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An Iterative Method with Six-Order Convergence for Solving Nonlinear Equations

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Abstract. Modification of Newtons method with higher-order convergence is pre- sented.The modification of Newtons method is based on Frontinis three-order method The new method requires two-step per iteration. Analysis of con- vergence demonstrates that the order of convergence is 6. Some numerical examples illustrate that the algorithm is more efficient and performs better than classical Newtons method and other methods.

Keywords: Nonlinear Equation, Iterative Method, Two-Step, Iterative Method, Convergence Order, Efficiency Index.

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1. Introduction

In this paper, we develop an iterative method to find a simple root a of the nonlinear equation $f(\alpha) = 0$, where $f : D \subset R \to R$ is a scalar function on an open interval D. It is well known that Newtons method is one of the best iterative methods for solving a single nonlinear equation by using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

which converges quadratically in some neighborhood of α. Many iterative methods have been developed by using various techniques including quadrature formulas, Taylor series and decomposition methods. For more details, see [3,7,9,10,11,14,17,19] and the references therein we applied Frontinis method by

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three-order, which is written as:[4,5]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(x_n - \frac{f(x_n)}{2f'(x_n)}\right)} \tag{2}$$

Also, we can obtain an approximation for $f'(y_n)$ [2]:

$$f'(y_n) \approx P_f(x_n, y_n) = \left[\frac{2f(x_n) - 5f(y_n)}{2f(x_n) - f(y_n)}\right] f'(x_n). \tag{3}$$

2. Development of Method and Convergence Analysis

Now we construct a two-step iterative method:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'\left(y_n - \frac{f(y_n)}{2P_f(x_n,y_n)}\right)}, \quad n = 0, 1, 2, \dots \end{aligned} \tag{4}$$

We prove the following convergence theorem for our new method presented by Equation (4).

THEOREM 2.1 *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to R$ for an open interval I. If x_0 is sufficiently close to α, then the three-step iterative method Equation (4) has Six-order convergence and satisfies the following error equation:*

$$e_{n+1} = -\frac{5}{4} c_3 c_2^3 e_n^6 + O(e_n^7). \tag{5}$$

where $e_n = x_n - \alpha$ for $n = 1, 2, \dots,$ and $c_n = \frac{f^{(n)}(\alpha)}{n! f'(\alpha)}$.

Proof Since f is sufficiently differentiable, by expanding $f(x_n)$ and $f'(x_n)$ about α, one obtains

$$f(x_n) = f'(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)). \tag{6}$$

$$f'(x_n) = f'(\alpha)(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^6)). \tag{7}$$

where $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$ for $k = 2, 3, \dots$. Furthermore, with using the Maple software we can get By expanding y_n about x_n, we obtain

$$\begin{aligned} y_n =& \alpha + c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 \\ &+ (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 \\ &+ (-10c_2c_4 - 6c_3^2 + 20c_3c_2^2 - 8c_2^4)^5 \\ &+ (-17c_4c_3 + 28c_4c_2^2 + 33c_2c_3^2 - 52c_3c_2^3 + 16c_2^5)e_n^6 \\ &+ O(e_n^7). \end{aligned} \tag{8}$$

Expanding $f(x_n - \frac{f(x_n)}{f'(x_n)})$ about x_n, we get

$$\begin{aligned} f(y_n) =& f'(\alpha)c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 \\ &+ (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 \\ &+ (-10c_2c_4 - 6c_3^2 + 24c_3c_2^2 - 12c_2^4)e_n^5 \\ &+ (-17c_4c_3 + 34c_4c_2^2 + 37c_2c_3^2 - 73c_3c_2^3 + 28c_2^5)e_n^6 \\ &+ O(e_n^7). \end{aligned} \tag{9}$$

Expanding $f'(y) = P_f(x_n, y_n)$ about x_n, we get

$$\begin{aligned} P_f(x_n, y_n) =& f'(\alpha)(1 + (-c_3 + c_2^2))e_n^2 \\ &+ (-2c_4 + 2c_2c_3 - (1/2)c_2^3)e_n^3 \\ &+ (2c_3c_2^2 + 2c_2c_4 - (7/4)c_2^4)e_n^4 \\ &+ (-2c_4c_3 + (7/2)c_4c_2^2 + 12c_2c_3^2 - (41/2)c_3c_2^3 + (63/8)c_2^5)e_n^5 \\ &+ O(e_n^6). \end{aligned} \tag{10}$$

Substituting Equation (8), Equation (9), and Equation (10), into the second formula of Equation (4), using Taylors expansion, and simplifying, we have

$$x_{n+1} = \alpha - (5/4)c_3c_2^3e_n^6 + O(e_n^7). \tag{11}$$

Thus, we have

$$e_{n+1} = -(5/4)c_3c_2^3e_n^6 + O(e_n^7). \tag{12}$$

This means the method defined by Equation (4) is of six-order. That completes the proof. ■

Remark 1 The order of convergence of the iterative method Equation (4) is 6. This method requires two evaluations of the function, namely, $f(x_n)$ and $f(y_n)$ and two evaluations of first derivatives $f'(x_n)$, $f'\left(y_n - \frac{f(y_n)}{2P_f(x_n,y_n)}\right)$. We take into account the definition of efficiency index [6,8] as $p^{1/w}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method. If we suppose that all the evaluations have the same cost, we have that the efficiency index of the method Equation (4) is $\sqrt{6} = 1.5650$.

3. Numerical Examples

In this section, the obtained theoretical results are confirmed by numerical experiments and compared with Algorithms 1 and 2 presented recently by Noor et al. [12] and Algorithm 3: [1] whose order of convergence of these methods is nine and with the existing three-step methods in [18] some nonlinear equations and compare them with XIA1, XIA2 and XIA3, whose order of convergence of these methods is eight Algorithm 1:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= y_n - \frac{2f(y_n)f'(y_n)}{2(f'(y_n))^2 - f(y_n)P_f(x_n, y_n)}, \\
x_{n+1} &= z_n - \left[\frac{f'(x_n) + f'(y_n)}{3f'(y_n) - f'(x_n)}\right] \frac{f(z_n)}{f'(x_n)}. \quad n = 0, 1, 2, \ldots
\end{aligned}
$$

Algorithm 2:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= y_n - \frac{2f(y_n)f'(y_n)}{2(f'(y_n))^2 - f(y_n)P_f(x_n, y_n)}, \\
x_{n+1} &= z_n - \left[\frac{2f'(x_n)^2}{f'(x_n)^2 - 4f'(x_n)f'(y_n) + f'(y_n)^2}\right] \frac{f(z_n)}{f'(x_n)}. \quad n = 0, 1, 2, \ldots
\end{aligned}
$$

Algorithm 3:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= y_n - \frac{2f(y_n)f'(y_n)}{2(f'(y_n))^2 - f(y_n)P_f(x_n, y_n)}, \\
x_{n+1} &= z_n - \frac{f(z_n)}{\frac{f(z_n)-f(y_n)}{z_n-y_n} + \frac{z_n-y_n}{z_n-x_n}\left[\frac{f(z_n)-f(x_n)}{z_n-x_n} - f'(x_n)\right]}. \quad n = 0, 1, 2, \ldots
\end{aligned}
$$

where $P_f(x_n, y_n) = f''(y_n)$

$$
P_f(x_n, y_n) = \frac{2}{y_n - x_n}\left(2f'(y_n) + f'(x_n) - \frac{3f(y_n) - f(x_n)}{y_n - x_n}\right).
$$

XIA1:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}, \\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}\left[1/2 + \frac{5f(x_n)^2 + 8f(x_n)f(y_n) + 2f(y_n)^2}{5f(x_n)^2 - 12f(x_n)f(y_n)}(1/2 + \frac{f(z_n)}{f(y_n)})\right].
\end{aligned}
$$

XIA2:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}\left[\frac{5f(x_n)^2 - 2f(x_n)f(y_n) + f(y_n)^2}{5f(x_n)^2 - 12f(x_n)f(y_n)} + (1 + 4\frac{f(y_n)}{f(x_n)})(\frac{f(z_n)}{f(y_n)})\right].$$

XIA3:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{4f(x_n)^2 - 5f(x_n)f(y_n) - f(y_n)^2}{4f(x_n)^2 - 9f(x_n)f(y_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}\left[1 + 4\frac{f(z_n)}{f(x_n)}\right]\left[\frac{8f(y_n)}{4f(x_n) - 11f(y_n)} + 1 + \frac{f(z_n)}{f(y_n)}\right].$$

All computations were done using Matlab. We use the following stopping criteria for computer programs: $|x_{n+1} - x_n| < \varepsilon$, $|f(x_{n+1})| < \varepsilon$ and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root a computed. For numerical illustrations in this section we used the fixed stopping criterion $\varepsilon = 10^{-15}$, where ε represents tolerance. We present some numerical test results with the following functions:

$$f_1(x) = x^3 + 4x^2 - 15, \qquad x_* = 1.93198055660636,$$
$$f_2(x) = xe^{x^2} - sin^2(x) + 3cos(x) + 5, \qquad x_* = -1.207647827130919,$$
$$f_3(x) = 10xe^{-x^2} - 1, \qquad x_* = 1.67963061042845,$$
$$f_5(x) = sin^2(x) - x^2 + 1, \qquad x_* = 1.4044916482153411,$$
$$f_4(x) = x^5 + x^4 + 4x^2 - 15, \qquad x_* = 1.347,$$

where x_* is the exact root. The absolute values of the function ($|f(x_n)|$) and ($|x_n - x_{n-1}|$), are shown in Table 1.

The computational results presented in Table 1 shows that in almost all of cases, the presented methods converge more rapidly than Algorithm 1, Algorithm 2 and Algorithm 3. This means that the new methods have better efficiency in computing process than Algorithm 1, Algorithm 2 and Algorithm 3 as the compared other methods, and furthermore, the formula Equation (4) produces the sixth-order methods. For most of the functions we tested, the obtained methods behave at least equal performance compared to the other known methods of the same order.

The computational results presented in Table 1 shows that in almost all of cases, the presented methods converge more rapidly than XIA1, XIA2, and XIA3. This means that the new methods have better efficiency in computing process than XIA1, XIA2, and XIA3 as the compared other methods, and furthermore, the formula Equation (4) produces the sixth-order methods. For most of the functions we tested, the obtained methods behave at least equal performance compared to

Table 1. Comparison of iterative methods.

Function	Method	$\lvert x_k - x_{k-1}\rvert$	$f(x_k)$
$f_1(x), x_0 = 2$	Matinfar	0	1.776356839400251e-015
	XIA1	2.489597032973023e-007	3.552713678800501e-015
	XIA2	2.489597035193469e-007	3.552713678800501e-015
	XIA3	2.489597035193469e-007	3.552713678800501e-015
$f_2(x), x_0 = -1$	Matinfar	0	2.664535259100376e-015
	XIA1	6.661338147750939e-016	1.509903313490213e-014
	XIA2	1.798561299892754e-014	2.664535259100376e-015
	XIA3	1.798561299892754e-014	3.632649736573512e-013
$f_3(x), x_0 = 1.5$	Matinfar	0	2.664535259100376e-015
	XIA1	0	2.220446049250313e-016
	XIA2	0	2.220446049250313e-016
	XIA3	0	2.220446049250313e-016
$f_4(x), x_0 = 1.5$	Matinfar	0	2.220446049250313e-016
	XIA1	0	3.330669073875470e-016
	XIA2	2.220446049250313e-016	4.440892098500626e-016
	XIA3	2.220446049250313e-016	4.440892098500626e-016
$f_5(x), x_0 = 1.2$	Matinfar	0	1.776356839400251e-015
	XIA1	4.280989683049796e-004	1.776356839400251e-015
	XIA2	4.280989683049796e-004	1.776356839400251e-015
	XIA3	4.280989683049796e-004	1.776356839400251e-015

the other known methods of the same order. All numerical tests have been written in Matlab. The numerical results imply that our three-step method has a good performance, and despite being simple and requiring a small number of calculations compared to other existing methods, delivers good numerical results.

4. Conclusions

In this work we presented an approach which can be used to constructing of sixthorder iterative methods that do not require the computation of second or higher derivatives. Also, we proposed a new two-step iterative method for solving nonlinear equations. We showed that the new two-step iterative method has sixth-order convergence. Numerical examples also show that the numerical results of our new two-step method, in equal iterations, improve the results of other existing three-step methods with Ninth-order and Eight-order convergence.

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