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# Numerical Solutions of Second Order Boundary Value Problem by Using Hyperbolic Uniform B-Splines of Order 4

O. El Khayyari[a] and A. Lamnii[a,*]

[a] *Faculty of Science and Technology, University Hassan first, Settat, Morocco.*

**Abstract.** In this paper, using the hyperbolic uniform spline of order 4 we develop the classes of methods for the numerical solution of second order boundary value problems with Dirichlet, Neumann and Cauchy types boundary conditions. The second derivative is approximated by the three-point central difference scheme. The approximate results, obtained by the proposed method, confirm the convergence of numerical solutions. Numerical results are given to illustrate the efficiency of our methods.

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**Index to information contained in this paper**



## 1. Introduction

Second order two-point boundary value problems are encountered in many engineering applications including the study of beam deflections, heat flow, and various dynamic systems. Much attention have been given to solve the second-order boundary value problems (2VBP) with Dirichlet, Neumann and Cauchy types boundary conditions, which have application in various branches of applied sciences. These problems are generally arise in the mathematical modeling of viscoelastic flows [4]. A spline has been widely applied for the numerical solutions of some ordinary and

*Corresponding author. Email: a_lamnii@yahoo.fr.


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partial differential equations in the numerical analysis. Many authors have used numerical approaches and approximate methods to solve second and third BVPs. In [6] Bhatti and Bracken solved linear and non-linear differential equation numerically by Galerkin method in a Bernstein polynomials basis. Lima and Carpentier in [11] have obtained a numerical solution of a singular boundary value problem in non-Newtonian fluid mechanics. Recently in [13] Feng and Li solved a second order Neumann boundary value problem with singular nonlinearity for exactly three positive solutions.Khan in [7] have obtained a parametric cubic spline solution of two point boundary value problems. In a series of paper by Caglar et al. [2, 3], BVPs of order two, three, four and five were solved using third, fourth and sixth-degree splines. Lamnii et al. [8–10] discussed a boundary-value problems based on spline interpolation and quasi-interpolation with second order convergence. Generally in solving boundary value problems we use the polynomial splines, but if the solution is hyperbolic, this approximation is non effective. This motivates us to use hyperbolic B-splines of order 4 (lower order) to solve these problems.
In this paper, where the second derivative is approximated by the three-point central difference scheme, we study a method based on the hyperbolic B-splines of order 4 for constructing numerical solutions to second-order boundary value problems (2BVPs) of the form:

$$y^{(2)}(\theta) + f(\theta)y'(\theta) + g(\theta)y(\theta) = p(\theta), \tag{1}$$

Subject to the three types boundary conditions:

$$Dirichlet : y(a) = a_0,\ y(b) = b_0, \tag{2}$$

$$Neumann : y'(a) = a_1,\ y'(b) = b_1, \tag{3}$$

$$Cauchy : y(a) = a_0,\ y'(b) = b_1, \tag{4}$$

where $f(\theta)$, $g(\theta)$ and $p(\theta)$) are given continuous functions defined in the bounded interval $[a, b]$, $a_i(i = 0, 1)$, and $b_i(i = 0, 1)$ are real constants.

The structure of this paper is organized as follows: in Section 2, we give a explicit representation of B-splines of order 4. The interpolation hyperbolic B-splines is developed in Section 3. Solutions of (2VBP) with Dirichlet, Neumann and Cauchy types boundary conditions are presented in Section 4. To illustrate our algorithm, various numerical examples are presented in Section 5 and a conclusion is given that summarizes the theoretical and numerical results.

## 2. Hyperbolic B-splines of Order 4

In this section, we briefly give a explicit representation of Uniform Algebrique Hyperbolic B-splines of order 4 and we give the interesting properties of UAH B-splines of order 4, for more details see [1, 12]. To do this, we need the following notations. Suppose $k$ one intergre such that $k \geqslant 1$. Let $m_k = 4 \times 2^k + 3$ and $h_k = \frac{b-a}{m_k-3}$. We consider the nodal points $\theta_i^k$ on the interval $I = [a, b]$ where

$$\begin{cases} \theta^k_{-3} = \theta^k_{-2} = \theta^k_{-1} = \theta^k_0 = a, \\ \theta^k_i = a + i \times h_k, i = 1...m_k - 4, \\ \theta^k_{m_k-3} = \theta^k_{m_k-2} = \theta^k_{m_k-1} = \theta_{m_k} = b, \end{cases} \tag{5}$$

The hyperbolic B-splines space of order 4 is defined as follows

$$\mathcal{V}_k = \{s \in \mathcal{C}^2(I) : \ s_{[\theta_i^k, \theta_{i+1}^k]} \in \Gamma_4\} \text{ where } \Gamma_4 = \{1, \theta, \cosh(\theta), \sinh(\theta)\}.$$

The dimension of $\mathcal{V}_k$ is $m_k$ and the fourth-order Uniform Algebrique Hyperbolic B-splines for $i = 0, 1, ..., m_k - 7$ are given by :

$$\nu_{i,k}(\theta) = C_k \begin{cases} -\theta + \theta_i^k + \sinh(\theta - \theta_i^k), & \theta_i^k \leqslant \theta < \theta_{i+1}^k; \\ \theta - \theta_{i+2}^k + 2(\theta - \theta_{i+1}^k)\cosh(h_k) + 2\sinh(\theta_{i+1}^k - \theta) + \sinh(\theta_{i+2}^k - \theta), & \theta_{i+1}^k \leqslant \theta < \theta_{i+2}^k; \\ -\theta + \theta_{i+2}^k + 2(\theta_{i+3}^k - \theta)\cosh(h_k) - \sinh(\theta_{i+2}^k - \theta) - 2\sinh(\theta_{i+3}^k - \theta), & \theta_{i+2}^k \leqslant \theta < \theta_{i+3}^k; \\ \theta - \theta_{i+4}^k + \sinh(\theta_{i+4}^k - \theta), & \theta_{i+3}^k \leqslant \theta < \theta_{i+4}^k; \\ 0, & \text{otherwise.} \end{cases}$$

where $C_k = \frac{1}{4h_k \sinh(\frac{h_k}{2})^2}$.

And the respective left and right hand side boundary hyperbolic B-splines are

$$\nu_{-3,k}(\theta) = \begin{cases} \frac{-h_k+\theta-a+\sinh(h_k-\theta-a)}{-h_k+\sinh(h_k)}, & \theta_0^k \leqslant \theta < \theta_1^k \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{-2,k}(\theta) = \begin{cases} \frac{\theta-a-\sinh(h_k)+\sinh(h_k-\theta+a)}{h_k-\sinh(h_k)} + \\ \frac{a-\theta-2\theta\sinh(h_k)+2a\sinh(h_k)+2\sinh(h_k)-2\sinh(h_k-\theta+a)+\sinh(\theta-a)}{2h_k\cosh(h_k)-2\sinh(h_k)}, & \theta_0^k \leqslant \theta < \theta_1^k \\ \frac{\theta-2h_k-a+\sinh(2h_k-\theta+a)}{2h_k\cosh(h_k)-2\sinh(h_k)}, & \theta_1^k \leqslant \theta < \theta_2^k \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{-1,k}(\theta) = \begin{cases} \frac{\theta-a-\sinh(\theta-a)}{-2h_k+2h_k\cosh(h_k)} + \\ \frac{\theta-a+2(\theta-a)\cosh(h_k)-2\sinh(h_k)+2\sinh(h_k-\theta+a)-\sinh(\theta-a)}{2h_k\cosh(h_k)-2\sinh(h_k)}, & \theta_0^k \leqslant \theta < \theta_1^k \\ 1 + \frac{h_k-\sinh(h_k)}{-2h_k+2h_k\cosh(h_k)} + \frac{2h_k-\theta+a-\sinh(2h_k-\theta+a)}{2h_k\cosh(h_k)-2\sinh(h_k)} + \\ \frac{2(h_k-\theta+a)\cosh(h_k)-\sinh(h_k)+\sinh(h_k-\theta+a)+\sinh(2h_k-\theta+a)}{4h_k\sinh(\frac{h_k}{2})^2}, & \theta_1^k \leqslant \theta < \theta_2^k \\ \frac{-3h_k+\theta-a+\sinh(3h_k-\theta+a)}{4h_k\sinh(\frac{h_k}{2})^2}, & \theta_2^k \leqslant \theta < \theta_3^k \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{m_k-6,k}(\theta) = \begin{cases} \frac{b-\theta-3h_k+\sinh(3h_k-b+\theta)}{4h_k\sinh(\frac{h_k}{2})^2}, & \theta_{m_k-6}^k \leqslant \theta < \theta_{m_k-5}^k \\ \frac{h_k-\sinh(h_k)}{2h_k-2h_k\cosh(h_k)} - \frac{b-2h_k-\theta+\sinh(2h_k-b+\theta)}{2h_k\cosh(h_k)-2\sinh(h_k)} + \\ \frac{2(b-2h_k-\theta)\cosh(h_k)+\sinh(h_k)+\sinh(-b+\theta+h_k)+\sinh(2h_k+\theta-b)}{4h_k\sinh(h_k)^2}, & \theta_{m_k-5}^k \leqslant \theta < \theta_{m_k-4}^k \\ \frac{h_k(b-\theta)\cosh(2h_k)+2h_k\cosh(\frac{3h_k}{2}-b+\theta)\sinh(\frac{h_k}{2})+2h_k\sinh(h_k)}{2h_k(-1+\cosh(h_k))(h_k\cosh(h_k)-\sinh(h_k))} + \\ \frac{-b\sinh(h_k)+\theta\sinh(h_k)-h_k\sinh(2h_k)+\sinh(h_k)\sinh(b-\theta)-h_k\sinh(h_k+b-\theta)}{2h_k(-1+\cosh(h_k))(h_k\cosh(h_k)-\sinh(h_k))}, & \theta_{m_k-4}^k \leqslant \theta < \theta_{m_k-3}^k \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{m_k-5,k}(\theta) = \begin{cases} \frac{b-\theta-2h_k+\sinh(2h_k-b+\theta)}{2h_k\cosh(h_k)-2\sinh(h_k)}, & \theta_{m_k-5}^k \leqslant \theta < \theta_{m_k-4}^k \\ \frac{-\sinh(h_k)(b-\theta-2h_k+2(h_k+\theta)\cosh(h_k)+\sinh(-\theta+b))}{2(h_k-\sinh(h_k))(h_k\cosh(h_k)-\sinh(h_k))} + \\ \frac{h_k(b+\theta)+b\sinh(2h_k)-h_k(2\sinh(-b+\theta+h_k)+\sinh(2h_k+\theta-b)}{2(h_k-\sinh(h_k))(h_k\cosh(h_k)-\sinh(h_k))}, & \theta_{m_k-4}^k \leqslant \theta < \theta_{m_k-3}^k \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{m_k-4,k}(\theta) = \begin{cases} \frac{h_k+\theta-b+\sinh(-h_k-\theta+b)}{h_k-\sinh(h_k)}, & \theta_{m_k-4}^k \leqslant \theta < \theta_{m_k-3}^k \\ 0, & \text{otherwise.} \end{cases}$$

The hyperbolic B-splines of order 4 are continuous and are normalized to sum to one, i.e. $\sum_{i=-3}^{m_k-4} \nu_{i,k}(\theta) = 1$. The hyperbolic B-splines of order 4 are linearly independent on $I = [a, b]$ and satisfying the following properties

- $\nu_{i,k}(\theta)$ is supported on the interval $[\theta_i^k, \theta_{i+1}^k]$;
- Positivity : $\nu_{i,k}(\theta) \geqslant 0, \quad \forall\theta \in [\theta_i^k, \theta_{i+1}^k]$.

According to the formulas of hyperbolic B-splines of order 4 given above, the values of $\nu_{i,k}$, $\nu_{i,k}^{'}$ and $\nu_{i,k}^{''}$ at the nodal points $\theta_i^k$ are given according to the following table:

Table 1. The values of $\nu_{i,k}(\theta)$, $\nu_{i,k}^{'}(\theta)$ and $\nu_{i,k}^{''}(\theta)$ at the knots

| | $\theta_i^k$ | $\theta_{i+1}^k$ | $\theta_{i+2}^k$ | $\theta_{i+3}^k$ | $\theta_{i+4}^k$ | *else* |
|---|---|---|---|---|---|---|
| $\nu_{i,k}(\theta)$ | 0 | $\frac{-h_k+\sinh(h_k)}{4h_k\sinh(\frac{h_k}{2})^2}$ | $\frac{h_k\cosh(h_k)-\sinh(h_k)}{2h_k\sinh(\frac{h_k}{2})^2}$ | $\frac{-h_k+\sinh(h_k)}{4h_k\sinh(\frac{h_k}{2})^2}$ | 0 | 0 |
| $\nu_{i,k}^{'}(\theta)$ | 0 | $\frac{1}{2h_k}$ | 0 | $\frac{-1}{2h_k}$ | 0 | 0 |
| $\nu_{i,k}^{''}(\theta)$ | 0 | $\frac{1}{2\tanh(\frac{h_k}{2})}$ | $\frac{-1}{\tanh(\frac{h_k}{2})}$ | $\frac{1}{2\tanh(\frac{h_k}{2})}$ | 0 | 0 |

## 3. Interpolation Method

In this section, we present the interpolation method using hyperbolic B-splines of order 4 is presented for the numerical solution of the second-order boundary value problems (2BVPs) given in (1)-(4). To construct such an approximate solution, we will give an approximate of $y^{(2)}(\theta_j^k)$ and $y'(\theta_j^k)$ by using Taylor series expansion.

According to Schoenberg-Whitney theorem (see [5]), for a given function $y(\theta)$ sufficiently smooth there exists a unique hyperbolic spline

$$s(\theta) = \sum_{i=-3}^{m_k-4} \mu_i \nu_{i,k}(\theta) \in \mathcal{V}_k,$$

satisfying the interpolation conditions:

$$s(\theta_j^k) = y(\theta_j^k), \quad j = 0, 1, \cdots, m_k - 3; \tag{6}$$

$$s'(a) = y'(a), \quad s'(b) = y'(b). \tag{7}$$

The $\mu_i$'s are unknown real coefficients.

For $j = 0, 1, \cdots, m_k - 3$, let

$$m_j = s'(\theta_j^k)$$

and for $j = 0, 1, \cdots, m_k - 3$, let

$$M_j = \frac{\mu_{j-3} - 2\mu_{j-2} + \mu_{j-1}}{2h_k \tanh(\frac{h_k}{2})}.$$

By using the Taylor series expansion we have:

$$m_j = s^{'}(\theta_j^k) = y^{'}(\theta_j^k) - \frac{1}{180}h_k^4 y^{(5)}(\theta_j^k) + O(h_k^6); \tag{8}$$

$$M_j = y^{''}(\theta_j^k) + \frac{1}{12}h_k^2 y^{(4)}(\theta_j^k) + \frac{1}{360}h_k^4 y^{(6)}(\theta_j^k) + O(h_k^6); \tag{9}$$

From Table 1 and equations (8)-(9), the approximation values of $y$, $y^{'}$ and $y^{''}$ at the nodal points $\theta_j^k$ are given according to the following table :

Table 2. The approximation values of $y(\theta_j^k)$, $y^{'}(\theta_j^k)$ and $y^{''}(\theta_j^k)$

| | $y(\theta_j^k)$ | $y^{'}(\theta_j^k)$ | $y^{''}(\theta_j^k)$ |
|---|---|---|---|
| Approximate value | $s(\theta_j^k)$ | $m_j$ | $M_j$ |
| Representation in $\mu_j$ | $C_k(-h_k + \sinh(h_k))(\mu_{j-1} + v_{h_k}\mu_{j-2} + \mu_{j-3})$ | $\frac{\mu_{j-1}-\mu_{j-3}}{2h_k}$ | $\frac{\mu_{j-1}-2\mu_{j-2}+\mu_{j-3}}{2h_k \tanh(\frac{h_k}{2})}$ |
| Error order | $O(h_k^4)$ | $O(h_k^4)$ | $O(h_k^2)$ |

where $v_{h_k} = \frac{2(h_k \cosh(h_k) - \sinh(h_k))}{\sinh(h_k) - h_k}$.

## 4. Hyperbolic B-splines Solutions of 2 BVP

### 4.1 *Hyperbolic solution with Dirichlet boundary condition*

Let $s(\theta)$ and $\widetilde{s}(\theta)$ be shape functions that satisfy the equation (1) and the boundary conditions (2). Then $s(\theta)$ and $\widetilde{s}(\theta)$ are written as a linear combination of $m_k$ shape functions given by

$$s(\theta) = \sum_{j=-3}^{m_k-4} \mu_j \nu_{j,k}(\theta) \tag{10}$$

and

$$\widetilde{s}(\theta) = \sum_{j=-3}^{m_k-4} \widetilde{\mu}_j \nu_{j,k}(\theta), \tag{11}$$

where $\mu_j$'s and $\widetilde{\mu}_j$'s are unknown real coefficients.

In order to solve the problem (1), hyperbolic B-spline in (10) is presumed to be its solution. Since the problem is defined on interval $[a, b]$ let $\theta_0^k = a$ and $\theta_{m_k}^k = b$, where $m_k$ is the number of partition. Thus, (1) becomes

$$y^{(2)}(\theta_j^k) + f(\theta_j^k)y'(\theta_j^k) + g(\theta_j^k)y(\theta_j^k) = p(\theta_j^k), \quad j = 0, 1, \cdots, m_k - 3. \tag{12}$$

Now, by using Table 1 and 2, the equation (12) becomes

$$\frac{\mu_{j-3} - 2\mu_{j-2} + \mu_{j-1}}{2h_k \tanh(\frac{h_k}{2})} + f_j \frac{\mu_{j-3} - \mu_{j-1}}{2h_k} + g_j C_k(-h_k + \sinh(h_k))(\mu_{j-3} + v_{h_k}\mu_{j-2} + \mu_{j-1}) = p_j + O(h_k^2) \tag{13}$$

where $f_j = f(\theta_j^k)$ , $g_j = g(\theta_j^k)$ and $p_j = p(\theta_j^k)$.
Consequently, by replacing $C_k = \frac{1}{4h_k \sinh(\frac{h_k}{2})^2}$ in the equation (13), we obtain

$$(\mu_{j-3} - 2\mu_{j-2} + \mu_{j-1}) + \alpha_{j,k}(\mu_{j-3} - \mu_{j-1}) + \beta_{j,k}(\mu_{j-1} + v_{h_k}\mu_{j-2} + \mu_{j-3}) = \gamma_{j,k} + O(2h_k^3 \tanh(\frac{h_k}{2})), \tag{14}$$

where $\alpha_{j,k} = f_j \tanh(\frac{h_k}{2})$, $\beta_{j,k} = g_j \frac{(-h_k+\sinh(h_k))}{\sinh(h_k)}$ and $\gamma_{j,k} = 2h_k \tanh(\frac{h_k}{2})p_j$.

By elimination $O(2h_k^3 \tanh(\frac{h_k}{2}))$ from (14), we obtain a linear system with $m_k - 2$ linear equations in $m_k$ unknowns $\mu_j,\ j = -3, -2, \cdots, m_k - 4$. So two more equations are needed.

The Dirichlet boundary conditions (2) are simplified using (14) resulting (15).

$$\begin{cases} y(a) = a_0; \\ y(b) = b_0. \end{cases} \quad \text{Thus,} \quad \begin{cases} \mu_{-3} = a_0; \\ \mu_{m_k-4} = b_0. \end{cases} \tag{15}$$

Take (14) , we get $m_k - 2$ linear equations with $\mu_i,\ i = -2, -1, 0, \cdots, m_k - 7, m_k - 6, m_k - 5$, as unknowns since $\mu_{-3}$, and $\mu_{m_k-4}$ have been yielded from (15).
Let $C = [\mu_{-2}, \mu_{-1}, \cdots, \mu_{m_k-6}, \mu_{m_k-5}]^T$, $\widetilde{C} = [\widetilde{\mu}_{-2}, \widetilde{\mu}_{-1}, \cdots, \widetilde{\mu}_{m_k-6}, \widetilde{\mu}_{m_k-5}]^T$, $D = [d_1, d_2, \cdots, d_{m_k-2}]^T$ and $E = [e_1, e_2, \cdots, e_{m_k-2}]^T$. In matrix form, the equation look like this (14):

$$(A_1 + \tanh(\frac{h_k}{2})FA_2 + \frac{(-h_k + \sinh(h_k))}{\sinh(h_k)}GB)C = D + E; \tag{16}$$

$$(A_1 + \tanh(\frac{h_k}{2})FA_2 + \frac{(-h_k + \sinh(h_k))}{\sinh(h_k)}GB)\widetilde{C} = D, \tag{17}$$

where $A_1$ and $A_2$ are the following $(m_k - 2) \times (m_k - 2)$ matrix:

$$A_1 = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \cdots & \cdots & & & \\ & & \cdots & \cdots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & \cdots & \cdots & & & \\ & & \cdots & \cdots & & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}$$

and where $F$,$G$, $B$ and $D$ are the following matrix

$$F = \begin{pmatrix} f_0 & & & & \\ & f_1 & & & \\ & & f_2 & & \\ & & & \ddots & \\ & & & & f_{m_k-3} \end{pmatrix}, \quad G = \begin{pmatrix} g_0 & & & & \\ & g_1 & & & \\ & & g_2 & & \\ & & & \ddots & \\ & & & & g_{m_k-3} \end{pmatrix},$$

$$B = \begin{pmatrix} v_{h_k} & 1 & & & & \\ 1 & v_{h_k} & 1 & & & \\ & 1 & v_{h_k} & 1 & & \\ & & & \ddots & & \\ & & & 1 & v_{h_k} & 1 \\ & & & & 1 & v_{h_k} \end{pmatrix}$$

$$D = \begin{pmatrix} 2h_k \tanh(\frac{h_k}{2})p_0 + f_0 \tanh(\frac{h_k}{2})\mu_{-3} - g_0 \frac{-h_k+\sinh(h_k)}{\sinh(h_k)} - \mu_{-3} \\ 2h_k \tanh(\frac{h_k}{2})p_1 \\ \vdots \\ 2h_k \tanh(\frac{h_k}{2})p_{m_k-4} \\ 2h_k \tanh(\frac{h_k}{2})p_{m_k-3} - f_{m_k-3} \tanh(\frac{h_k}{2})\mu_{m_k-4} - g_{m_k-3}\frac{-h_k+\sinh(h_k)}{\sinh(h_k)}\mu_{m_k-4} - \mu_{m_k-4} \end{pmatrix}$$

and $e_i = O(2h_k^3 \tanh(\frac{h_k}{2})), i = 1, 2, \cdots, m_k - 2$.

After solving the linear system (17), $\widetilde{\mu}_i,\ i = -2, -1, 0, \cdots, m_k-7, , m_k-6, , m_k-5$, $\widetilde{\mu}_{-3} = \mu_{-3}, ,$ and $\widetilde{\mu}_{m_k-4} = \mu_{m_k-4}$ will be used together to get the approximation spline solution $\widetilde{s}(\theta) = \displaystyle\sum_{i=-3}^{m_k-4} \widetilde{\mu}_i \nu_{i,k}(\theta)$.

### 4.2 *Hyperbolic solution with Neumann boundary condition*

The same reasoning that led us to consider the Dirichlet boundary conditions (2) can also be applied to Neumann boundary conditions (3) as well.

$$\begin{cases} y'(a) = a_1; \\ y'(b) = b_1. \end{cases} \quad \text{Thus,} \quad \begin{cases} \mu_{-1} - \mu_{-3} = 2h_k a_1; \\ \mu_{m_k-4} - \mu_{m_k-6} = 2h_k b_1. \end{cases} \tag{18}$$

Take (14) and (18), we yield :

$$(A_1 + \tanh(\frac{h_k}{2})FA_2 + \frac{(-h_k + \sinh(h_k))}{\sinh(h_k)}GB)C = D + E; \tag{19}$$

$$(A_1 + \tanh(\frac{h_k}{2})FA_2 + \frac{(-h_k + \sinh(h_k))}{\sinh(h_k)}GB)\widetilde{C} = D, \tag{20}$$

where $A_1$ and $A_2$ are the following $(m_k) \times (m_k)$ matrix:

$$A_1 = \begin{pmatrix} -1 & 0 & 1 & & & \\ 1 & -2 & 1 & & & \\ & & \cdots & \cdots & \cdots & \\ & & & \cdots & \cdots & \cdots \\ & & & 1 & -2 & 1 \\ & & & -1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & & & & \\ -1 & 0 & 1 & & & \\ & & \cdots & \cdots & \cdots & \\ & & & \cdots & \cdots & \cdots \\ & & & -1 & 0 & 1 \\ & & & & 0 & 0 \end{pmatrix}$$

and where $F$, $G$ ,$B$ ,$\widetilde{C}$ and $D$ are the following matrix

$$F = \begin{pmatrix} 0 & & & & & \\ & f_0 & & & & \\ & & f_1 & & & \\ & & & \ddots & & \\ & & & & f_{m_k-3} & \\ & & & & & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & & & & & \\ & g_0 & & & & \\ & & g_1 & & & \\ & & & \ddots & & \\ & & & & g_{m_k-3} & \\ & & & & & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & & & & & & \\ 1 & v_{h_k} & 1 & & & & \\ & 1 & v_{h_k} & 1 & & & \\ & & & \ddots & & & \\ & & & & 1 & v_{h_k} & 1 \\ & & & & & & 0 \end{pmatrix}, \quad \widetilde{C} = [\widetilde{\mu}_{-3}, \widetilde{\mu}_{-2}, \cdots, \widetilde{\mu}_{m_k-5}, \widetilde{\mu}_{m_k-4}]^T,$$

$$D = \begin{pmatrix} 2h_k a_1 \\ 2h_k \tanh(\frac{h_k}{2}) p_0 \\ \vdots \\ 2h_k \tanh(\frac{h_k}{2}) p_{m_k-3} \\ 2h_k b_1 \end{pmatrix}$$

### 4.3 *Hyperbolic solution with Cauchy boundary condition*

By a similar technique used in previous sections and by using Cauchy boundary condition (4) we get:

$$\begin{cases} y(a) = a_0; \\ y'(b) = b_1. \end{cases} \text{Thus, } \begin{cases} \mu_{-3} = a_0; \\ \mu_{m_k-4} - \mu_{m_k-6} = 2h_k b_1. \end{cases} \tag{21}$$

Take (14) and (21), we yield :

$$(A_1 + \tanh(\frac{h_k}{2}) F A_2 + \frac{(-h_k + \sinh(h_k))}{\sinh(h_k)} G B) C = D + E; \tag{22}$$

$$(A_1 + \tanh(\frac{h_k}{2}) F A_2 + \frac{(-h_k + \sinh(h_k))}{\sinh(h_k)} G B) \widetilde{C} = D, \tag{23}$$

where $A_1$ and $A_2$ are the following $(m_k - 1) \times (m_k - 1)$ matrix:

$$A_1 = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \\ & & & 1 & -2 & 1 \\ & & & -1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \\ & & & -1 & 0 & 1 \\ & & & & 0 & 0 \end{pmatrix}$$

and where $F$, $G$ ,$B$ ,$\widetilde{C}$ and $D$ are the following matrix

$$F = \begin{pmatrix} f_0 & & & & & \\ & f_1 & & & & \\ & & f_2 & & & \\ & & & \ddots & & \\ & & & & f_{m_k-3} & \\ & & & & & 0 \end{pmatrix}, \quad G = \begin{pmatrix} g_0 & & & & & \\ & g_1 & & & & \\ & & g_2 & & & \\ & & & \ddots & & \\ & & & & g_{m_k-3} & \\ & & & & & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} v_{h_k} & 1 & & & & & \\ 1 & v_{h_k} & 1 & & & & \\ & 1 & v_{h_k} & 1 & & & \\ & & & \ddots & & & \\ & & & & 1 & v_{h_k} & 1 \\ & & & & & & 0 \end{pmatrix}, \quad \widetilde{C} = [\widetilde{\mu}_{-2}, \widetilde{\mu}_{-1}, \cdots, \widetilde{\mu}_{m_k-5}, \widetilde{\mu}_{m_k-4}]^T,$$

$$D = \begin{pmatrix} 2h_k \tanh(\frac{h_k}{2})p_0 + f_0 \tanh(\frac{h_k}{2})\mu_{-3} - g_0 \frac{-h_k+\sinh(h_k)}{\sinh(h_k)} - \mu_{-3} \\ 2h_k \tanh(\frac{h_k}{2})p_1 \\ \vdots \\ 2h_k \tanh(\frac{h_k}{2})p_{m_k-3} \\ 2h_k b_1 \end{pmatrix}$$

## 5. Numerical Examples

In this section, seven numerical examples will be presented to assess the efficiency of the interpolation hyperbolic B-spline method for solving the second-order boundary value problems (2BVPs) of the form ((1),(2)), ((1),(3)) and ((1),(4)). For the sake of comparison, We will use the absolute error defined as

$$Errors = |exact\ solution - approximate\ solution|_\infty$$

to verify the accuracy.

*Example 5.1* We consider the following Dirichlet boundary-value problem

$$\begin{cases} y^{(2)}(x) = -2\sinh(x) + (1-x)\cosh(x),\ x \in [0,1]; \\ y(0) = 1,\ \ y(1) = 0. \end{cases} \tag{24}$$

The exact solution is $y(x) = (1-x)\cosh(x)$.
Result has been shown for different values of $k$ in Table 3.

Table 3. Maximum absolute error for Problem (24).

| $k$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| Error | 9.475e-003 | 4.688e-003 | 2.332e-003 | 1.163e-003 | 5.807e-004 | 2.901e-004 |

*Example 5.2* We consider the following Dirichlet boundary-value problem

$$\begin{cases} y^{(2)}(x) + y'(x) = -x\sinh(x) - (1+x)\cosh(x)),\ x \in [0,1]; \\ y(0) = y(1) = 0. \end{cases} \tag{25}$$

The exact solution is $y(x) = (1-x)\sinh(x)$.
Result has been shown for different values of $k$ in Table 4.

Table 4. Maximum absolute error for Problem (25).

| $k$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| Error | 3.255e-002 | 1.529e-002 | 6.714e-003 | 2.435e-003 | 7.695e-004 | 2.983e-004 |

*Example 5.3* We consider the following Dirichlet boundary-value problem

$$\begin{cases} y^{(2)}(x) + xy(x) = (-3x - x^3)\exp(x), \, x \in [0,1]; \\ y(0) = y(1) = 0. \end{cases} \tag{26}$$

The exact solution is $y(x) = x(1-x)\exp(x)$.
Result has been shown for different values of $k$ in Table 5.

Table 5. Maximum absolute error for Problem (26).

| $k$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| Error | 6.495e-002 | 3.179e-002 | 1.572e-002 | 7.821e-003 | 3.900e-003 | 1.947e-003 |

*Example 5.4* We consider the following Neumann boundary-value problem

$$\begin{cases} -y^{(2)}(x)) = (2 - 4x^2)y, \quad x \in [0,1]; \\ y'(0) = 0, \; y'(1) = \frac{-2}{exp(1)}. \end{cases} \tag{27}$$

The exact solution is $y(x) = \cosh(x^2) - \sinh(x^2)$.
Result has been shown for different values of $k$ in Table 6.

Table 6. Maximum absolute error for Problem (27).

| $k$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| Error | 5.610e-003 | 2.845e-003 | 1.528e-003 | 7.625e-004 | 3.807e-004 | 1.902e-004 |

*Example 5.5* We consider the following Neumann boundary-value problem

$$\begin{cases} y^{(2)}(x) + y(x) = \cos(x), \, x \in [0,5]; \\ y'(0) = 0, \; y'(5) = 0. \end{cases} \tag{28}$$

The exact solution is

$$y(x) = \frac{(-6\cos^2(5) + 2 + 10\cot(5)\sin(10) + 2\cos(10))\cos(x) + 2\cos^3(x)}{4} + \frac{2x\sin(x) + \sin(x)\sin(2x)}{4}$$

Result has been shown for different values of $k$ in Table 7.

Table 7. Maximum absolute error for Problem (28).

| $k$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| Error | 6.192e-002 | 2.412e-002 | 1.033e-002 | 4.732e-003 | 2.258e-003 | 1.101e-003 |

*Example 5.6* We consider the following Neumann boundary-value problem

$$\begin{cases} y^{(2)}(x) + y(x) = x^2 \exp(-x),\ x \in [0, 10]; \\ y'(0) = 0,\ \ y'(10) = 0. \end{cases} \tag{29}$$

The exact solution is $y(x) = -\frac{\cos(10)+99\exp(-10)}{2\sin(10)} \cos(x) - \frac{\sin(x)}{2} + \frac{\exp(-x)(1-x)^2}{2}$.
Result has been shown for different values of $k$ in Table 8.

Table 8. Maximum absolute error for Problem (29).

| $k$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| Error | 2.559e-001 | 6.775e-002 | 2.310e-002 | 9.130e-003 | 3.984e-003 | 1.849e-003 |

*Example 5.7* We consider the following Cauchy boundary-value problem

$$\begin{cases} y^{(2)}(x) + y(x) = -x,\ x \in [0, 2]; \\ y(0) = 0,\ \ y'(2) = 0. \end{cases} \tag{30}$$

The exact solution is $y(x) = \frac{\sin(x)}{\cos(2)} - x$.
Result has been shown for different values of $k$ in Table 9.

Table 9. Maximum absolute error for Problem (30).

| $k$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| Error | 3.677e-001 | 1.964e-001 | 1.016e-001 | 5.176e-002 | 2.612e-002 | 1.312e-002 |

## 6. Conclusion

The objective of this paper is to present a new simple numerical method, suitable and accurate to solve (2VBP) boundary value problem with Dirichlet, Neumann and Cauchy conditions. In particular, the approach is simple and efficient and can be extended to other classes of systems of boundary value problems. The numerical examples indicates that our method can be also applied to solve higher order boundary value problems by using the hyperbolic (tension) B-splines of higher order.

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