Non-polynomial quartic spline solution of boundary - value problem

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Abstract. Quartic non-polynomial spline function approximation in off step points is developed,for the solution of fourth-order boundary value problems.Using consistency relation of such spline and suitable choice of parameter,we have obtained second, fourth and sixth-orders methods.Convergence analysis of sixth-order method has been given.The methods are illustrated by some examples,to verify the order of accuracy of the presented methods. The computed results are compared with other exiting methods,collocation,decomposition and spline methods.Computed result verify the applicability and accuracy of our presented methods.

Keywords: Non polynomial quartic spline, Boundary value problems, Boundary formula, Convergence analysis.

1. Introduction

We consider the fourth- order boundary value problem of the form:

$$y^{(4)}(x) + f(x)y(x) = g(x), \quad a \leqslant x \leqslant b \tag{1}$$

with boundary conditions

$$y^{(i)}(a) = A_i, \quad y^{(i)}(b) = B_i, \quad i = 0, 2 \tag{2}$$

where A_i, B_i, $(i = 0, 2)$ are finite real constants. These problems always arise in study of plate deflection theory. Details of the solution of (1) are given in [1,2,3].

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Finite difference methods can be used for the existence and uniqueness numerical approximation of the solution of (1), [3-6]. Fyfe [7] has used cubic splines for the smooth approximation of the solution of a certain fourth order boundary value problem. Usmani [8] has developed second and fourth orders methods based on quintic and sextic spline functions. Later on Usmani [9] has developed second order method based on quartic polynomial spline functions.

In this paper we used non-polynomial quartic spline to develop a family of new numerical methods to obtain smooth approximations to the solution of fourth-order differential equation. The new methods are of order two, four, and six.

The new methods perform better accuracy in comparison with other existing methods collocation, decomposition and conventional spline methods and thus represent an improvement over existing methods. The spline functions proposed in this paper in each subinterval $x_{i-\frac{1}{2}} \leqslant x \leqslant x_{i+\frac{1}{2}}$, has the following form:

$$\text{span}\{1, \text{x}, \text{x}^2, \sin(|\text{k}|\text{x}), \cos(|\text{k}|\text{x})\},$$

where k is the frequency of the trigonometric part of the spline function, which can be real or pure imaginary, and $\text{span}\{1, \text{x}, \text{x}^2, \text{x}^3, \text{x}^4\}$ when $k \to 0$. In Section 2, numerical methods based on non-polynomial spline are developed for solving Eq.(1) along with boundary condition (2).

Development of the boundary formulas are considered in Section 3. The class of methods are given in section 4. convergence analysis of the methods are given in section 5, Section 6 is devoted to numerical experiments, discussion and comparison with other known methods.

2. Numerical methods

To develop the smooth approximation for solution of boundary value problem (1) along with the boundary conditions (2), we define a grid of n equally spaced points over $[a, b]$, $x_0 = a$, $x_{i-\frac{1}{2}} = a + (i - \frac{1}{2})h$, and $h = \frac{b-a}{n}$, $i = 1, \ldots, n$, $x_n = b$.

For each segment, the non-polynomial spline function $p_i(x)$ can be define as :

$$p_i(x) = a_i \cos k(x - x_i) + b_i \sin k(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i) + e_i \quad i = 0, 1, \ldots, n \tag{3}$$

where, a_i, b_i, c_i, d_i, e_i are real finite constants and k is arbitrary parameter to be determine.

Let y_i be the spline approximation to $y(x_i)$, obtained by the segment $p_i(x)$ of the mixed splines function passing through the points $(x_{i-\frac{1}{2}}, y_{i-\frac{1}{2}})$ and $(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$. To obtain necessary conditions for determination of coefficients introduced in (3), we do not only required that $p_i(x)$ satisfies interpolatory conditions, at $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$, but also continuity condition of first, second, and third derivatives at the common nodes (x_{i+j}, y_{i+j}). We first denote the following expressions:

$$p_i(x_{i+j}) = y_{i+j}, \quad p_i'(x_{i+j}) = D_{i+j}, \quad p_j''(x_{i+j}) = M_{i+j},$$
$$p_i'''(x_{i+j}) = L_{i+j}, \quad p_i^{(4)}(x_{i+j}) = F_{i+j} \tag{4}$$

(for the relevant values of $j = \pm\frac{1}{2}$).

Algebraic manipulation yields the following expressions, where by $\theta = kh$ and $i = 0, 1, \ldots, n-1$.

$$a_i = \frac{F_{i+\frac{1}{2}}}{k^4 \cos(\frac{\theta}{2})} + \frac{L_i \tan(\frac{\theta}{2})}{k^3}, \quad b_i = -\frac{L_i}{k^3}, \quad c_i = \frac{M_{i+\frac{1}{2}}}{2} + \frac{F_{i+\frac{1}{2}}}{2k^2},$$

$$d_i = D_i + \frac{L_i}{k^2}, \quad e_i = y_{i+\frac{1}{2}} + F_{i+\frac{1}{2}}(\frac{-1}{k^4} - \frac{h^2}{8k^2}) - h^2 \frac{M_{i+\frac{1}{2}}}{8} - h\frac{L_i}{2k^2} - h\frac{D_i}{2}$$

(5)

Continuity condition of the first, second, and third derivatives at knots, $p_{i-1}^{(\mu)}(x_i) = p_i^{(\mu)}(x_i)$, where $\mu = 0, 1, 2, 3$ yields the following relations:

$$\frac{h}{2}(D_i - D_{i-1}) = (y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}}) + (\frac{-\cos\theta}{k^4 \cos(\frac{\theta}{2})} - \frac{3h^2}{8k^2} + \frac{1}{k^4})F_{i-\frac{1}{2}}$$

$$+ \left(\frac{1}{k^4 \cos(\frac{\theta}{2})} - \frac{1}{k^4} - \frac{h^2}{8k^2}\right) F_{i+\frac{1}{2}} - \frac{3h^2}{8} M_{i-\frac{1}{2}} \qquad (6)$$

$$- \frac{h^2}{8} M_{i+\frac{1}{2}} + (L_i + L_{i-1}) \left(\frac{\tan(\frac{\theta}{2})}{k^3} - \frac{h}{2k^2}\right)$$

$$D_i - D_{i-1} = hM_{i-\frac{1}{2}} + \left(-\frac{\sin(\theta)}{k^3 \cos(\frac{\theta}{2})} + \frac{h}{k^2}\right) F_{i-\frac{1}{2}} \qquad (7)$$

$$\frac{\tan(\frac{\theta}{2})}{k}(L_i + L_{i-1}) = \left(\frac{\cos\theta}{k^2 \cos(\frac{\theta}{2})} - \frac{1}{k^2}\right) F_{i-\frac{1}{2}} + \left(\frac{-1}{k^2 \cos(\frac{\theta}{2})} + \frac{1}{k^2}\right) F_{i+\frac{1}{2}}$$

$$+ M_{i+\frac{1}{2}} - M_{i-\frac{1}{2}} \qquad (8)$$

$$L_i - L_{i-1} = \frac{2 \sin(\frac{\theta}{2})}{k} F_{i-\frac{1}{2}} \qquad (9)$$

Using Eqs. (6)-(8), it follows that

$$hD_i = (y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}}) + \left(\frac{h}{2k^3 \tan(\frac{\theta}{2})} - \frac{h}{2k^3 \sin(\frac{\theta}{2})} + \frac{h^2}{8k^2}\right) (F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}})$$

$$+ \left(\frac{1}{k^2} - \frac{h}{2k \tan(\frac{\theta}{2})} - \frac{h^2}{8}\right) (M_{i+\frac{1}{2}} - M_{i-\frac{1}{2}})$$

(10)

Also from Eqs. (8), and (9), it follows that

$$L_i = \left(\frac{1}{2k \sin(\frac{\theta}{2})} - \frac{1}{2k \tan(\frac{\theta}{2})}\right) (F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}}) + \frac{k}{2 \tan(\frac{\theta}{2})}(M_{i+\frac{1}{2}} - M_{i-\frac{1}{2}}) \quad (11)$$

Eliminating L's from Eqs. (9), (11), and D's from the Eqs. (7), (10), it follows that

$$\left(\frac{1}{2k\tan(\frac{\theta}{2})} - \frac{1}{2k\sin(\frac{\theta}{2})}\right)(F_{i+\frac{1}{2}}+F_{i-\frac{3}{2}}) + \left(\frac{1}{k\sin(\frac{\theta}{2})} - \frac{1}{k\tan(\frac{\theta}{2})} + \frac{2\sin(\frac{\theta}{2})}{k}\right)F_{i-\frac{1}{2}}$$

$$= \frac{k}{2\tan(\frac{\theta}{2})}\left(-M_{i+\frac{1}{2}} + 2M_{i-\frac{1}{2}} - M_{i-\frac{3}{2}}\right) \quad (12)$$

$$h^2 M_{i-\frac{1}{2}} = y_{i-\frac{3}{2}} - 2y_{i-\frac{1}{2}} + y_{i+\frac{1}{2}} + (F_{i-\frac{3}{2}} + F_{i+\frac{1}{2}})\left(\frac{1}{k^4\cos(\frac{\theta}{2})} - \frac{h^2}{8k^2\cos(\frac{\theta}{2})} - \frac{1}{k^4}\right)$$

$$+ F_{i-\frac{1}{2}}\left(\frac{4\tan(\frac{\theta}{2})\sin(\frac{\theta}{2})}{k^4} + \frac{h^2}{4k^2\cos(\frac{\theta}{2})}\right) \quad (13)$$

$$- \frac{h^2\sin(\frac{\theta}{2})\tan(\frac{\theta}{2})}{2k^2} + \frac{2}{k^4} - \frac{2}{k^4\cos(\frac{\theta}{2})} - \frac{h^2}{k^2}\right)$$

Eliminating of M's from the Eqs. (12) and (13), gives the main consistency relation

$$y_{i-\frac{5}{2}} - 4y_{i-\frac{3}{2}} + 6y_{i-\frac{1}{2}} - 4y_{i+\frac{1}{2}} + y_{i+\frac{3}{2}} = h^4\left[\alpha(F_{i-\frac{5}{2}} + F_{i+\frac{3}{2}}) + \beta(F_{i-\frac{3}{2}} + F_{i+\frac{1}{2}}) + \gamma F_{i-\frac{1}{2}}\right]$$

$$i = 3,\ldots, n-2, \quad (14)$$

where

$$\alpha = \frac{-8 + \theta^2 + 8\cos(\frac{\theta}{2})}{8\theta^4\cos(\frac{\theta}{2})}, \quad \beta = \frac{8 + 3\theta^2 - 16\cos(\frac{\theta}{2}) - (-8 + \theta^2)\cos\theta}{4\theta^4\cos(\frac{\theta}{2})}$$

$$\gamma = \frac{-8 + \theta^2 + 24\cos(\frac{\theta}{2}) + 2(8 + 3\theta^2)\cos\theta}{4\theta^4\cos(\frac{\theta}{2})} \quad i = 3,\ldots, n-2,$$

where $\lim_{k\to 0}(\alpha, \beta, \gamma) = \frac{1}{384}(1, 76, 230)$, which is the consistency relation based on quartic polynomial spline functions derived in [9].

Now from equation (1) we have

$$y^{(4)}(x_{i-\frac{1}{2}}) = -f_{(x_{i-\frac{1}{2}})}y_{(x_{i-\frac{1}{2}})} + g_{(x_{i-\frac{1}{2}})}, \quad i = 1,\ldots, n, \quad (15)$$

By using (15) in (14) we can obtain:

$$(1 + \alpha h^4 f_{i-\frac{5}{2}})y_{i-\frac{5}{2}} + (-4 + \beta h^4 f_{i-\frac{3}{2}})y_{i-\frac{3}{2}} + (6 + \gamma h^4 f_{i-\frac{1}{2}})y_{i-\frac{1}{2}}$$

$$+ (-4 + \beta h^4 f_{i+\frac{1}{2}})y_{i+\frac{1}{2}} + (1 + \alpha h^4 f_{i+\frac{3}{2}})y_{i+\frac{3}{2}}$$

$$= h^4[\alpha(g_{i-\frac{5}{2}} + g_{i+\frac{3}{2}}) + \beta(g_{i-\frac{3}{2}} + g_{i+\frac{1}{2}}) + \gamma(g_{i-\frac{1}{2}})], \quad i = 3,\ldots, n-2, \quad (16)$$

The relation given in Eq.(16), gives $(n-4)$ linear equations in n unknowns y_i, $i = 1,\ldots, n$. Four more equations, two at each end of the range of integration are

needed to be associated with (16), these boundary equations would be discussed in next section.

3. Development of the boundary formulas

To obtain unique solution we need four more equations to be associate with (16) so that we use the following boundary conditions. In order to obtain the sixth-order boundary formula we define the following identities:

$$a_0 y_0 + \sum_{i=1}^{4} a_i y_{i-\frac{1}{2}} + c_0 h^2 y_0'' = h^4 \left(\sum_{i=1}^{5} b_i y_{i-\frac{1}{2}}^{(4)} \right) + t_1 \tag{17}$$

$$a_0' y_0 + \sum_{i=1}^{5} a_i' y_{i-\frac{1}{2}} + c_0' h^2 y_0'' = h^4 \left(\sum_{i=1}^{6} b_i' y_{i-\frac{1}{2}}^{(4)} \right) + t_2 \tag{18}$$

$$a_0' y_n + \sum_{i=1}^{5} a_{6-i}' y_{n+i-\frac{11}{2}} + c_0' h^2 y_n'' = h^4 \left(\sum_{i=1}^{6} b_{7-i}' y_{n+i-\frac{13}{2}}^{(4)} \right) + t_{n-1} \tag{19}$$

$$a_0 y_n + \sum_{i=1}^{4} a_{5-i} y_{n+i-\frac{9}{2}} + c_0 h^2 y_n'' = h^4 \left(\sum_{i=1}^{5} b_{6-i} y_{n+i-\frac{11}{2}}^{(4)} \right) + t_n \tag{20}$$

where a_0, a_0', a_i, a_i', c_0, c_0', b_i, b_i' are arbitrary parameters to be determined. In order to obtain the sixth - order method we find that:

$$(a_0, a_1, a_2, a_3, a_4) = \left(1, -\frac{178787}{105664}, \frac{36547}{40640}, -\frac{117537}{528320}, \frac{8041}{528320} \right)$$

$$(b_1, b_2, b_4) = \left(-\frac{177528097}{3043123200}, -\frac{94231181}{3043123200}, -\frac{6262211}{3043123209} \right)$$

$$(b_3, b_5) = \left(\frac{9564917}{1014374400}, \frac{432400}{1521561600} \right)$$

$$c_0 = -\frac{52353}{264160}, \quad c_0' = \frac{8653677}{3643552}$$

$$(a_0', a_1', a_2', a_3', a_4', a_5') = \left(1, -\frac{5192821}{7287104}, \frac{99289925}{7287104}, -\frac{5548377}{662464}, -\frac{5548377}{662464}, \frac{910165}{7287104}, 1 \right)$$

$$(b_1', b_2', b_4') = \left(1, \frac{21296617201}{6677637120}, \frac{34188749633}{97938677760} \right)$$

$$(b_3', b_5', b_6') = \left(\frac{308276689813}{293816033280}, -\frac{20495398853}{293816033280}, \frac{3224877857}{293816033280} \right)$$

4.   Class of the methods

Expanding the given method in(16) about, x_i, by Taylor's series, the local trunca-
tion errors t_i, $i = 3, \ldots, n - 2$, associated with our method in Eq.(16) is

$$
\begin{aligned}
t_i =& (1 - 2(\alpha + \beta) + \gamma)h^4 y_i^{(4)} + \frac{1}{2}(-1 + 2\alpha + 2\beta + \gamma)h^5 y_i^{(5)} \\
&+ \frac{1}{24}(7 - 3(34\alpha + 10\beta + \gamma))h^6 y_i^{(6)} + \frac{1}{48}(-5 + 98\alpha + 26\beta + \gamma)h^7 y_i^{(7)} \qquad (21) \\
&+ \frac{1}{1920}(69 - 5(706\alpha + 82\beta + \gamma))h^8 y_i^{(8)} + \frac{1}{11520}(-115 + 8646\alpha + 726\beta + 3\gamma)h^9 y_i^{(9)} \\
&+ \frac{1}{967680}(2497 - 21(16354\alpha + 730\beta + \gamma)h^{10} y_i^{(10)} + O(h^{11}), \quad 3 \leqslant i \leqslant n - 2.
\end{aligned}
$$

By choosing different values of α, β, and γ provided that $\gamma = 1 - 2(\alpha + \beta)$, we will
obtain the following methods,
   Remark 1. By choosing $\alpha = \frac{4}{900}$, $\beta = \frac{3}{20}$, and $\gamma = \frac{622}{900}$ we obtain the second -
order method with truncation error $t_i = -\frac{1}{900}h^6 y_i^{(6)} + O(h^7)$.
   Remark 2. If we use $\alpha = 0$, $\beta = \frac{1}{6}$, and $\gamma = \frac{4}{6}$ we will obtain the fourth - order
method with truncation error $t_i = -\frac{1}{720}h^8 y_i^{(8)} + O(h^9)$.
   Remark 3. For $\alpha = -\frac{1}{720}$, $\beta = \frac{124}{720}$, and $\gamma = \frac{474}{720}$ we obtain the sixth - order
method with truncation error $t_i = \frac{1}{3024}h^{10} y_i^{(10)} + O(h^{11})$.

5.   Convergence analysis

We investigate the convergence analysis of the presented method describe in previ-
ous section.The resulting linear system of equations may be writing in matrix for
as

$$AY = C + T \qquad (22)$$
$$A\widetilde{Y} = C \qquad (23)$$
$$AE = T, \qquad (24)$$

   where

$$
\begin{aligned}
&Y = (y_{i+\frac{1}{2}}), \quad \widetilde{Y} = (\widetilde{y}_{i+\frac{1}{2}}), \\
&C = (c_i), \quad T = (t_i), \qquad\qquad (25) \\
&E = (\widetilde{e}_{i+\frac{1}{2}}) = (y_{i+\frac{1}{2}} - \widetilde{y}_{i+\frac{1}{2}})
\end{aligned}
$$

are n-dimensional column vectors.The coefficient matrix of A is define as

$$A = M_0 + D + h^4 BF, \quad F = diag(f_{i+\frac{1}{2}}), \ i = 0, 1, ..., n - 1 \qquad (26)$$

and M_0, B are five-band matrices. The five-band symmetric matrix M_0 has the

following form

$$
M_0 = \begin{bmatrix}
10 & -5 & 1 & & & & & & \\
-5 & 6 & -4 & 1 & & & & & \\
1 & -4 & 6 & -4 & 1 & & & & \\
 & & . & . & . & & & & \\
 & & & . & . & . & & & \\
 & & & & . & . & . & & \\
 & & & & 1 & -4 & 6 & -4 & 1 \\
 & & & & & 1 & -4 & 6 & -5 \\
 & & & & & & 1 & -5 & 10
\end{bmatrix} = P^2,
\tag{27}
$$

where the tridiagonal matrix P is define as

$$
p_{ij} = \begin{cases}
3, & i = g = 1, n \\
2, & i = j = 2, 3, \ldots, n - 1 \\
-1, & |i - j| = 1 \\
0, & otherwise
\end{cases}
$$

and

$$
B = \begin{bmatrix}
b_1 & b_2 & b_3 & b_4 & b_5 & & & & \\
b'_1 & b'_2 & b'_3 & b'_4 & b'_5 & b'_6 & & & \\
\alpha & \beta & \gamma & \beta & \alpha & & & & \\
 & \alpha & \beta & \gamma & \beta & \alpha & & & \\
 & & . & . & . & & & & \\
 & & & . & . & . & & & \\
 & & & \alpha & \beta & \gamma & \beta & \alpha & \\
 & & b'_6 & b'_5 & b'_4 & b'_3 & b'_2 & b'_1 & \\
 & & & b_5 & b_4 & b_3 & b_2 & b_1 &
\end{bmatrix}
\tag{28}
$$

$$
D = \begin{bmatrix}
d_1 & d_2 & d_3 & d_4 & & & & \\
d'_1 & d'_2 & d'_3 & d'_4 & d'_5 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & \\
 & 0 & 0 & 0 & 0 & 0 & & \\
 & & . & . & . & & & \\
 & & & . & . & . & & \\
 & & 0 & 0 & 0 & 0 & 0 & \\
 & & d'_5 & d'_4 & d'_3 & d'_2 & d'_1 & \\
 & & & d_4 & d_3 & d_2 & d_1 &
\end{bmatrix}
\tag{29}
$$

where

$$
(d_1, d_2, d_3, d_4) = \left(-\frac{1235427}{105664}, \frac{239747}{40640}, -\frac{64857}{528320}, \frac{8041}{528320} \right)
$$

$$
(d'_1, d'_2, d'_3, d'_4, d'_5) = \left(\frac{31242699}{7287104}, \frac{55567301}{7287104}, -\frac{2898521}{662464}, -\frac{8197269}{7287104}, 1 \right)
$$

And the known vector C is given as

$$\widetilde{c}_i = \begin{cases} -a_0 A_0 - c_0 h^2 B_0 + h^4 \left(\sum_{i=1}^{5} b_i g_{i-\frac{1}{2}} \right), & i = 1; \\ -a_0' A_0 - c_0' h^2 B_0 + h^4 \left(\sum_{i=1}^{6} b_i' g_{i-\frac{1}{2}} \right), & i = 2; \\ h^4 [\alpha(g_{i-\frac{5}{2}} + g_{i+\frac{3}{2}}) + \beta(g_{i-\frac{3}{2}} + g_{i+\frac{1}{2}}) + \gamma g_{i-\frac{1}{2}}], & i = 3, ..., n-2; \\ -a_0' A_2 - c_0' h^2 B_2 + h^4 \left(\sum_{i=1}^{6} b_{7-i}' g_{n+i-\frac{13}{2}} \right), & i = n-1; \\ -a_0 A_2 - c_0 h^2 B_2 + h^4 \left(\sum_{i=1}^{5} b_{6-i} g_{n+i-\frac{11}{2}} \right), & i = n. \end{cases} \quad (30)$$

Our purpose is to derive a bound on $\|E\|$. but from Eq (26), since M_0 is monotone matrix, then M_0^{-1} is exists. We have to show $A = M_0(I + M_0^{-1}(D + h^4 BF))$ is invertabel if $\|M_0^{-1}(D + h^4 Bf)\| < 1$, (by using Numman Lemma). Then from equation (24)

$$E = A^{-1}T = (M_0 + D + h^4 BF)^{-1}T = [M_0(I + M_0^{-1}(D + h^4 BF))]^{-1}T$$

$$\|E\| \leqslant \|[I + M_0^{-1}(D + h^4 Bf)]^{-1}\| . \|M_0^{-1}\| . \|T\|, \quad (31)$$

Following Usmani in [9] we have

$$\|M_0^{-1}\| \leqslant \frac{1}{384 h^4}(5(b-a)^4 + 10(b-a)^2 h^2 + 9h^4), \quad (32)$$

and also we have $\|T\| \leqslant \frac{793}{100000} h^{10} M_{10}$ where $M_{10} = max_{a \leqslant \zeta \leqslant b} |y^{10}(\zeta)|$, $\|B\| \leqslant \frac{566828}{100000}$ thus by using Numman Lemma we have

$$\|E\| \leqslant \frac{\|M_0^{-1}\| \|T\|}{1 - \|M_0^{-1}(D + h^4 BF)\|}, \quad (33)$$

Therefore we obtain

$$\|E\| \leqslant \frac{793 h^{10} w M_{10}}{-1882901 w + h^4(38400000 - 566828 w \|F\|)} \equiv O(h^6) \quad (34)$$

provided that

$$\|F\| \leqslant \frac{63398400000 h^4 - 3108670000 w}{935760384 h^4 w}. \quad (35)$$

where

$$w = (5(b-a)^4 + 10(b-a)^2 h^2 + 9h^4), \quad \|F\| = Max|f(x)|, \quad a \leqslant x \leqslant b$$

.

6. Numerical results

Example 1. We consider the following boundary-value problem

$$y^{(4)} + xy = -(8 + 7x + x^3)e^x, \quad 0 \leqslant x \leqslant 1$$
$$y(0) = y(1) = 0, \quad y''(0) = 0, \quad y''(1) = -4e$$

The exact solution for this problem is $y(x) = x(1-x)e^x$.

This problem has been solved by many authors [5,6,8,9]. We applied our methods described in Section 2,3 to solve this problem with different value of h and parameters α, β, and γ as stated in table 1.

Table 1.   Observed maximum absolute errors for second order for example (1).

	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$	$h = \frac{1}{64}$
Second order method for $\alpha = \frac{4}{900}, \beta = \frac{3}{20}, \gamma = \frac{622}{900}$.				
Our method	0.86642×10^{-5}	0.27155×10^{-5}	0.71029×10^{-6}	0.17947×10^{-6}
method in [9]	0.424×10^{-3}	0.108×10^{-3}	0.270×10^{-4}	0.675×10^{-5}
method in [8]	0.867×10^{-3}	0.216×10^{-3}	0.540×10^{-4}	0.135×10^{-4}
method in[6]	0.174×10^{-2}	0.433×10^{-3}	0.108×10^{-3}	$\ldots$
Fourth order method for $\alpha = 0, \beta = \frac{1}{6}, \gamma = \frac{4}{6}$.				
Our method	0.276244×10^{-6}	0.238054×10^{-7}	0.15997×10^{-8}	0.999846×10^{-10}
method in [6]	0.903×10^{-5}	0.579×10^{-6}	0.364×10^{-7}	$\ldots$
Sixth order method for $\alpha = -\frac{1}{720}, \beta = \frac{124}{720}, \gamma = \frac{474}{720}$.				
Our method	0.161189×10^{-8}	0.216506×10^{-10}	0.111561×10^{-11}	0.447919×10^{-12}
method in [5]	0.191×10^{-6}	0.312×10^{-8}	0.498×10^{-10}	$\ldots$

The computed solution are compared with the exact solution at grid points. The observed maximum absolute errors are given in Table 1 and compared with the methods in [5,6,8,9]. It has been observed that our methods are more efficient.

Example 2. We consider the following boundary -value problem

$$y^{(4)} - xy = -(11 + 9x + x^2 - x^3)e^x, \quad -1 \leqslant x \leqslant 1$$
$$y(-1) = y(1) = 0, \quad y''(-1) = \frac{2}{e}, \quad y''(1) = -6e$$

The exact solution for this problem in $y(x) = (1 - x^2)e^x$.

Table 2.   Observed maximum absolute errors for example (2).

h	out method for $\alpha = \frac{4}{900}, \beta = \frac{3}{20}, \gamma = \frac{622}{900}$	our method for $\alpha = 0, \beta = \frac{1}{6}, \gamma = \frac{4}{6}$	our method for $\alpha = -\frac{1}{720}, \beta = \frac{124}{720}, \gamma = \frac{474}{720}$
$\frac{1}{4}$	0.115081×10^{-1}	0.459256×10^{-4}	0.109133×10^{-5}
$\frac{1}{8}$	0.415441×10^{-2}	0.4009×10^{-5}	0.131425×10^{-7}
$\frac{1}{16}$	0.113626×10^{-2}	0.27385×10^{-6}	0.363915×10^{-9}
$\frac{1}{32}$	0.291195×10^{-3}	0.175456×10^{-7}	0.156519×10^{-11}

7.   Conclusion

The new methods of orders 2,4, and 6 based on non-polynomial spline are developed for the solution of fourth - order boundary -value problems. The new methods enable us to approximate the solution at every point of the range of integration. Table 1 and 2 show that our methods produced better in the sense that max $|e_i|$ is minimum in comparison with the existing methods [5,6,8,9].

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