The Use of a Runge-Kutta Scheme for an ODE-PDE Model of Supply Chains

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Abstract. Integrating various suppliers to satisfy market demand is of great importance for effective supply chain management. In this paper, we consider the ODE-PDE model of supply chain and apply a classical explicit fourth-order Runge-Kutta scheme for the related ODE model of suppliers. Also, the convergence of the proposed method is proved. Finally a numerical example is studied to demonstrate the accuracy of the proposed method with different choices of time and space meshes.

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1. Introduction

A supply chain is a network, which consists of all stages (e.g order processing, purchasing, inventory control, manufacturing, and distribution) involved in producing and delivering a final product or service. The entire chain connects customers, retailers, distributors, manufacturins and/or suppliers, beginning with the creation of raw material or component parts by suppliers and ending with consumption of the product by customers. Supply chain management (SCM) is related to the coordination of materials, products and information flows among suppliers, manufacturers, distributors, retailers and customers (Simchi-Levi, Kaminsky, & Simchi-Levi,2000). Mean while, supply chain management (SCM) has been a great importance in

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competitive strategy to enhance organizational productivity and profitability. Researchers have paid much attention to issuse concerning supply chain due to demand of market [11] it is obvious that increase of company competition between supply chains highlights the importance of using a proper system for evaluating its performance to recognize competition improvement opportunity. Recently, numerical analysis is of great importance for effective supply chain management.

For instance, [8] and [4] ODE-PDE approaches can be used to existence supply chains. describe a scheme based on upwind and explicit Euler method [4].

The ODE-PDE model for supply chain first proposed in [5], beside the conservation laws formulation proposed in [4] by difference method Euler, in order to increase the accuracy of the results in this work, we used classical explicit fourth-order Runge-Kutta method and provide convergence of the method with Lipschitz condition.

This study is fundamental in order to reduce some unwished for phenomena (bottlenecks, dead times, and so on). The outline of the paper is the following. In section 2, we present the ODE-PDE model. then section 3 describes the convergence and the results of discussions are presented in section 4.

2.   Successive Processors

In this section, the supply chain network model introduced and investigated in [5] and extend the existence results obtained therein.

First, we conside n supply chains with j suppliers, where every supplier j is only linked with the previous supplier $j - 1$. see also Figure 1.
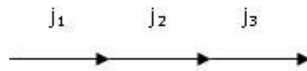

Figure 1. supply chain network model

2.1   *The Model*

In this section, we present an ODE-PDE model for supply chains first proposed in [5] a supply chain is a directed graph consisting of arcs, $\zeta = 1, \cdots, N$ and vertices $\nu = 1, \cdots, N - 1$. Each arc $j \in \zeta$, parameterized by an interval $[a_j, b_j]$, models a supplier. Each vertex is connected to one incoming arc and one outgoing arc, and we assume that arcs are consecutively labeled; i.e., arc j is connected to arc $j + 1$ and $a_j = b_{j-1}$. such model includes time-dependent queues describing the transition of parts among suppliers. Moreover a simulation algorithm using Godunov scheme with boundary conditions at junctions was implemented and tested. A mathematical model describing supply chains where every supplier consists of a processor characterized by its processing time $T_j > 0$, its maximal processing rate $\mu_j > 0$ and $\rho_j(x, t)$ densites of parts in the supplier j at point x and time t. For computing the time evolution of every part the modelling of queues is essential. By assuming FIFO (=First In First Out) policy two cases of queue states can be distinguished: either the queue is empty or non-empty. If the queue is empty, part

x is directly given into the processor j and is produced with time $q_j(t)$. Otherwise the queue is non-empty, so part x and time t has to wait its time of waiting is the inverse of the processing rate. The rate $\frac{L_j}{T_j}$ represents the processing velocity. Now, the dynamics of each processor on an arc j by a conservation law,

$$\partial_t \rho_j(x,t) + \partial_x f_j(\rho_j(x,t)) = 0 \quad \forall x \in [a_j, b_j], t > 0, \tag{1}$$

where the flux function $f_j(\rho_j(x,t))$ is given by

$$f_j : [0, \infty) \to [0, \mu_j] \qquad , f_j(\rho_j(x,t)) = \min\{\mu_j, \frac{L_j}{T_j}\rho_j(x,t)\}$$

Each queue buffer occupancy is a time-dependent function $t \to q_j(t)$ [4].
we have

$$\frac{d}{dt}q_j(t) = f_{j-1}(\rho_{j-1}(b_{j-1},t)) - f_j(\rho_j(a_j,t)) \tag{2}$$

where $f_{j-1}(\rho_{j-1}(b_{j-1},t))$ is defined by the evolution on supplier $j-1$, while the flux on the outgoing arc j is defined [4] as

$$f_j(\rho_j(a_j,t)) = \begin{cases} \min\{f_{j-1}(\rho_{j-1}(b_{j-1},t)), \mu_j\}, & q_j = 0, \\ \mu_j, & q_j > 0. \end{cases} \tag{3}$$

This allows for the following interpretation: If the outgoing buffer is empty, we process as many parts as possible but at most μ_j. notice that the flux $f_j(_j(a_j,t))$ depends on the capacity of the queue: If the queue buffer is empty, the inflow to supplier j is equal to the outflow from supplier $j-1$.
Finally, the complete system of equations is

$$\partial_t \rho(x,t) + \partial_x \min\{\mu_j, \frac{L_j}{T_j}\rho_j(x,t)\} = 0 \qquad , \forall x \in [a_j, b_j], T > 0, j \in \zeta, \tag{4}$$

$$\rho_j(0,x) = \rho_{j,0}(x) \geqslant 0, \qquad \forall x \in [a_j, b_j], \tag{5}$$

$$\frac{d}{dt}q_j(t) = f_{j-1}(\rho_{j-1}(b_{j-1},t)) - f_j(\rho_j(a_j,t)), \qquad j = 2, ..., N, \tag{6}$$

$$q_j(0) = q_{j,0} \geqslant 0, \tag{7}$$

$$f_j(\rho_j(a_j,t)) = \begin{cases} \min\{f_{j-1}(\rho_{j-1}(b_{j-1},t)), \mu_j\}, & q_j(t) = 0, \\ \mu_j, & q_j(t) > 0. \end{cases} \tag{8}$$

LEMMA 2.1 *Consider a supply chain evolution $\rho_j(x,t)$, $q_j(t)$, i.e., a solution to (4)-(8). Then for every $j \in \zeta$, $t \geqslant 0$, and x, it holds that $\rho_j(x,t) \geqslant 0$, $q_j(t) \geqslant 0$ [4].*

2.1.1 Discretization upwind method:

We describe the upwind method with various possible discretizations for the equation (1) by the suppose $(x,t) = (x_i, t_n)$ and for each arc $j \in \zeta$, define a numerical grid in $[0, L_j] \times [0, T]$ using the following notation [4]:

- $\Delta x_j = \frac{L_j}{N_j}$ is the space grid mesh, where Nj is the number of segments in to which we divide the jth supplier;

- $\Delta t_j = \frac{T}{\eta_j}$ is the time grid mesh, where η_j denotes the number of segments into which $[0,T]$ is divided;

- $(x_i, t^n)_j = (i\Delta x_j, n\Delta t_j), i = 0, \cdots, N_j, n = 0, \cdots, \eta_j$, are the grid points;

- $^j\rho_i^n$ is the value taken by the approximated density at the point $(x_i, t^n)_j$;

- q_j^n is the value taken by the approximate queue buffer occupancy at time t^n.

  Now, discretization (1):

$$\frac{\rho_j(x_i,t_{n+1})-\rho_j(x_i,t_n)}{\Delta t} + \frac{\frac{L_j}{T_j}[\rho_j(x_i,t_n)-\rho_j(x_{i-1},t_n)]}{\Delta x} = 0 \quad j \in \zeta, i = 0\cdots, N_j, n = 0, \cdots \eta_j.$$

$$\frac{\rho_j(x_i,t_{n+1})-\rho_j(x_i,t_n)}{\Delta t} + \frac{L_j}{T_j}\frac{[\rho_j(x_i,t_n)-\rho_j(x_{i-1},t_n)]}{\Delta x} = 0,$$

$$\rho_j(x_i,t_{n+1}) - \rho_j(x_i,t_n) + \frac{L_j\Delta t}{\Delta x T_j}[\rho_j(x_i,t_n) - \rho_j(x_{i-1},t_n)] = 0,$$

with the strengthening factor condition given by

$$\Delta t \leqslant \frac{\Delta x}{max_j \frac{L_j}{T_j}}. \tag{9}$$

Therefore, we have:

$$^j\rho_i^{n+1} =^j \rho_i^n - \frac{\Delta t}{\Delta x}\frac{L_j}{T_j}(^j\rho_i^n -^j \rho_{i-1}^n), \quad j \in \zeta, i = 0\cdots, N_j, n = 0, \cdots \eta_j. \tag{10}$$

2.1.2   classical explicit fourth- order Runge-Kutta method

The classical explicit fourth-order Runge-Kutta method has been proposed in [3] we apply this method for (2).
Let $t = t_n$

$$L_1 = \Delta t[f_{j-1}(\rho_{j-1}(b_{j-1}, t_n)) - f_j(\rho_j(a_j, t_n))], \tag{11}$$

$$L_2 = \Delta t[f_{j-1}(\rho_{j-1}(b_{j-1}, t_n) + \frac{L_1}{2}) - (f_j(\rho_j(a_j, t_n) + \frac{\Delta t}{2})], \tag{12}$$

$$L_3 = \Delta t[f_{j-1}(\rho_{j-1}(b_{j-1}, t_n) + \frac{L_2}{2}) - (f_j(\rho_j(a_j, t_n) + \frac{\Delta t}{2})], \tag{13}$$

$$L_4 = \Delta t[f_{j-1}(\rho_{j-1}(b_{j-1}, t_n) + (f_j(\rho_j(a_j, t_n) + L_3)]. \tag{14}$$

Then we have,

$$q_j(t_{n+1}) = q_j(t_n) + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4). \tag{15}$$

2.1.3   Correction of numerical fluxes in the case of negative queues

The ODE numerical scheme does not necessarily maintain the positivity properties of the true solutions given by Lemma 2.1 we thus modify the Runge-Kutta scheme so as to accomplish positivity of queue buffer occupancies. consider a supplier j and a time interval $[t^n, t^{n+1}[$ so that $q^{n+1} < 0 < q^n$ then we define $q_j(t)$ for every time t by linear interpolation namely,

$$q_j(t) = \frac{q_j^{n+1} - q_j^n}{\Delta t}t + \frac{q_j^n t^{n+1} - q_j^{n+1} t^n}{\Delta t}, \quad t \in [t^n, t^{n+1}[. \tag{16}$$

Then $q(0)$ vanishes at some time $\bar{t} > t^n$, which is computed by (16) correcting to zero $q_j(t)$, $t \in [\bar{t}, t^{n+1}]$ the following numerical correction for the entering flux $f_{j,inc}^n$ is needed [4]:

$$f_{j,inc}^n = \frac{1}{\Delta t}[\Delta t' \mu_j + (\Delta t - \Delta t')f_{j-1,out}^n]. \tag{17}$$

We define,

$$\bar{t} = t^n + \Delta t',$$

   which is computed by (15) and (16):

$$\Delta t' = \frac{\Delta t q_j^n}{q_j^{n+1} - q_j^n} = \frac{\Delta t q_j^n}{\frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4)}. \tag{18}$$

The correction (18) on the boundary incoming data for the supplier j influences the approximation an alternative classical explicit Runge-Kutta numerical method for avoiding negativity of queues is the use of adaptive time meshes, where Δt is replaced by $\Delta t'$ computed in (18).

3.   Convergence

The aim of this section is to study the convergence of the upwind- classical explicit Runge-Kutta numerical method.
   We also consider the following single-step method:

$$q_j(t_{n+1}) = q_j(t_n) + \Delta t \Phi(x_j, q_j, \Delta t), \tag{19}$$

   Comparing (15) and (19) we conclude that the fourth-order Runge-Kutta method term:

$$\Phi(x_i, q_j, \Delta t) = \frac{1}{6\Delta t}[L_1 + 2L_2 + 2L_3 + L_4]. \tag{20}$$

   On the other hand, we have the following equation (2) if flux f_{j-1}, f_j and $L_i, i =$

$1, \cdots, 4$ satisfies the Lipschitz condition,

$$|(f_{j-1}(\rho_{j-1}(b_{j-1}, t))) - f_j(\rho_j(a_j, t))) - (f_{j-2}(\rho_{j-2}(b_{j-2}, t)) - f_j(\rho_j(a_j, t)))|$$
$$\leqslant L|f_{j-1}(\rho_{j-1}(b_{j-1}, t))) - f_{j-2}(\rho_{j-2}(b_{j-2}, t))|.$$

then, we apply the flux f_{j-1}, f_j and $L_i, i = 1, \cdots, 4$ to Runge-Kutta method.

So we taylor series expansion have the following differential equation

$$q(t_{j+1}) = q(t_j + \Delta t) = q(t_j) + \Delta t q'(t_j) + \frac{\Delta t^2}{2} q''(t_j) + \cdots$$

Then we have, the following equation (11), (12), (13), (14)

$$|L_1 - L_1^*| = \Delta t |(f_{j-1}(\rho_{j-1}(b_{j-1}, t)) - f_j(\rho_j(a_j, t))) - (f_{j-1}^*(\rho_{j-1}(b_{j-1}, t)) - f_j(\rho_j(a_j, t)))|$$
$$\leqslant \Delta t L|(f_{j-1}(\rho_{j-1}(b_{j-1}, t))) - f_{j-1}^*(\rho_{j-1}(b_{j-1}, t))|, \tag{21}$$

$$|L_2 - L_2^*| = \Delta t |(f_{j-1}(\rho_{j-1}(b_{j-1}, t)) + \tfrac{1}{2}L_1) - (f_j(\rho_j(a_j, t)) + \tfrac{1}{2}\Delta t) -$$
$$((f_{j-1}^*(\rho_{j-1}(b_{j-1}, t)) + \tfrac{1}{2}L_1^*) - (f_j(\rho_j(a_j, t)) + \tfrac{1}{2}\Delta t)|$$
$$\leqslant \Delta t L\{|(f_{j-1}(\rho_{j-1}(b_{j-1}, t))) - f_{j-1}^*(\rho_{j-1}(b_{j-1}, t))| + \tfrac{1}{2}|L_1 - L_1^*|\}$$
$$\leqslant \Delta t L\{|(f_{j-1}(\rho_{j-1}(b_{j-1}, t))) - f_{j-1}^*(\rho_{j-1}(b_{j-1}, t))| +$$
$$\tfrac{1}{2}(\Delta t L|(f_{j-1}(\rho_{j-1}(b_{j-1}, t))) - f_{j-1}^*(\rho_{j-1}(b_{j-1}, t))|\}$$
$$\leqslant \Delta t L(1 + \tfrac{1}{2}\Delta t L)|(f_{j-1}(\rho_{j-1}(b_{j-1}, t))) - f_{j-1}^*(\rho_{j-1}(b_{j-1}, t))|, \tag{22}$$

$$|L_3 - L_3^*| = \Delta t L|(f_{j-1}(\rho_{j-1}(b_{j-1}, t)) + \tfrac{1}{2}L_2) - (f_j(\rho_j(a_j, t)) + \tfrac{1}{2}\Delta t) -$$
$$|(f_{j-1}^*(\rho_{j-1}(b_{j-1}, t)) + \tfrac{1}{2}L_2^*) - (f_j(\rho_j(a_j, t)) + \tfrac{1}{2}\Delta t)$$
$$\leqslant |\Delta L\{|f_{j-1}(\rho_{j-1}(b_{j-1}, t)) - f_{j-1}^*(\rho_{j-1}(b_{j-1}, t))| + \tfrac{1}{2}|L_2 - L_2^*| \tag{23}$$
With replacing (21) and (22):
$$\leqslant \Delta t L\{1 + \tfrac{1}{2}\Delta t L(1 + \tfrac{1}{2}\Delta t L)\}|(f_{j-1}(\rho_{j-1}(b_{j-1}, t))) -$$
$$f_j^*(\rho_j(a_j, t))|,$$

$$|L_4 - L_4^*| \leqslant \Delta t L\{1 + \frac{1}{2}\Delta t L(1 + \frac{1}{2}\Delta t L)\}|(f_{j-1}(\rho_{j-1}(b_{j-1}, t) - f_j^*(\rho_j(a_j, t))|. \tag{24}$$

Now we can prove $\Delta t \to 0$ (20), (21), (22), (23) and (24) applies Lipschitz condition

$$|\Phi(x_j, q_j, \Delta t) - \Phi^*(x_j, q_j, \Delta t)| = \tfrac{1}{\Delta t}|(\tfrac{1}{6}L_1 + \tfrac{2}{6}L_2 + \tfrac{2}{6}L_3 + \tfrac{1}{6}L_4) - \tfrac{1}{6}(L_1^* + 2L_2^* + 2L_3^* + L_4^*)|$$
$$= \tfrac{1}{6}|(L_1 - L_1^*) + 2(L_2 - L_2^*) + 2(L_3 - L_3^*) + (L_4 - L_4^*)|$$
$$\leqslant \tfrac{1}{\Delta t}\{\tfrac{1}{6}|L_1 - L_1^*| + \tfrac{2}{6}|L_2 - L_2^*| + \tfrac{2}{6}|L_3 - L_3^*| + \tfrac{1}{6}|L_4 - L_4^*|\},$$

by (21),(22), (23) and (24):

$$|\Phi - \Phi^*| \leqslant L\{1 + \frac{1}{6}\Delta t L + \frac{1}{6}\Delta t\}|(f_{j-1}(\rho_{j-1}(b_{j-1}, t))) - f_{j-1}^*(\rho_{j-1}(b_{j-1}, t))|,$$

if $\Delta t \to 0$

$$|\Phi - \Phi^*| \leqslant L|(f_{j-1}(\rho_{j-1}(b_{j-1}, t))) - f^*_{j-1}(\rho_{j-1}(b_{j-1}, t))|.$$

Therefore, the convergence of the method is proved.

4.   Numerical Results

The exmaple we consider is similar to the one in [1] a supply chain with $N = 4$ suppliers and the following parameters: $L_j = T_j = 1$, $j = 1, \cdots, 4$.

We discretization the system (15) using an upwind-scheme for upwind- classical explicit Runge-Kutta method. numerical for negative queue buffer occupancies. therein, each arc j could have different space increments, namely $\Delta x_j = \frac{L_j}{N_j}$, where N_j is the number of space discretization points. the time steps Δt are constant and satisfy the amplification factor condition on each arc.

The program has been provided with Matlab 7.12 to compare the results with [4], we conclude the following: consisting of maximal fluxes, processing times, and lengths of each supplier are reported in Table 1.

Table 1. A supply chain with N=4 suppliers.

Supplier	μ_j	T_j	L_j
1	99	1	1
2	12	1	1
3	10	1	1
4	8	1	1

A total simulation time $T = 300$ and for the first arc the inflow profile is given by [4] to fix notation, let us define the following [4]:

Different Spatial Steps Method (DSSM): Upwind scheme for densities, equation (10) the numerical grid is defined by choosing a fixed time grid mesh Δt; then different space grid meshes are necessary.

Different Temporal Steps Method (DTSM): Upwind method for densities, equation (10) the numerical grid is defined by choosing a fixed space grid mesh Δx; then different time grid meshes are necessary.

Table 2.   CPU times and $L^1 errors$ using   DSSM

Δt	Δx_1	Δx_2	Δx_3	Δx_4	CPU	proposed method $L^1 errors$	Method in[4] $L^1 errors$
0.00625	0.00625	0.00125	0.00125	0.00125	1.671	0.0021	0.0029
0.0125	0.0125	0.0025	0.0025	0.0025	0.140	0.0115	0.027847
0.025	0.025	0.005	0.005	0.005	0.046	0.01982	0.132399

Table 3.   CPU times and $L^1 errors$ using   DTSM

Δx	Δt_1	Δt_2	Δt_3	Δt_4	CPU	proposed method $L^1 errors$	Method in[4] $L^1 errors$
0.00625	0.0025	0.01875	0.0125	0.00625	0.828	0.00162	0.003435
0.0125	0.05	0.00375	0.0025	0.00125	0.140	0.03132	0.059411
0.025	0.1	0.075	0.05	0.025	0.046	0.07581	0.127714

5.   Conclusion

A manufacturer can determine few suppliers to build a long-term close relationship with them. The proposed method has been applied to find approximate solutions of classical explicit fourth-order Runge-Kutta method. The method is based on computing the coefficients in the upwind-Euler scheme for an ODE-PDE model of supplier by classical explicit fourth-order Runge-Kutta method. It is observed that the method has the best advantage when the known functions in an equation can be expanded to using iterative methods which converge rapidly. The method can be developed and applied to supply chain.

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