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On the Function of Block Anti Diagonal Matrices and Its Applications

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Abstract. The matrix functions appear in several applications in engineering and sciences. The computation of these functions almost involve complicated theory. Thus, improving the concept theoretically seems unavoidable to obtain some new relations and algorithms for evaluating these functions. The aim of this paper is proposing some new formulas to the function of block anti diagonal matrices. Moreover, some theorems will be proven and applications will be given.

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1. Introduction

Let $f(z)$ is an analytic function on a closed contour Γ which encircles $\sigma(A)$, whenever $\sigma(A)$ denotes the set of eigenvalues (spectrum) of matrix A. A function of an square matrix can be defined by considering Cauchy integral definition as follows [2, 3]:

$$f(A) = \frac{1}{2\pi i} \oint_{\Gamma} f(\xi)(\xi I - A)^{-1} d\xi. \tag{1}$$

The entries of $(\xi I - A)^{-1}$ are analytic on Γ, and $f(A)$ is analytic in a neighborhood of $\sigma(A)$. It is noticed that the integral of an arbitrary matrix W could be understood as the matrix whose entries are the integrals of the entries of W. Moreover,

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this mathematically appealing formula for computing the matrix function is very complicated and requires complex analysis to be fully understandable. Therefore, several applicable other techniques for evaluating matrix functions are investigated in depth, such as Jordan canonical form, interpolation methods, series methods and iterative procedures. The application of matrix functions have arisen in differential equation, Markov models, Control theory, non linear matrix equations, nuclear magnetic resonance, nonsymmetric eigenvalue problem, boundary value problems and several other area. For more readings, one may refer to [3, 6].

In this research article, the main contribution is the computation of block anti diagonal matrices and its applications. Firstly, the concepts of the block anti diagonal and also central symmetric X-form matrices will be explained. Secondly, several new formulations for computing function of block anti diagonal by proving some theorems will be given. Thirdly, we will deduce the applications in differential equation and also in Control theory. Finally, conclusions of the article will be drown.

2. New Formulations

In this section, the theory of the matrix functions will be modified. In order to tackle this problem, definition of block anti diagonal matrix which has significant role in our theory will be expressed as following. Let $A_{11}, \ldots, A_{mm}$ are $n_k \times n_k$ complex matrices with the same dimensions, then A is called block anti diagonal matrix if it is in the following form:

$$A = \begin{pmatrix} & & A_{11} \\ & \iddots & \\ A_{mm} & & \end{pmatrix}_{n\times n} = \text{blockadiag}_n(A_{11}, \ldots, A_{mm}). \tag{2}$$

whenever $\sum_{k=1}^{m} n_k = n$. Alternatively, assume A is an $n \times n$ complex matrix. If $\alpha_1, \ldots, \alpha_n$ are complex scalars, then A is called anti diagonal matrix if it is in the following form:

$$A = \begin{pmatrix} & & \alpha_1 \\ & \iddots & \\ \alpha_n & & \end{pmatrix}_{n\times n} = \text{adiag}_n(\alpha_1, \ldots, \alpha_n). \tag{3}$$

Example 2.1 A trivial anti diagonal matrix is called "reversal identity matrix" and denoted by J_n. This matrix is defined element wise as the matrix:

$$J_n = (\mathbf{j}_{ij})_{n\times n} = \begin{cases} 1, & j = n - i + 1, \\ 0, & j \neq n - i + 1, \end{cases} \tag{4}$$

that is,

$$J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}_{n\times n} = \text{adiag}_n(1, \ldots, 1). \tag{5}$$

Furthermore, properties of the reversal matrix as pointed out in [4], include as following

(1) $J_n^T = J_n$.

(2) $J_n^{-1} = J_n$.

(3) $J_n^p = I_n$ for even values of p, and $J_n^p = J_n$ for odd values of p.

(4) $\mathrm{Tr}(J_n) = 1$ if n is odd; and $\mathrm{Tr}(J_n) = 0$ if n is even.

(5) Any matrix A satisfying the condition $AJ_n = J_nA$ is said to be "centrosymmetric".

(6) Any matrix A satisfying the condition $AJ_n = J_nA^T$ is said to be "persymmetric".

Another applicable matrix in the text is central symmetric X-form matrix which is investigated in depth in [5, 7]. The block version of this matrix is expressed in the following definition. Assume $A_{11}, \dots, A_{mm}$ and $B_{11}, \dots, B_{mm}$ are $n_k \times n_k$ complex matrices. We define the block central symmetric X-form matrix in the following odd dimension form

$$A = \begin{pmatrix} A_{mm} & & & & & & B_{mm} \\ & \ddots & & & & \iddots & \\ & & A_{22} & & B_{22} & & \\ & & & A_{11} & & & \\ & & B_{22} & & A_{22} & & \\ & \iddots & & & & \ddots & \\ B_{mm} & & & & & & A_{mm} \end{pmatrix} \in \mathbb{C}^{(2n_k-1)\times(2n_k-1)}, \tag{6}$$

and the block central symmetric X-form matrix in the even dimension form

$$B = \begin{pmatrix} A_{mm} & & & & & & B_{mm} \\ & \ddots & & & & \iddots & \\ & & A_{22} & & & B_{22} & \\ & & & A_{11} & B_{11} & & \\ & & & B_{11} & A_{11} & & \\ & & B_{22} & & & A_{22} & \\ & \iddots & & & & \ddots & \\ B_{mm} & & & & & & A_{mm} \end{pmatrix} \in \mathbb{C}^{(2n_k)\times(2n_k)}. \tag{7}$$

It should be emphasized that the particular case of the matrices (6) and (7) are the scalar central symmetric X-form matrices that have been defined in [5, 7] in the following form:

$$A = \begin{pmatrix} \alpha_m & & & & & & \beta_m \\ & \ddots & & & & \iddots & \\ & & \alpha_2 & & \beta_2 & & \\ & & & \alpha_1 & & & \\ & & \beta_2 & & \alpha_2 & & \\ & \iddots & & & & \ddots & \\ \beta_m & & & & & & \alpha_m \end{pmatrix}, \quad B = \begin{pmatrix} \alpha_m & & & & & & \beta_m \\ & \ddots & & & & \iddots & \\ & & \alpha_2 & & & \beta_2 & \\ & & & \alpha_1 & \beta_1 & & \\ & & & \beta_1 & \alpha_1 & & \\ & & \beta_2 & & & \alpha_2 & \\ & \iddots & & & & \ddots & \\ \beta_m & & & & & & \alpha_m \end{pmatrix}. \tag{8}$$

Now, we are interested to depict a discipline for the function of block anti diagonal matrix. Before that, we investigate the function of block diagonal matrices briefly. [3] Suppose f has the following Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k, \tag{9}$$

with radius of convergence r. If $A \in \mathbb{C}^{n\times n}$ and $a_k = \frac{f^{(k)}(\alpha)}{k!}$, then $f(A)$ is defined and is given by

$$f(A) = \sum_{k=0}^{\infty} a_k (A - \alpha I)^k, \tag{10}$$

if and only if each of the distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ of A satisfies one of the conditions:

(1) $|\lambda_i - \alpha| < r$,

(2) $|\lambda_i - \alpha| = r$, and the series for $f^{(n_i-1)}$ (whenever n_i is the index of λ_i) is convergent at the point $\lambda = \lambda_i$ for $i = 1, \ldots, n$. According to Theorem 2.1, by setting $\alpha = 0$, for diagonal matrix $A = \mathrm{diag}(\zeta_1, \ldots, \zeta_n)$, it is obvious that

$$f(A) = \mathrm{diag}\left(\sum_{k=0}^{\infty} a_k \zeta_1^k, \ldots, \sum_{k=0}^{\infty} a_k \zeta_n^k\right) = \mathrm{diag}\left(f(\zeta_1), \ldots, f(\zeta_n)\right), \tag{11}$$

Simultaneously, if $A = \mathrm{diag}(A_{11}, \ldots, A_{mm})$ is block diagonal matrix, then it is clear that

$$f(A) = \mathrm{blockdiag}\left(\sum_{k=0}^{\infty} a_k A_{11}^k, \ldots, \sum_{k=0}^{\infty} a_k A_{mm}^k\right) = \mathrm{blockdiag}(f(A_{11}), \ldots, f(A_{mm})), \tag{12}$$

Some basic matrix functions that have received many attraction, are defined by using Taylor series as following [3]:

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}, \tag{13}$$

$$\sin(A) = \sum_{k=0}^{\infty} \frac{(-1)^k A^{2k+1}}{(2k+1)!}, \quad \cos(A) = \sum_{k=0}^{\infty} \frac{(-1)^k A^{2k}}{(2k)!}, \tag{14}$$

$$\log(I + A) = \sum_{k=0}^{\infty} \frac{(-1)^k A^{k+1}}{(k+1)}, \qquad \rho(A) < 1 \tag{15}$$

$$\sinh(A) = \sum_{k=0}^{\infty} \frac{A^{2k+1}}{(2k+1)!}, \quad \cosh(A) = \sum_{k=0}^{\infty} \frac{A^{2k}}{(2k)!}, \tag{16}$$

$$\log(A) = -2\sum_{k=0}^{\infty} \frac{1}{2k+1}((I - A)(I + A)^{-1})^{2k+1}, \qquad \min_i \mathrm{Re}\lambda_i(A) < 1. \tag{17}$$

A fundamental relationship between exponential dependent matrix functions is the matrix analogue of "Euler's formula" which is defined as

$$e^{iA} = \cos(A) + \sin(A), \tag{18}$$

and consequently yields that

$$\sin(A) = \frac{e^{iA} - e^{-iA}}{2i}, \quad \cos(A) = \frac{e^{iA} + e^{-iA}}{2}. \tag{19}$$

Now, we present following basic theorem. Let $A = \text{blockantidiag}_n(A_{11}, \ldots, A_{mm}) \in \mathbb{C}^{n\times n}$, where $A_{11}, \ldots, A_{mm} \in \mathbb{C}^{n_k \times n_k}$. Then $f(A)$ will be obtained as the even dimension form

$$f(A) = \begin{pmatrix} \sum_{k=0}^{\infty} a_{2k}(A_{11}A_{2m,2m})^k & & & & & A_{11}\sum_{k=0}^{\infty} a_{2k+1}(A_{2m,2m}A_{11})^k \\ & \ddots & & & \iddots & \\ & & \sum_{k=0}^{\infty} a_{2k}(A_{m,m}A_{m+1,m+1})^k & A_{m,m}\sum_{k=0}^{\infty} a_{2k+1}(A_{m+1,m+1}A_{m,m})^k & & \\ & & A_{m+1,m+1}\sum_{k=0}^{\infty} a_{2k+1}(A_{m,m}A_{m+1,m+1})^k & \sum_{k=0}^{\infty} a_{2k}(A_{m,m}A_{m+1,m+1})^k & & \\ & \iddots & & & \ddots & \\ A_{2m,2m}\sum_{k=0}^{\infty} a_{2k+1}(A_{11}A_{2m,2m})^k & & & & & \sum_{k=0}^{\infty} a_{2k}(A_{2m,2m}A_{11})^k \end{pmatrix}, \tag{20}$$

and, the odd dimension form

$$f(A) = \begin{pmatrix} \sum_{k=0}^{\infty} a_{2k}(A_{11}A_{2m-1,2m-1})^k & & & & & & A_{11}\sum_{k=0}^{\infty} a_{2k+1}(A_{2m-1,2m-1}A_{11})^k \\ & \ddots & & & & \iddots & \\ & & \sum_{k=0}^{\infty} a_{2k}(A_{m-1,m-1}A_{m+1,m+1})^k & & A_{m-1,m-1}\sum_{k=0}^{\infty} a_{2k+1}(A_{m+1,m+1}A_{m-1,m-1})^k & & \\ & & & A_{mm}\sum_{k=0}^{\infty} a_{2k}(A_{mm})^k & & & \\ & & A_{m+1,m+1}\sum_{k=0}^{\infty} a_{2k+1}(A_{m-1,m-1}A_{m+1,m+1})^k & & \sum_{k=0}^{\infty} a_{2k}(A_{m+1,m+1}A_{m-1,m-1})^k & & \\ & \iddots & & & & \ddots & \\ A_{2m-1,2m-1}\sum_{k=0}^{\infty} a_{2k+1}(A_{11}A_{2m-1,2m-1})^k & & & & & & \sum_{k=0}^{\infty} a_{2k}(A_{2m-1,2m-1}A_{11})^k \end{pmatrix}. \tag{21}$$

Proof In order to prove this theorem, the block numbers of the matrix A divided to two cases:

(Case 1: m is even number) In this case, suppose that m (the number of blocks) of A is even. Therefore, A is given by

$$A = \text{blockadiag}_n(A_{11}, \ldots, A_{mm}, A_{m+1,m+1}, \ldots, A_{2m,2m}).$$

It is clear that the powers of A for $p \in \{0, 2, 4, 6, \ldots\}$ can be easily obtained by

$$A^p = \text{blockdiag}_n\left((A_{11}A_{2m,2m})^p, \ldots, (A_{m,m}A_{m+1,m+1})^p, (A_{m+1,m+1}A_{m,m})^p, \ldots, (A_{2m,2m}A_{11})^p\right), \tag{22}$$

and for $p \in \{1, 3, 5, 7, \ldots\}$, it can be computed as

$$A^p = \text{blockadiag}_n((A_{11}A_{2m,2m})^p A_{11} \ldots (A_{mm}A_{m+1,m+1})^p A_{mm}$$
$$(A_{m+1,m+1}A_{m,m})^p A_{m+1,m+1} \ldots (A_{2m,2m}A_{11})^p A_{2m,2m}).$$

Now, after employing series definition of the matrix function in the form

$$f(A) = \sum_{k=0}^{\infty} a_k A^k = \sum_{k \in \{2n\}} a_k A^k + \sum_{k \in \{2n+1\}} a_k A^k, \tag{23}$$

and simplifying the relations, we attain the block X-form matrix (20).

(Case 2: m is odd number) In this case, the dimension of the number of blocks is assumed to be odd. Therefore, A is given by:

$$A = \text{blockadiag}_n(A_{11}, \ldots, A_{m-1,m-1}, A_{m,m}, A_{m+1,m+1}, \ldots, A_{2m-1,1m-1}).$$

Then, the powers of block anti diagonal A for $p \in \{0, 2, 4, 6, \ldots\}$ are expressed by

$$A^p = \text{blockdiag}_n((A_{11}A_{2m-1,2m-1})^p, \ldots, (A_{m-1,m-1}A_{m+1,m+1})^p, A^p_{m,m},$$
$$(A_{m-1,m-1}A_{m+1,m+1})^p, \ldots, (A_{11}A_{2m-2,2m-1})^p),$$

and also the powers of block anti diagonal A for $p \in \{1, 3, 5, \ldots\}$ are presented by

$$A^p = \text{blockadiag}_n(A_{11}(A_{11}A_{2m-1,2m-1})^p, \ldots, A_{m-1,m-1}(A_{m-1,m-1}A_{m+1,m+1})^p\ A^p_{m,m},$$
$$A_{m+1,m+1}(A_{m-1,m-1}A_{m+1,+1})^p, \ldots, A_{m,m}(A_{11}A_{2m-1,2m-1})^p).$$

Now, once again employing series definition of matrix function (23), the block X form matrix (21) can be yielded. This completes the proof. ■

The structure of $f(A)$ in Theorem 2 is characterized in Figure 1 and Figure 2. We have plotted the $f(A)$ either in even dimension or odd dimension by using MATLAB software. The blocks are 5×5 the matrix $\mathbf{1}_5 = (1)_{ij}$.

Example 2.2. Let $A, B, C \in \mathbb{C}^{n_k \times n_k}$ and block anti diagonal matrix M is in the form

$$M = \begin{pmatrix} & & A \\ & B & \\ C & & \end{pmatrix} \in \mathbb{C}^{n \times n}. \tag{24}$$

It is known that on of the most fundamental matrix function with potential application is matrix exponential. We are interested to obtain an explicit formula for $\exp(M)$. According to the relation (21), we obtain

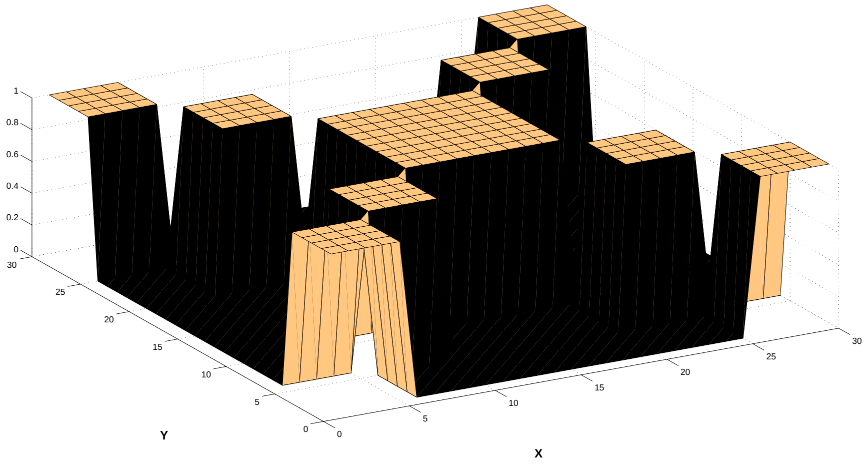

Figure 1. Even blocks ($m = 6, n_k = 5, n = 30$)

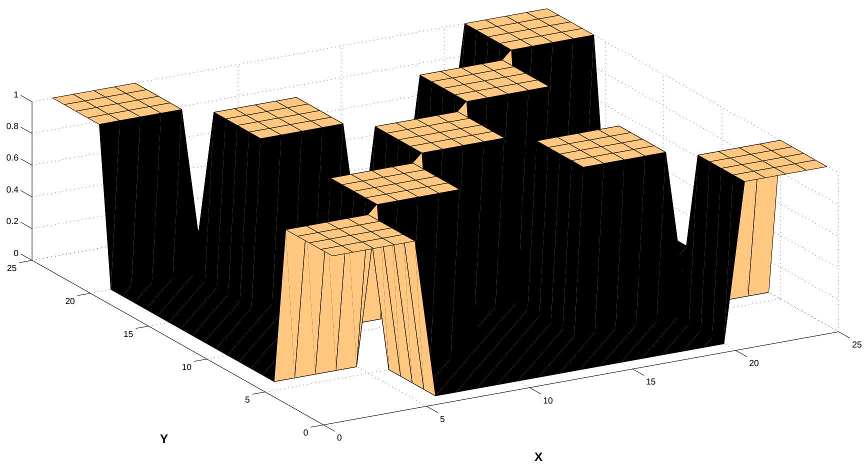

Figure 2. Odd blocks ($m = 5, n_k = 5, n = 25$)

$$e^M = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(AC)^k}{(2k)!} & & A\sum_{k=0}^{\infty} \frac{(CA)^k}{(2k+1)!} \\ & \sum_{k=0}^{\infty} \frac{B^k}{k!} & \\ C\sum_{k=0}^{\infty} \frac{(AC)^k}{(2k+1)!} & & \sum_{k=0}^{\infty} \frac{(CA)^k}{(2k)!} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(\sqrt{AC})^{2k}}{(2k)!} & & A(\sqrt{CA})^{-1}\sum_{k=0}^{\infty} \frac{(\sqrt{CA})^{2k+1}}{(2k+1)!} \\ & \sum_{k=0}^{\infty} \frac{B^k}{k!} & \\ C(\sqrt{AC})^{-1}\sum_{k=0}^{\infty} \frac{(\sqrt{AC})^{2k+1}}{(2k+1)!} & & \sum_{k=0}^{\infty} \frac{(\sqrt{CA})^{2k}}{(2k)!} \end{pmatrix}.$$

Hence, after some algebraic manipulation and employing the relations $\sinh(A) = \sum_{k=0}^{\infty} \frac{A^{2k+1}}{(2k+1)!}$ and $\cosh(A) = \sum_{k=0}^{\infty} \frac{A^{2k}}{(2k)!}$, it is brought forward as

$$e^M = \begin{pmatrix} \cosh(\sqrt{AC}) & & A(\sqrt{CA})^{-1}\sinh(\sqrt{CA}) \\ & \exp(B) & \\ C(\sqrt{AC})^{-1}\sinh(\sqrt{AC}) & & \cosh(\sqrt{CA}) \end{pmatrix} \tag{25}$$

Let $A = \mathrm{adiag}(\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^{m\times m}$. Then $f(A)$ for the even dimension of A can be determined by

$$f(A) = \begin{pmatrix} \sum_{k=0}^{\infty} a_{2k}(\alpha_1\alpha_{2m})^k & & & \alpha_1 \sum_{k=0}^{\infty} a_{2k+1}(\alpha_1\alpha_{2m})^k \\ & \ddots & \iddots & \\ & \sum_{k=0}^{\infty} a_{2k}(\alpha_m\alpha_{m+1})^k & \alpha_m \sum_{k=0}^{\infty} a_{2k+1}(\alpha_m\alpha_{m+1})^k & \\ & \alpha_{m+1} \sum_{k=0}^{\infty} a_{2k+1}(\alpha_m\alpha_{2m+1})^k & \sum_{k=0}^{\infty} a_{2k}(\alpha_m\alpha_{m+1})^k & \\ & \iddots & \ddots & \\ \alpha_{2m} \sum_{k=0}^{\infty} a_{2k+1}(\alpha_1\alpha_{2m})^k & & & \sum_{k=0}^{\infty} a_{2k}(\alpha_1\alpha_{2m})^k \end{pmatrix}, \tag{26}$$

and for the odd dimension of A can be obtained by

$$f(A) = \begin{pmatrix} \sum_{k=0}^{\infty} a_{2k}(\alpha_1\alpha_{2m-1})^k & & & & \alpha_1 \sum_{k=0}^{\infty} a_{2k+1}(\alpha_1\alpha_{2m-1})^k \\ & \ddots & & \iddots & \\ & \sum_{k=0}^{\infty} a_{2k}(\alpha_{m-1}\alpha_{m+1})^k & & \alpha_{m-1} \sum_{k=0}^{\infty} a_{2k+1}(\alpha_{m-1}\alpha_{m+1})^k & \\ & & \sum_{k=0}^{\infty} a_k\alpha_m^k & & \\ & \alpha_{m+1} \sum_{k=0}^{\infty} a_{2k+1}(\alpha_{m-1}\alpha_{m+1})^k & & \sum_{k=0}^{\infty} a_{2k}(\alpha_{m-1}\alpha_{m+1})^k & \\ & \iddots & & \ddots & \\ \alpha_{2m-1} \sum_{k=0}^{\infty} a_{2k}(\alpha_1\alpha_{2m-1})^k & & & & \sum_{k=0}^{\infty} a_{2k}(\alpha_1\alpha_{2m-1})^k \end{pmatrix}. \tag{27}$$

The matrices $f(A)$ in (26) and (27) are central symmetric X-form while the matrices (20) and (21)are not necessarily symmetric.

Example 2.3. This example made considering the scalar antidiagonal matrix $M_n = \mathrm{adiag}_n(\zeta, \ldots, \zeta)$ where $\zeta \in \mathbb{R}$. Then, according to Corollary 2.1, for $n = 2k$ we obtain

$$\exp(M_n) = \begin{pmatrix} \cosh(\zeta)I_n & \sinh(\zeta)J_n \\ \sinh(\zeta)J_n & \cosh(\zeta)I_n \end{pmatrix}, \tag{28}$$

$$\exp(-M_n) = \begin{pmatrix} \cosh(\zeta)I_n & -\sinh(\zeta)J_n \\ -\sinh(\zeta)J_n & \cosh(\zeta)I_n \end{pmatrix},$$

and for $n = 2k - 1$ we attain

$$\exp(M_n) = \begin{pmatrix} \cosh(\zeta)I_n & & \sinh(\zeta)J_n \\ & \exp(\zeta) & \\ \sinh(\xi)J_n & & \cosh(\zeta)I_n \end{pmatrix},$$

$$\exp(-M_n) = \begin{pmatrix} \cosh(\zeta)I_n & & -\sinh(\zeta)J_n \\ & \exp(-\zeta) & \\ -\sinh(\zeta)J_n & & \cosh(\zeta)I_n \end{pmatrix}.$$

Consequently, for both cases we obtain that

$$\cosh(M_{2k}) = \cos(\zeta)I_{2k}, \quad \sin(M_{2k}) = \sin(\zeta)J_{2k}, \tag{29}$$

$$\cosh(M_{2k-1}) = \cosh(\zeta)I_{2k+1}, \quad \sinh(M_{2k-1}) = \sinh(\zeta)J_{2k-1}. \tag{30}$$

3. Applications

This section is devoted to provide some applications for the modified theory.

3.1 *Matrix Differential Equation*

It is well known that the differential equation has important role in engineering and alive phenomena. One of the most repetitive equation which is appears in coupled Spring-Mass systems is second order linear matrix differential equation [1, 3]. Let us consider second order linear initial value problem in the form

$$\frac{\partial^2 y(t)}{\partial t^2} + Ay(t) = 0, \qquad y(0) = y_0, \quad y'(0) = y_0', \tag{31}$$

where $A \in \mathbb{C}^{n\times n}$ and $y(t), y'(t) \in \mathbb{C}^n$. If the change variable $z = \frac{\partial}{\partial t}y(t)$ will be applied, then the following system can be obviously obtained:

$$\frac{\partial}{\partial t}\begin{pmatrix} z \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} & -tA \\ tI & \end{pmatrix}}_{M}\begin{pmatrix} z \\ y \end{pmatrix}, \qquad \begin{pmatrix} z(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} y_0' \\ y_0 \end{pmatrix}. \tag{32}$$

As it it is known that the vector initial value problem

$$\frac{\partial y(t)}{\partial t} + Ay(t) = 0, \qquad y(0) = y_0, \tag{33}$$

where $A \in \mathbb{C}^{n\times n}$ and $y(t) \in \mathbb{C}^n$ has the solution $y(t) = e^{At}y_0$. Hence, according to modified theory which has been expressed in the text, the solution of (31) will be yielded via

$$e^{tM} = \begin{pmatrix} \cosh(\sqrt{-A}t) & -At(\sqrt{-A}t)^{-1}\sinh(\sqrt{-A}t) \\ I(\sqrt{-A}t)^{-1}\sinh(\sqrt{-A}t) & \cosh(\sqrt{-A}t) \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(i\sqrt{A}t) & -At(i\sqrt{A}t)^{-1}\sinh(i\sqrt{A}t) \\ (i\sqrt{A}t)^{-1}\sinh(i\sqrt{A}t) & \cosh(i\sqrt{A}t) \end{pmatrix}.$$

Since $\sinh(iA) = i\sin(A)$ and $\cosh(iA) = \cos(A)$, we then have

$$e^{tM} = \begin{pmatrix} \cos(\sqrt{A}) & A(\sqrt{A})^{-1}\sin(\sqrt{A}) \\ (\sqrt{A})^{-1}\sin(\sqrt{A}) & \cos(\sqrt{A}) \end{pmatrix}. \tag{34}$$

Thus, we attain

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{A}t) & A(\sqrt{A})^{-1}\sin(\sqrt{A}t) \\ (\sqrt{A}t)^{-1}\sin(\sqrt{A}t) & \cos(\sqrt{A}t) \end{pmatrix} \begin{pmatrix} y_0' \\ y_0 \end{pmatrix}.$$

Consequently, we obtain the following solution for the equation (31):

$$y(t) = (\sqrt{A}t)^{-1}\sin(\sqrt{A}t)y_0' + \cos(\sqrt{A}t)y_0. \tag{35}$$

It should be noticed that $\sqrt{A}$ is any (non principal or principal) square root of A which is the matrix satisfied in the matrix equation $X^2 - A = 0$. The solution (35) is obtained in many test by using other approaches like inverse laplace transformation.

3.2 *Computing Block Exponential Dependent Functions*

Another application of modified theory is computing the block form of exponential dependent functions such as $\sin(A)$, $\cosh(A)$ and t^A. Assume

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{36}$$

wherein the blocks are in dimension $n_k \times n_k (k = 1, 2)$, and $n_1 + n_2 = n$. Now, according to relation (20), the exponential of M can be easily computed as following

$$e^M = e^{\begin{pmatrix} A & \\ & D \end{pmatrix} + \begin{pmatrix} & B \\ C & \end{pmatrix}} = e^{\begin{pmatrix} A & \\ & D \end{pmatrix}} e^{\begin{pmatrix} & B \\ C & \end{pmatrix}}$$

$$= \begin{pmatrix} e^A & \\ & e^D \end{pmatrix} \begin{pmatrix} \cosh(\sqrt{BC}) & B(\sqrt{CB})^{-1}\sinh(\sqrt{CB}) \\ C(\sqrt{BC})^{-1}\sinh(\sqrt{BC}) & \cosh(\sqrt{CB}) \end{pmatrix}.$$

It is remarked that in last simplification, we must have

$$\begin{pmatrix} A & \\ & D \end{pmatrix} \begin{pmatrix} & B \\ C & \end{pmatrix} = \begin{pmatrix} & B \\ C & \end{pmatrix} \begin{pmatrix} A & \\ & D \end{pmatrix}.$$

Hence, we can conclude that

$$e^M = \begin{pmatrix} e^A \cosh(\sqrt{BC}) & e^A B(\sqrt{CB})^{-1} \sinh(\sqrt{CB}) \\ e^D C(\sqrt{BC})^{-1} \sinh(\sqrt{BC}) & e^D \cosh(\sqrt{CB}) \end{pmatrix}. \quad (37)$$

Thus, we obtain following explicit relations to the block exponential dependent functions by combination of modified theory and equations (12), (13), (15), and (18) as follows:

$$\cos(M) = \begin{pmatrix} \cos(A)\cos(\sqrt{BC}) & \sin(A)B(\sqrt{CB})^{-1}\sin(\sqrt{CB}) \\ \sin(D)C(\sqrt{BC})^{-1}\sin(\sqrt{BC}) & \cos(D)\cos(\sqrt{CB}) \end{pmatrix}, \quad (38)$$

$$\sin(M) = \begin{pmatrix} \sin(A)\cos(\sqrt{BC}) & \cos(A)B(\sqrt{CB})^{-1}\sin(\sqrt{CB}) \\ \cos(D)C(\sqrt{BC})^{-1}\sin(\sqrt{BC}) & \sin(D)\cos(\sqrt{CB}) \end{pmatrix}, \quad (39)$$

$$\cosh(M) = \begin{pmatrix} \cosh(A)\cosh(\sqrt{BC}) & \sinh(A)B(\sqrt{CB})^{-1}\sinh(\sqrt{CB}) \\ \sinh(D)C(\sqrt{BC})^{-1}\sinh(\sqrt{BC}) & \cosh(D)\cosh(\sqrt{CB}) \end{pmatrix}, \quad (40)$$

$$\sinh(M) = \begin{pmatrix} \sinh(A)\cosh(\sqrt{BC}) & \cosh(A)B(\sqrt{CB})^{-1}\sinh(\sqrt{CB}) \\ \cosh(D)C(\sqrt{BC})^{-1}\sinh(\sqrt{BC}) & \sinh(D)\cosh(\sqrt{CB}) \end{pmatrix}, \quad (41)$$

$$t^M = \begin{pmatrix} t^A \cosh(\sqrt{BC}\ln(t)) & t^A B(\sqrt{CB})^{-1}\sinh(\sqrt{CB}\ln(t)) \\ t^D C(\sqrt{BC})^{-1}\sinh(\sqrt{BC}\ln(t)) & t^D \cosh(\sqrt{CB}\ln(t)) \end{pmatrix}. \quad (42)$$

It is straightforward that t^M is defined by $e^{M\ln(t)}$ for $t > 0$. This function is very important, particularly in computing matrix Gamma an Beta functions that are defined as [7]:

$$\Gamma(A) = \int_0^1 e^{-t} t^{A-I_n} dt, \quad (43)$$

$$\mathcal{B}(A, B) = \int_0^1 t^{A-I_n}(1-t)^{B-I_n} dt. \quad (44)$$

It is emphasized that the Gamma and Beta functions are important in solving matrix differential equation.

3.3 *Control Theory*

For the third application of the modified theory, we mention an application in Control theory. For this purpose, let A, B, C, D, E, F are square $n \times n$ matrices, and further $X(t)$ and $Y(t)$ are $n \times n$ diagonal matrices. Then the following homogeneous coupled matrix differential equations with initial condition $X(0) = E$ and $Y(0) = F$ is arisen in Control theory [1]:

$$\frac{\partial X(t)}{\partial t} = AX(t)B + CY(t)D,$$

$$\frac{\partial Y(t)}{\partial t} = CX(t)D + AY(t)B. \tag{45}$$

This equation by using vector operator and Hadamard product can be rewritten as:

$$\frac{\partial}{\partial t}\begin{pmatrix} \text{vec}X(t) \\ \text{vec}Y(t) \end{pmatrix} = \begin{pmatrix} (B^T \circ A)t & (D^T \circ C)t \\ (D^T \circ C)t & (B^T \circ A)t \end{pmatrix}\begin{pmatrix} \text{vec}X(t) \\ \text{vec}Y(t) \end{pmatrix}.$$

Therefore, if $(B^T \circ A)(D^T \circ C) = (D^T \circ C)(B^T \circ A)$ then the general solution is

$$\begin{pmatrix} \text{vec}X(t) \\ \text{vec}Y(t) \end{pmatrix} = \exp\begin{pmatrix} (B^T \circ A)t & (D^T \circ C)t \\ (D^T \circ C)t & (B^T \circ A)t \end{pmatrix}\begin{pmatrix} \text{vec}E \\ \text{vec}F \end{pmatrix}.$$

According to the relations in the text, we obtain the following solution

$$\begin{pmatrix} \text{vec}X(t) \\ \text{vec}Y(t) \end{pmatrix} = \begin{pmatrix} e^{(B^T \circ A)t}\sinh(D^T \circ C)t & e^{(B^T \circ A)t}\cosh(D^T \circ C)t \\ e^{(B^T \circ A)t}\cosh(D^T \circ C)t & e^{(B^T \circ A)t}\sinh(D^T \circ C)t \end{pmatrix}\begin{pmatrix} \text{vec}E \\ \text{vec}F \end{pmatrix}.$$

Consequently, after simplification the explicit will be yielded as following

$$\text{vec}X(t) = e^{(B^T \circ A)t}\sinh(D^T \circ C)t.\text{vec}E + e^{(B^T \circ A)t}\cosh(D^T \circ C)t.\text{vec}F,$$

$$\text{vec}Y(t) = e^{(B^T \circ A)t}\cosh(D^T \circ C)t.\text{vec}E + B^T e^{(B^T \circ A)t}\sinh(D^T \circ C)t.\ \text{vec}F. \tag{46}$$

4. Conclusion

In this work, the computation of function of block anti diagonal matrices is described by employing the series definition. Moreover, several explicit formulas have been proposed in order to obtain simple way for computing some matrix functions that have Taylor series. Eventually, applications in initial value problems, computing exponential dependent function of block matrices, and matrix differential equations appear in Control theory are given by introducing some examples.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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