International Journal of
Mathematical Modelling & Computations
Vol. 01, No. 01, January 2011, 09- 14

Orthogonal zero interpolants and applications

M. A. Bokhari* and H. Al-Attas
Dept of Math & Stat, KFUPM, Dhahran, Saudi Arabia

Abstract. Orthogonal zero interpolants (OZI) are polynomials which interpolate the "zero-function" at a finite number of pre-assigned nodes and satisfy orthogonality condition. OZI's can be constructed by the 3-term recurrence relation. These interpolants are found useful in the solution of constrained approximation problems and in the structure of Gauss-type quadrature rules. We present some theoretical and computational aspects of OZIs and also discuss their structure and significance at the multiple nodes.

Keywords: Ortogonal zero interpolant, 3-term recurrence relation, constrained least squares approximation, Parseval equality, Jacobi matrix, Gauss-Radau/Lobatto rules.

Index to information contained in this paper

1. Introduction

Orthogonal polynomials form a constructive tool for representing general functions or data sets. In certain cases, a representative curve is required to match certain characteristics of a given function either in terms of its values or monotonicity or even curvature at a finite number of points. We define a class of orthogonal polynomials that preserve such properties. These polynomials prove useful in certain constrained least square approximation [1], [3] and optimal control problems [11]. In addition, the proposed interpolants have some relevance with the Parseval equality and also in the structure of Gauss-type quadrature rules.

2. Orthogonal zero interpolants

Let $S_k = \{x_1, x_2, \ldots, x_k\} \subset \mathbb{R}$. We shall say that a real valued function g is an $S_k - zero$ interpolant if $g(x_i) = 0$, $i = 1, 2, \ldots, k$, i.e., g interpolates the zero-

*Corresponding author. Email: mbokhari@kfupm.edu.sa

function at the pre-assigned points x_i, $i = 1, 2, \ldots, k$. Let π_j denote the class of all polynomials up to degree j. We set

$$\pi_n(S_k) := \{p \in \pi_{n+k} : p \text{ is an } S_k - zero \text{ interpolant}\} \tag{1}$$

and describe some of its properties:

(a) $\pi_n(S_k)$ is an $n+1$ dimensional linear subspace of π_{n+k}. Each $S_k - OZI$ $\psi \in \pi_n(S_k)$ is of the form $\psi = qP_{S_k}$ for some $q \in \pi_n$ where

$$P_{S_k}(x) := \Pi_{i=1}^{k}(x - x_i). \tag{2}$$

(b) $\bigcup_{n=0}^{\infty} \pi_n(S_k)$ is uniformly dense in $C([a,b], S_k) := \{g \in C[a,b] : g \text{ is an } S_k - zero \text{ interpolant}\}$[1].

(c) By use of the standard 3-term recurrence relation [8], we can determine an orthogonal basis "$\psi_0, \psi_1, \ldots, \psi_n$" of the space $\pi_n(S_k)$ for a given weight function w on $[a,b]$ as follows:

$$\psi_{i+1}(x) = (x - \alpha_i)\psi_i(x) - \beta_i\psi_{i-1}(x), \qquad i = 1, 2, \ldots \tag{3}$$

where

$$\psi_0(x) = P_{S_k}(x), \qquad \psi_1(x) = (x - \alpha_0)\psi_0(x); \qquad \alpha_0 = \langle\psi_0, \psi_0\rangle_w. \tag{4}$$

The recursion coefficients in (3) are given by

$$\alpha_i = \frac{\langle x\psi_i, \psi_i\rangle_w}{\langle\psi_i, \psi_i\rangle_w}, \qquad \beta_i = \frac{\langle\psi_i, \psi_i\rangle_w}{\langle\psi_{i-1}, \psi_{i-1}\rangle_w}, \qquad i = 1, 2, \ldots \tag{5}$$

where the inner products $\langle ., . \rangle_w$ are defined as

$$\langle f, g\rangle_w := \int_a^b f(x)g(x)w(x)d. \tag{6}$$

definition 1. With $S_k = \{x_1, x_2, \ldots, x_k\} \subset \mathbb{R}$, the set of k pre-assigned nodes, the polynomial ψ_i, $j = 0, 1, 2, \ldots$, generated by the recurrence relation (3) will be called S_k-orthogonal zero interpolant (OZI) with respect to w on $[a,b]$.

Remark 1. Each $\psi_n(x)$, $n = 1, 2, 3, \ldots (cf(c))$ can be expressed as $\psi_n(x) = P_{S_k}(x)q_n(x)$. Here, $q_n \in \pi_n$ is a monic orthogonal polynomial for the weight function $P_{S_k}^2(x)w(x)$ over the interval $[a,b]$. Therefore, $\psi_n(x)$ has n distinct real zeros in the open interval (a,b). These zeros are referred to as the internal zeros of $\psi_n(x)$. $S_k = \{x_1, x_2, \ldots, x_k\}$ is the set of fixed zeros of $\psi_n(x)$. The computation of the internal zeros is based on the work of Wilf [12] and Golub et al [5] as described in

Theorem 1. (Internal zeros of $S_k - OZI$). Let $\psi_n(x)$ be the $S_k - OZI$ for the weight function w on the interval $[a,b]$. Then its internal zeros are exactly the n eigenvalues of the n^{th} order Jacobi matrix:

$$J_n(S_k^2 w) := \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & 0 & . & . & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & 0 & & \\ 0 & \sqrt{\beta_2} & \alpha_2 & \sqrt{\beta_3} & . & \\ . & 0 & \sqrt{\beta_3} & . & . & 0 \\ . & & . & . & . & \sqrt{\beta_{n-1}} \\ 0 & & & 0 & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix} \tag{7}$$

where α_i, $i = 0, 1, 2, \ldots, n-1$; β_j, $j = 1, 2, \ldots, n-1$, are defined in (4) and (5).

Proof. As noted above, $\psi_n = q_n P_{S_k}$ where $q_n \in \pi_n$ is a monic orthogonal polynomial for the weight function "$P_{S_k}^2 w$" on $[a, b]$. Thus, $q_k \in \pi_k$, $k = 0, 1, 2, \ldots$ can be defined by the 3-term recurrence relation $q_{k+1}(x) = (x - \alpha_k)q_k(x) - \beta_k q_{k-1}(x)$ where $q_0(x) = 1$ and $q_1(x) = (x - \alpha_0)q_0(x)$. Note that $\langle \psi_i, \psi_i \rangle_w = \int_a^b \psi_i^2(x)w(x)dx = \int_a^b q_i^2(x)P_{S_k}^2 w(x)dx = \langle q_i, q_i \rangle_{p_{S_k}^2 w}$. Therefore, the recursion coefficients α_k and β_k are the same as those stated in (4)-(5). Thus, n eigenvalues of the tri-diagonal matrix (7) are the n internal zeros of $q_n \in \pi_n$ [5] and by Remark 1 these are the n internal zeros of the $S_k - OZI\psi_n$.

3. Zeros of $S_k - OZI$ and Gauss-type quadrature rules

In this section we shall illustrate that the nodes of Gauss-type quadrature rules directly arise from the zeros of the OZI. We set $S_1 = x - b$ in (2) and consider the fixed zero b and the internal $n-1$ zeros "$t_1, t_2, \ldots, t_{k-1}$" of the resultant $S_1 - OZI\psi_{n-1}$ for the weight function $w(x)$ on $[a, b]$. Then with a modified weight function $w_1(x) := S_1(x)w(x)$, we have

Theorem 2. The nodes of the right end w_1-weighted n-point Gauss-Radau formula [5], [10]

$$\int_a^b f(t)w_1(t)d \approx \sum_{i=1}^{n-1} f(t_i)\omega_i^b + f(b)\omega_n^b. \tag{8}$$

are exactly the n zeros of $S_1 - OZI\psi_{n-1}$.

Proof. We note that the nodes of quadrature rule (8) are computed by the tri-diagonal matrix [5]

$$J_{n,R}^b = \begin{bmatrix} J_{n-1}(w_1) & \sqrt{\beta_{n-1}}e_{n-1} \\ \sqrt{\beta_{n-1}}e_{n-1}^t & \alpha_n^b \end{bmatrix} \text{with} \quad \alpha_n^b = b - \frac{p_{n-2}(b)}{p_{n-1}(b)} \tag{9}$$

where (i) $J_{n-1}(w_1)$ is a tri-diagonal matrix of order $n-1$ obtained from (7) by setting $P_{S_k}(x) \equiv 1$ and $w = w_i$, (ii) $e_{n-1}^T = [1, 0, 0, \ldots, 0] \in \mathbb{R}^{n-1}$, (iii) p_k is the k^{th} degree monic orthogonal polynomial for the weight function w_1 on $[a, b]$.

In fact, the eigenvalues of the tridiagonal matrix $J_{n,R}^b$ are b and the internal zeros of monic orthogonal polynomial $p_{n-1} \in \pi_{n-1}$ for the weight function w_1 on $[a, b]$. The resultant zeros are indeed the zeros of the $S_1 - OZI\psi_{n-1}$ for the weight function $w(x)$ on $[a, b]$.

Remark 2. The weights ω_i^b in formula (8) are given by $\omega_i^b = \beta_0 u_{i,1}$, $i = 1, 2, \ldots, n$ where $\int_a^b (b-t)w(t)dt$ and each $u_{i,1}$ is the first component of the normalized eigenvector of the matrix $J_{n,R}^b$ corresponding to its ith eigenvalue. ,

Remark 3. Note that formula (8) is based on the weight function $w_1(x) := (b-x)w(x)$. If g is differentiable at b, then the integral $\int_a^b g(x)dx$ can be approximate by (8) as follows. Define

$$f(x) := \begin{cases} -\frac{g(b)-g(x)}{x-b}, & if a \leqslant x < b \\ g'(b), & if x = b \end{cases}. \tag{10}$$

Then $\int_a^b g(x)dx = (b-a)g(b) - \int_a^b f(x)(b-x)dx = (b-a)g(b) - \sum_{i=1}^{n-1} \frac{g(b)-g(a)}{b-t_i}\omega_i^b + g'(b)\omega_n^b$. Note that the quadrature rule in this special case involves the derivative of the integrand at b [4].

Remark 4. If we set $P_{S_1}(x) = x - a$ and $P_{S_2}(x) = (x-a)(b-x)$, then the similar observation (cf Theorem 2) holds for the left hand Gauss-Radau and the Gauss-Lobatto quadrature rules [9], [10].

4. Constrained L_2-approximation problem

Application of OZI is found useful in the solution of certain constrained approximation problems. More precisely, for an $f \in L_w^2[a,b]$ and the finite set $S_k = \{x_1, x_2, \ldots, x_n\} \subset [a,b]$ consider

Theorem 3. Let $N = n + k - 1$ with $n \geqslant 0$. Then ψ_k, $k = 0, 1, 2, \ldots$, the w-weighted S_k-OZI's, determine the optimal solution of the problem [1]

$$\min_{\substack{p\in\pi_N \\ p(x_i)=f(x_i) \\ i=1,\ldots,k}} \parallel f - p \parallel_w . \tag{11}$$

In addition, similar to Parseval equality [8]

$$\sum_{i=0}^{\infty} \frac{\langle f_L, \psi_i\rangle_w^2}{\langle \psi_i, \psi_i\rangle_w^2} = \langle f_L, f_L\rangle_w, \qquad \forall f \in C[a,b] \tag{12}$$

holds for $f_L = f - L(., S_k, f)$ with $L(., S_k, f)$ as the Lagrange interpolant to f at the points of S_k.

Proof. In case of $n = 0$, $L(., S_k, f)$ will be the optimal solution of the problem. Else, we transform (10) to an equivalent unconstrained problem

$$\min_{\phi\in\pi_n(S_k)} \parallel f_L - \phi \parallel_w \tag{13}$$

with $n = N - k + 1$. To solve (13), we follow a standard technique [8] and fix an orthonormal basis $\{\phi_0, \phi_1, \ldots, \phi_n\}$ for $\pi_n(S_k)$ where $\phi_k = \frac{\psi_k}{||\psi_k||_w}$, $k = 0, 1, \ldots, n$, (cf (3)). Writing $\phi \in \pi_n(S_k)$ as $\phi(x) = \sum_{i=0}^n b_i\phi_i(x)$ we obtain

$$\parallel f_L - \phi \parallel_w^2 = \int_a^b [f_L(x) - \phi(x)]^2 w(x)dx = \langle f_L, f_L\rangle_w + \sum_{i=0}^n (b_i - c_i)^2 - \sum_{i=0}^n c_i^2 \tag{14}$$

where $c_i = \langle \phi_i, f_L \rangle_w$. Then, of all linear combinations of the form $\phi(x) = \sum_{i=0}^{n} b_i \phi_i(x)$, the minimum in (14) is attained when $b_i = c_i$, $i = 0, 1, \ldots, n$. Thus, $p_n^* = L(., S_k, f) + \phi_n^*$, with $\phi_n^*(x) = \sum_{i=0}^{n} c_i \phi_i(x)$ gives the optimal solution of constrained minimization problem (11).

Next note that $0 \leqslant \| f_L - \phi_n^* \|_w^2 = \langle f_L, f_L \rangle_w - \sum_{i=0}^{n} c_i^2$ (cf (14)). It is known that $\lim_{n \to \infty} \| f_L - \phi_n^* \|_w^2 = 0$ when $f \in C[a, b]$ [1]. This leads to the Parseval type equality $\sum_{i=0}^{n} \langle \phi_i, f_L \rangle_w^2 = \langle f_L, f_L \rangle_w$ over the S_k-orthogonal zero interpolants.

Remark 5. From the details provided above, we have like the Bessels inequality [8]

$$\sum_{i=0}^{n} \frac{\langle \psi_i, f_L \rangle^2}{\langle \psi_i, \psi_i \rangle_w^2} \leqslant \langle f_L, f_L \rangle_w, \qquad \forall f \in L_w^2[a, b]. \tag{15}$$

5. Interpolants subject to multiple nodes

So far we have discussed OZIs related to a finite simple pre-assigned nodes (cf (3)). In certain approximation problems, the use of data that includes some derivative(s) information provides an improvement in the approximation. To implement this, we construct OZIs with multiple pre-assigned nodes: if each ith node x_i is required to be of multiplicity n_i, $i = 0, 1, \ldots, k$, we replace P_{S_k} in (2) by

$$P_{S_k^{(n_i)}} := \Pi_{i=1}^{k} (x - x_i)^{n_i}. \tag{16}$$

With $\psi_0(x) = P_{S_k^{(n_i)}}(x)$, we generate $S_k^{(n_i)}$-OZIs via recurrence relation (3). These OZIs can be used to solve a derivative related constrained minimization problem given by [2]

$$\min_{\substack{p \in \pi_N \\ p^{(j)}(x_i) = f^{(i)}(x_i) \\ j=0,1,\ldots,n_i \\ i=1,\ldots,k}} \| f - p \|_w . \tag{17}$$

Here, we can also generalize the result on Parsevel equality (cf (11)). In addition, one can notice that the nodes of the generalized Gauss-type quadrature rules [6], [7] arise from the multiple zeros of certain $S_k^{(n_i)}$-OZI, $k = 1, 2$, in a similar manner as discussed in Theorem 2 and Remark 4.

Acknowledgement

The authors acknowledges the research facilities availed at KFUPM, Saudi Arabia during the preparation of this paper.

References

[1] M. A. Bokhari and M. Iqbal, *L_2-Approximation of real valued functions with interpolatory constraints*, J. Comp. Appl Math, 70 (1996) 201-205.

[2] M. A. Bokhari, *On Hermite interpolating L_2- approximants*, Dynamical Systems and Applications Vol 16 (2), (2007) 203-216.

[3] M. A. Bokhari , Asghar Qadir and H. Al-Attas, *On Gauss-type quadrature rules*, Numerical Functional Analysis and Optimization, 1532-2467, Vol 31(10), (2010) 1120-1134.

[4] G. Golub and G. Meurant, Matrices, *moments and quadrature, in D. F. Griffiths and G. A. Watson (EDs.), Numerical analysis 1993 (Dundee, 1993), Pitman Res.* Notes Math Ser., Vol 303, Harlows: Longman Sci. Tech (1994) 105-156.

[5] W. Gautschi, *Gauss-Radau formulae for Jacobi and Laguerre weight functions.* 1999 International Symposium on Computational Sciences, to honor John R. Rice (West Lafayette, IN). Math Comp. Simulation 54, (2000) 403-412.

[6] W. Gautschi, *High-order Gauss-Lobbato formula. Mathematical journey through analysis, matrix theory and scientific computation.* Numerical Algorithm 25 (2000) 213-222.

[7] D. Kincaid and W. Cheney, *Numerical Analysis: Mathematics of Scientific Computing, 3rd Ed.*, The Brooks/Cole Series in Advanced Mathematics, (2002).

[8] R. Lobatto, *Lessen over de Differentiaal-en Iintegraal-Rekening. Part II. Iintegraal- Rekening*, The Hague: Van Cleef, (1852).

[9] R Radau, *Etude sur le formules dapproximation qui servent a calculer la valeur numerique dune integrale definie*, J. Math Pures Appl. 6, (1880) 283-336.

[10] I. Sadek and M. A. Bokhari, *Optimal control of a parabolic distributed parametric system via interpolating orthogonal polynomials*, Optimal Control, Applications & Methods, Vol. 19, (1998) 205-213.

[11] H. Wilf, *Mathematics for the physical science*, Wiley (1962).