog of a well-known result in abstract frame theory. The similar characterization of fusion frames has been proved in [3].

THEOREM 2.1 *Let $\mathcal{W} = \{W_i\}_{i\in I}$ be a family of closed subspaces in $\mathcal{H}$, and let $\mathcal{A} = \{\alpha_i\}_{i\in I}$ be a family of weights. Then the following conditions are equivalent:*

(*i*) *$\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$.*
(*ii*) *The synthesis operator $T_{\mathcal{W}_\alpha}$ is bounded, linear and onto.*
(*iii*) *The analysis operator $T^*_{\mathcal{W}_\alpha}$ is injective with closed range.*

Proof This claim holds in an analogous way as in frame theory. ■

THEOREM 2.2 *Let $\{e_j\}_{j\in J}$ be an orthonormal basis for $\mathcal{H}$, and let $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i\in I}$ be a Parseval fusion frame for $\mathcal{H}$. Then there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an orthonormal fusion basis $\{N_i\}_{i\in I}$ for $\mathcal{K}$ such that $P(N_i) = W_i$ $(i \in I)$, where P is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$.*

Proof Let $\mathcal{K} = \ell^2(\mathcal{H}, I)$ and let $\Theta : \mathcal{H} \to \mathcal{K}$ defined by

$$\Theta(f) = \{\alpha_k \pi_{W_k}(f)\}_{k\in I} = \sum_{i\in I}\sum_{j\in J} < \alpha_i \pi_{W_i}(f), e_j > e_{ij}$$

for all $f \in \mathcal{H}$. Since $\mathcal{W}_\alpha$ is a Parseval fusion frame for $\mathcal{H}$, we have

$$\|\Theta(f)\|^2 = \sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(f)\|^2 = \|f\|^2.$$

Thus Θ is well- defined and is an isometry. So we can embed $\mathcal{H}$ into $\mathcal{K}$. By identifying $\mathcal{H}$ with $\Theta(\mathcal{H})$, we can regard $\mathcal{H}$ as a closed subspace of $\mathcal{K}$. Let $P : \mathcal{K} \to \Theta(\mathcal{H})$ be the orthogonal projection. Then for each $i \in I, j \in J$ and $f \in \mathcal{H}$ we obtain.

$$\begin{aligned} < \Theta(f), e_{ij} > &=< \sum_{m\in I}\sum_{n\in J} < \alpha_m \pi_{W_m}(f), e_n > e_{mn}, e_{ij} > \\ &= \sum_{m\in I}\sum_{n\in J} < \alpha_m \pi_{W_m}(f), e_n >< e_{mn}, e_{ij} > \\ &=< \alpha_i \pi_{W_i}(f), e_j >=< f, \alpha_i \pi_{W_i}(e_j) > \end{aligned}$$

Further

$$\begin{aligned} < \Theta(f), P(e_{ij}) > &=< P\Theta(f), e_{ij} >=< \Theta(f), e_{ij} > \\ &=< f, \alpha_i \pi_{W_{i(e_j)}} >=< \Theta(f), \Theta(\alpha_i \pi_{W_i}(e_j)) > \end{aligned}$$

Thus $P(e_{ij}) - \Theta(\alpha_i \pi_{W_i}(e_j)) \perp \Theta(\mathcal{H})$. But $\mathcal{R}_P = \Theta(\mathcal{H})$ hence

$$P(e_{ij}) = \Theta(\alpha_i \pi_{W_i}(e_j)).$$

If for each $i \in I$ we take $N_i = \overline{\text{span}}\{e_{ij}\}_{j\in J}$, then $\{N_i\}_{i\in I}$ is an orthonormal fusion basis for $\mathcal{K}$. We claim that $P(N_i) = \Theta(W_i)$ $(i \in I)$. Let $f \in N_i$ and write $f = \sum_{j\in J} c_j e_{ij}$, then

$$\begin{aligned} P(f) &= \sum_{j\in J} c_j P(e_{ij}) = \sum_{j\in J} c_j \Theta(\alpha_i \pi_{W_i}(e_j)) \\ &= \Theta(\sum_{j\in J} c_j \alpha_i \pi_{W_i}(e_j)) \in \Theta(W_i) \end{aligned}$$

For the converse let $f \in W_i$, then

$$\begin{aligned} \Theta(f) = \Theta(\pi_{W_i}(f)) &= \Theta\pi_{W_i}(\sum_{j\in J} < f, e_j > e_j) = \sum_{j\in J} \frac{< x, e_j >}{\alpha_i} \Theta(\alpha_i \pi_{W_i}(e_j)) \\ &= \sum_{j\in J} \frac{< x, e_j >}{\alpha_i} P(e_{ij}) = P(\sum_{j\in J} \frac{< f, e_j >}{\alpha_i} e_{ij}) \in P(N_i) \end{aligned}$$

It follows that $P(N_i) = W_i$. ■

THEOREM 2.3 *Let $\mathcal{W}_\alpha$ be a fusion frame for $\mathcal{H}$. Then there is an orthonormal fusion basis $\mathcal{N} = \{N_i\}_{i\in I}$ for $\ell^2(\mathcal{H}, I)$ such that $T_{\mathcal{W}_\alpha}(N_i) = W_i$ for every $i \in I$.*

Proof Let $\{e_j\}_{j\in J}$ be an orthonormal basis for $\mathcal{H}$, then $\mathcal{F}_i = \{\pi_{W_i}(e_j)\}_{j\in J}$ is a Parseval frame for W_i and hence $W_i = \overline{\text{span}}\{\pi_{W_i}(e_j)\}_{j\in J}$. Let $\{e_{ij}\}_{i\in I, j\in J}$ be the associated orthonormal basis to $\{e_j\}_{j\in J}$ for $\ell^2(\mathcal{H}, I)$, and let $N_i = \overline{\text{span}}\{e_{ij}\}_{j\in J}$. Then $\mathcal{N} = \{N_i\}_{i\in I}$ is an orthonormal fusion basis for $\ell^2(\mathcal{H}, I)$. Now if $f \in N_i$, then we can write $f = \sum_{j\in J} < f, e_{ij} > e_{ij}$, thus

$$T_{\mathcal{W}_\alpha}(f) = \sum_{j\in J} < f, e_{ij} > T_{\mathcal{W}_\alpha}(e_{ij}) = \sum_{j\in J} \alpha_i < f, e_{ij} > \pi_{W_i}(e_j),$$

this shows that $T_{\mathcal{W}_\alpha}(f) \in W_i$. Finally if $g \in W_i$, then we have

$$\begin{aligned} g &= \sum_{j\in J} < g, \pi_{W_i}(e_j) > \pi_{W_i}(e_j) = \sum_{j\in J} \frac{1}{\alpha_i} < g, e_j > T_{\mathcal{W}_\alpha}(e_{ij}) \\ &= T_{\mathcal{W}_\alpha}\big(\sum_{j\in J} \frac{1}{\alpha_i} < g, e_j > e_{ij}\big). \end{aligned}$$

Thus $g \in T_{\mathcal{W}_\alpha}(N_i)$. Altogether we have $T_{\mathcal{W}_\alpha}(N_i) = W_i$. ■

THEOREM 2.4 *Let $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$, and let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for each subspace W_i. Then there exists an orthonormal fusion basis $\mathcal{N} = \{N_i\}_{i\in I}$ for $\mathcal{H}$ and a bounded, surjective operator $U : \mathcal{H} \longrightarrow \mathcal{H}$ such that $U(N_i) = W_i$.*

Proof According [3, Theorem 3.2] $\{\alpha_i e_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$. Let $\{u_{ij}\}_{i\in I, j\in J_i}$ be an arbitrary orthonormal basis for $\mathcal{H}$. By [7, Theorem 5.5.5] there is a bounded, surjective operator $U : \mathcal{H} \longrightarrow \mathcal{H}$ such that $U(u_{ij}) = \alpha_i e_{ij}$ for all $i \in I, j \in J_i$. Define $N_i = \overline{span}\{u_{ij}\}_{j\in J_i}$, then $\mathcal{N} = \{N_i\}_{i\in I}$ is an orthonormal fusion basis for $\mathcal{H}$ and $U(N_i) = W_i$. ■

THEOREM 2.5 *Let $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i\in I}$ and $\mathcal{Z}_\beta = \{(Z_j, \beta_j)\}_{j\in J}$ be fusion frame sequences and let P_1, P_2 denote the orthogonal projections of $\mathcal{H}$ onto $\overline{\text{span}}\{W_i\}_{i\in I}$, $\overline{\text{span}}\{Z_j\}_{j\in J}$, respectively. Then*

(*i*) *The family $\{(W_i, Z_j, \alpha_i, \beta_j)\}_{i\in I, j\in J}$ is a Bessel fusion sequence for $\mathcal{H}$.*
(*ii*) *The family $\{(W_i, Z_j, \alpha_i, \beta_j)\}_{i\in I, j\in J}$ is a fusion frame sequence if and only if there exists a $K > 0$ such that for all $f \in \overline{\text{span}}\{W_i, Z_j\}_{i\in I, j\in J}$*

$$\|P_1(f)\|^2 + \|P_2(f)\|^2 \geqslant K\|f\|^2.$$

Proof Let C_1, D_1 be the frame bounds for $\mathcal{W}_\alpha$ and let C_2, D_2 be the bounds for $\mathcal{Z}_\beta$ respectively. Then

(i) For each $f \in \mathcal{H}$ we have

$$\begin{aligned} \sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(f)\|^2 + \sum_{j\in J} \beta_j^2 \|\pi_{Z_j}(f)\|^2 &= \sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(P_1 f)\|^2 + \sum_{j\in J} \beta_j^2 \|\pi_{Z_j}(P_2 f)\|^2 \\ &\leqslant D_1\|P_1(f)\|^2 + D_2\|P_2(f)\|^2 \leqslant 2\max\{D_1, D_2\}\|f\|^2. \end{aligned}$$

(ii) Let $\{(W_i, Z_j, \alpha_i, \beta_j)\}_{i\in I, j\in J}$ be a fusion frame sequence with fusion frame

bounds C, D. Then for all $f \in \overline{\text{span}}\{W_i, Z_j\}_{i\in I, j\in J}$ we have

$$C\|f\|^2 \leqslant \sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(f)\|^2 + \sum_{j\in J} \beta_j^2 \|\pi_{Z_j}(f)\|^2 = \sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(P_1 f)\|^2 + \sum_{j\in J} \beta_j^2 \|\pi_{Z_j}(P_2 f)\|^2$$
$$\leqslant D_1\|P_1(f)\|^2 + D_2\|P_2(f)\|^2 \leqslant \max\{D_1, D_2\}(\|P_1(f)\|^2 + \|P_2(f)\|^2).$$

To prove the converse implication suppose there exist some $K > 0$ such that for all $f \in \overline{\text{span}}\{W_i, Z_j\}_{i\in I, j\in J}$

$$\|P_1(f)\|^2 + \|P_2(f)\|^2 \geqslant K\|f\|^2.$$

Then we have

$$\begin{aligned}\sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(f)\|^2 + \sum_{j\in J} \beta_j^2 \|\pi_{Z_j}(f)\|^2 &= \sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(P_1 f)\|^2 + \sum_{j\in J} \beta_j^2 \|\pi_{Z_j}(P_2 f)\|^2 \\ &\geqslant C_1\|P_1(f)\|^2 + C_2\|P_2(f)\|^2 \\ &\geqslant \min\{C_1, C_2\}(\|P_1(f)\|^2 + \|P_2(f)\|^2) \\ &\geqslant K\min\{C_1, C_2\}\|f\|^2.\end{aligned}$$

for all $f \in \overline{\text{span}}\{W_i, Z_j\}_{i\in I, j\in J}$. The upper frame bound follows immediately from(i). ■

In the next theorem we consider direct sum of fusion frames which is a fusion frame for their direct sum space.

THEOREM 2.6 *Let* $\big\{\{(W_{ij}, \alpha_i)\}_{i\in I} : j = 1, 2, \ldots, k\big\}$ *be a* k*-tuple of fusion frames for Hilbert spaces* $\mathcal{H}_j (1 \leqslant j \leqslant k)$*, respectively. Then* $\{(W_{i1} \oplus W_{i2} \oplus \cdots \oplus W_{ik}, \alpha_i)\}_{i\in I}$ *is a fusion frame for* $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_k$.

Proof It is enough to prove the theorem for $k = 2$. Let C_j, D_j be the frame bounds for $\{(W_{ij}, \alpha_i)\}_{i\in I}$ $(j = 1, 2)$. Since $\pi_{W_{i1}\oplus W_{i2}} = \pi_{W_{i1}} \oplus \pi_{W_{i2}}$ for all $i \in I$, then we have

$$\begin{aligned}\min\{C_1, C_2\}\|f \oplus g\|^2 &= \min\{C_1, C_2\}(\|f\|^2 + \|g\|^2) \leqslant C_1\|f\|^2 + C_2\|g\|^2 \\ &\leqslant \sum_{i\in I} \alpha_i^2 \|\pi_{W_{i1}}(f)\|^2 + \sum_{i\in I} \alpha_i^2 \|\pi_{W_{i2}}(g)\|^2 \leqslant D_1\|f\|^2 + D_2\|g\|^2 \\ &\leqslant \max\{D_1, D_2\}(\|f\|^2 + \|g\|^2) = \max\{D_1, D_2\}\|f \oplus g\|^2,\end{aligned}$$

for all $f \in \mathcal{H}_1$. $g \in \mathcal{H}_2$. Now we observe that

$$\sum_{i\in I} \alpha_i^2 \|\pi_{W_{i1}}(f)\|^2 + \sum_{i\in I} \alpha_i^2 \|\pi_{W_{i2}}(g)\|^2 = \sum_{i\in I} \alpha_i^2 \|\pi_{W_{i1}\oplus W_{i2}}(f \oplus g)\|^2.$$

This shows that $\{(W_{i1} \oplus W_{i2}, \alpha_i)\}_{i\in I}$ is a fusion frame with fusion frame bounds $\min\{C_1, C_2\}$ and $\max\{D_1, D_2\}$. ■

COROLLARY 2.7 *Let* $\big\{\{(W_{ij}, \alpha_i)\}_{i\in I} : j = 1, 2, \ldots, k\big\}$ *be a* k*-tuple of Parseval fusion frames for* $\mathcal{H}_j$ *(*$1 \leqslant j \leqslant k$*), respectively. Then* $\{(W_{i1} \oplus W_{i2} \oplus \cdots \oplus W_{ik}, \alpha_i)\}_{i\in I}$ *is a Parseval fusion frame for* $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_k$.

Proof This follows immediately from Theorem 2.6. ■

3. Erasures of Subspaces

Our purpose of this section is to study the conditions which under to remove an element from a fusion frame, again we obtain another fusion frame. We say that $\mathcal{W}_v$ is an exact fusion frame, if it ceases to be fusion frame whenever anyone of its element is removed.

Now we state an useful result of fusion frames that proved in [2].

THEOREM 3.1 *Let $\mathcal{W}_v = \{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$, and let $i \in I$ and $T_i \in B(\mathcal{H}, W_i)$. If $f \in \mathcal{H}$ and $f = \sum_{i\in I} v_i^2 T_i(f)$. Then we have*

(i) $\sum_{i\in I} v_i^2 \parallel T_i(f) \parallel^2 = \sum_{i\in I} v_i^2 \parallel \pi_{W_i} S^{-1}_{\mathcal{W}_v}(f) - T_i(f) \parallel^2 + \sum_{i\in I} v_i^2 \parallel \pi_{W_i} S^{-1}_{\mathcal{W}_v}(f) \parallel^2$

(ii) $\sum_{i\in I} v_i^2 \parallel \pi_{W_i} S^{-1}_{\mathcal{W}_v}(f) - \pi_{W_i}(f) \parallel^2 + \sum_{i\in I} v_i^2 \parallel \pi_{W_i} S^{-1}_{\mathcal{W}_v}(f) - T_i(f) \parallel^2 = \sum_{i\in I} v_i^2 \parallel T_i(f) - \pi_{W_i}(f) \parallel^2 .$

Proof See [2, Theorem 2.2]. ■

COROLLARY 3.2 *Let $\mathcal{W}_v$ be a fusion frame for $\mathcal{H}$, and let $T_{\mathcal{W}_v}$ be associated synthesis operator. Then the pseudo-inverse operator $T^{\dagger}_{\mathcal{W}_v} : \mathcal{H} \longrightarrow \ell^2(\mathcal{H}, I)$ is given by*

$$T^{\dagger}_{\mathcal{W}_v}(f) = \{v_i \pi_{W_i} S^{-1}_{\mathcal{W}_v}(f)\}_{i\in I} \qquad \forall f \in \mathcal{H}. \tag{5}$$

Proof Let $f \in \mathcal{H}$, then by [8, Theorem 2.1] the equation $T_{\mathcal{W}_v}(\{f_i\}_{i\in I}) = f$ has exactly one solution with minimal norm, this solution is $T^{\dagger}_{\mathcal{W}_v}(f)$. The result now follows by combining (4) and Theorem 3.1(i). ■

The following theorem gives the conditions which under that with remove an element from a fusion frame, again we obtain either another fusion frame or an incomplete set.

THEOREM 3.3 *Let $\mathcal{W}_v$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds C, D and let $j \in I$ and $\mathcal{W}_v^{\mathcal{J}} = \{(W_i, v_i)\}_{i\in I, i\neq j}$ then.*

(i) If there is some $g \in W_j - \{0\}$ such that $\pi_{W_j} S^{-1}_{\mathcal{W}_v}(g) = \frac{1}{v_j^2} g$, then $\mathcal{W}^{\mathcal{J}} = \{W_i\}_{i\in I, i\neq j}$ is an incomplete set in $\mathcal{H}$.

(ii) If $Id_{\mathcal{H}} - v_j^2 \pi_{W_j} S^{-1}_{\mathcal{W}_v}$ is a bounded, invertible operator on $\mathcal{H}$. Then $\mathcal{W}_v^{\mathcal{J}}$ is a fusion frame with fusion frame bounds $\frac{C^2}{C + v_j^2 \|(Id_{\mathcal{H}} - v_j^2 \pi_{W_j} S^{-1}_{\mathcal{W}_v})^{-1}\|^2}$ and D.

Proof (i) Define $T_i : \mathcal{H} \longrightarrow W_i$ by $T_i = v_i^{-2} \delta_{ij} \pi_{W_i}$ for all $i \in I$, where δ_{ij} is the Kronecker delta. Then we have $\sum_{i\in I} v_i^2 T_i(g) = \sum_{i\in I} \delta_{ij} \pi_{W_i}(g) = \pi_{W_j}(g) = g$. Now by Theorem 3.1$(i)$ we compute

$$\sum_{i\in I} v_i^2 \parallel v_i^{-2} \delta_{ij} \pi_{W_i}(g) \parallel^2 = \sum_{i\in I} v_i^2 \parallel \pi_{W_i} S^{-1}_{\mathcal{W}_v}(g) - v_i^{-2} \delta_{ij} \pi_{W_i}(g) \parallel^2 + \sum_{i\in I} v_i^2 \parallel \pi_{W_i} S^{-1}_{\mathcal{W}_v}(g) \parallel^2 .$$

Consequently,

$$\frac{1}{v_j^2} \parallel g \parallel^2 = \frac{1}{v_j^2} \parallel g \parallel^2 + 2 \sum_{\substack{i\in I \\ i\neq j}} v_i^2 \parallel \pi_{W_i} S^{-1}_{\mathcal{W}_v}(g) \parallel^2 .$$

Hence $S^{-1}_{\mathcal{W}_v}(g) \in \big(\overline{span}\{W_i\}_{i\in I, i\neq j}\big)^{\perp}$, since $S^{-1}_{\mathcal{W}_v}(g) \neq 0$, it follows that $\mathcal{W}^{\mathcal{J}}$ is an incomplete set in $\mathcal{H}$.
(ii) By (4) for every $f \in \mathcal{H}$, we have $f = \sum_{i\in I} v_i^2 S^{-1}_{\mathcal{W}_v} \pi_{W_i}(f)$, hence

$$\pi_{W_j}(f) = \sum_{i\in I} v_i^2 \pi_{W_j} S^{-1}_{\mathcal{W}_v} \pi_{W_i}(f).$$

Consequently

$$\big(Id_H - v_j^2 \pi_{W_j} S^{-1}_{\mathcal{W}_v}\big)\pi_{W_j}(f) = \sum_{\substack{i\in I \\ i\neq j}} v_i^2 \pi_{W_j} S^{-1}_{\mathcal{W}_v} \pi_{W_i}(f)\ .$$

Now by using the Schwarz inequality we compute

$$\begin{aligned}
\| \big(Id_H - v_j^2 \pi_{W_j} S^{-1}_{\mathcal{W}_v}\big)\pi_{W_j}(f) \|^2 &= \| \sum_{\substack{i\in I \\ i\neq j}} v_i^2 \pi_{W_j} S^{-1}_{\mathcal{W}_v} \pi_{W_i}(f) \|^2 \\
&= \sup_{\|g\|=1} |< g\ ,\ \sum_{\substack{i\in I \\ i\neq j}} v_i^2 \pi_{W_j} S^{-1}_{\mathcal{W}_v} \pi_{W_i}(f) >|^2 \\
&= \sup_{\|g\|=1} |\sum_{\substack{i\in I \\ i\neq j}} v_i^2 < \pi_{W_i} S^{-1}_{\mathcal{W}_v} \pi_{W_j}(g)\ ,\ \pi_{W_i}(f) >|^2 \\
&\leqslant \sup_{\|g\|=1} \big(\sum_{\substack{i\in I \\ i\neq j}} v_i^2 \| \pi_{W_i} S^{-1}_{\mathcal{W}_v} \pi_{W_j}(g) \| \| \pi_{W_i}(f) \| \big)^2 \\
&\leqslant \sup_{\|g\|=1} \big(\sum_{\substack{i\in I \\ i\neq j}} v_i^2 \| \pi_{W_i} S^{-1}_{\mathcal{W}_v} \pi_{W_j}(g) \|^2 \big)\big(\sum_{\substack{i\in I \\ i\neq j}} v_i^2 \| \pi_{W_i}(f) \|^2 \big) \\
&\leqslant \frac{1}{C}\big(\sum_{\substack{i\in I \\ i\neq j}} v_i^2 \| \pi_{W_i}(f) \|^2 \big),
\end{aligned}$$

which implies that

$$\begin{aligned}
C \| f \|^2 &\leqslant \sum_{\substack{i\in I \\ i\neq j}} v_i^2 \| \pi_{W_i}(f) \|^2 + v_j^2 \| \pi_{W_j}(f) \|^2 \\
&\leqslant \sum_{\substack{i\in I \\ i\neq j}} v_i^2 \| \pi_{W_i}(f) \|^2 + v_j^2 \| \big(Id_H - v_j^2 \pi_{W_j} S^{-1}_{\mathcal{W}_v}\big)^{-1} \|^2 \ \frac{1}{C} \sum_{\substack{i\in I \\ i\neq j}} v_i^2 \| \pi_{W_i}(f) \|^2 \\
&= \big(1 + \frac{v_j^2}{C} \| \big(Id_H - v_j^2 \pi_{W_j} S^{-1}_{\mathcal{W}_v}\big)^{-1} \|^2 \big) \sum_{\substack{i\in I \\ i\neq j}} v_i^2 \| \pi_{W_i}(f) \|^2\ .
\end{aligned}$$

Hence $\mathcal{W}_v^{\mathcal{J}}$ satisfies the lower fusion frame condition with lower bound as required. Clearly $\mathcal{W}_v^{\mathcal{J}}$ also holds the upper fusion frame condition. ■

Corollary 3.4 *Suppose that $\mathcal{W}_v$ is a fusion frame for $\mathcal{H}$, and let $j \in I$. If $\| S^{-1}_{\mathcal{W}_v} \| < \frac{1}{v_j^2}$, then $\mathcal{W}_v^{\mathcal{J}}$ is a fusion frame for $\mathcal{H}$, with same fusion frame bounds in Theorem* 3.3(ii).

Proof This claim follows immediately from the fact that we have

$$\| v_j^2 \pi_{W_j} S_{\mathcal{W}_v}^{-1} \| \leqslant v_j^2 \| S_{\mathcal{W}_v}^{-1} \| < 1.$$

■

The following corollary is proved in [4, Corollary 3.3 (iii)]. We give another proof of this corollary with extra information about the bounds.

COROLLARY 3.5 *Let $\mathcal{W}_v$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds C, D and let $j \in I$. If $v_j^2 < C$, then $\mathcal{W}^{\mathcal{J}}$ is a fusion frame with same fusion frame bounds in Theorem* 3.3(ii).

Proof The result follows from Corollary 3.4 and the following fact,

$$\| S_{\mathcal{W}_v}^{-1} \| \leqslant \frac{1}{C} < \frac{1}{v_j^2}.$$

■

Remark 1 In Corollary 3.5 the inequality is strict. To see this, let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$, and define $W_i = span\{e_i\}$ for all $i \in I$. Then $\mathcal{W}_1 = \{(W_i, 1)\}_{i\in I}$ is an orthonormal fusion basis for $\mathcal{H}$, and so $\mathcal{W}_1$ is an exact Parseval fusion frame. Indeed for each $j \in I$ we have $\| S_{\mathcal{W}_1}^{-1} \| = \frac{1}{v_j^2} = 1$.

COROLLARY 3.6 *Suppose that $\mathcal{W}_v$ is a fusion frame for $\mathcal{H}$, and let $j \in I$. If $W_j = \mathcal{H}$ and $\| S_{\mathcal{W}_v}^{-1} \| \neq \frac{1}{v_j^2}$, then $\mathcal{W}_v^{\mathcal{J}}$ is a fusion frame for $\mathcal{H}$, with same fusion frame bounds in Theorem* 3.3(ii).

Proof Since $W_j = \mathcal{H}$, we have $\pi_{W_j} = Id_{\mathcal{H}}$. Define

$$T_i : H \longrightarrow W_i \qquad T_i(f) = v_i^{-2} \delta_{ij} \pi_{W_i}(f) \qquad \forall f \in \mathcal{H}.$$

Then by Theorem 3.1(i) we also have

$$< S_{\mathcal{W}_v}^{-1}(f) \ , \ f > = \sum_{i\in I} v_i^2 \| \pi_{W_i} S_{\mathcal{W}_v}^{-1}(f) \|^2 \leqslant \sum_{i\in I} v_i^2 \| T_i(f) \|^2 = \frac{1}{v_j^2} \| f \|^2,$$

which implies that $\| S_{\mathcal{W}_v}^{-1} \| \leqslant \frac{1}{v_j^2}$. This show that the implication holds. ■

COROLLARY 3.7 *Let $\mathcal{W}_v$ be an exact fusion frame for $\mathcal{H}$. Then for every $i \in I$ we have $\| S_{\mathcal{W}_v}^{-1} \| \geqslant \frac{1}{v_i^2}$.*

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