International Journal of
Mathematical Modelling & Computations
Vol. 13, No. 03, Summer 2023, 1- 14
DOI:10.30495/ijm2c.2024.1999587.1287

An explicit method for numerical solution of the equation governing the motion of a particle under arbitrary force fields

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Abstract. In this paper, an implicit second order integro-differential equation governing unsteady motion of a solid particle submerged in a fluid medium and, affected by an arbitrary force field is solved numerically. It is assumed that the particle Reynolds number is quite small to use the well-known Basset kernel for the history force. The implicitness and singularity of the equation are removed by using a hybrid quadrature rule (HQR) and a generalized quadrature rule (GQR), respectively. A recursive plan is used to reduce the required CPU time. Two schemes along with the associated numerical solution algorithms are presented. It is described how the accuracy of the method can be increased in a systematic way. The results obtained by several examples show the effectiveness of the method.

Received: 23 June 2023, Revised: 10 August 2023, Accepted: 17 September 2023.

Keywords: Particle motion; Basset history force; Integro-differential equation; Creeping flow; Numerical solution;

AMS Subject Classification: 45JXX, 94DXX.

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1. Introduction

Particle dynamics in a fluid medium is an important subject in different industries. Material separation systems [17, 22], particulate fluid flows [8, 16], heat transfer enhancement by surface bombardment [4, 18], drug delivery systems [6, 11], separation of biological samples [9] and clean room technology [1] are some examples

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in which the particle behavior in a fluid medium should be analyzed precisely. To this end, the governing equation for the particle dynamics, has been the subject of many applied and theoretical researches in the past years [12, 13, 24, 25]. One of the most challenging issues in analyzing the equation of particle motion is the so-called Basset/history force. This term causes the equation to change its type from a differential to an integro-differential one.

The well-known integro-differential governing the particle motion suffers from two drawbacks [15]. Firstly, its numerical solution is very time consuming due to the history force which requires an integration from 0 to final time, t, in every time step. The second issue pertains to the kernel of the history force, which is singular at t = 0. Due to these drawbacks, often, the history force is neglected in the problem analysis. When the particle to fluid density ratio is large enough or the particle experiences almost stable motion, the history force can be ignored without causing a significant error [5, 10]. However, in many practical cases, these assumptions are not held and, the history force must be taken into account to obtain accurate results for the particle dynamics [23, 27].

Some efforts have been made to tackle the issues and give a solution for the governing equation. A method named window model was introduced by [2] to decrease the computation time. This method is based on the physics of the problem rather than being a mathematical approach. Also, some implicit methods for solving special cases, often in linear form, are available in the literature [3, 14].

In this paper we derive an explicit and simple method for numerical solution of the general form of the particle motion equation. In contrary to the most available methods, in the present work, the particle would be under any arbitrary force originating from various factors such as electric or magnetic fields. In other words, the presented method is capable to solve both linear and nonlinear forms of the particle motion equation. By using HQR and GQR, the implicitness and singularity of the equation are removed. In order to reduce the time consuming cost, recursive schemes are designed to compute the history force.

The paper is organized as follows: in Section 2 the integro-differential equation governing the particle motion in a fluid medium is introduced. Section 3 is devoted to the numerical method designed for solving the equation under-study. This section is divided into two subsections each containing a solution algorithm. Finally, some numerical examples are given in Section 4.

2. Problem Description

In an unsteady stokes flow, the total hydrodynamic force exerted on a moving particle in a viscous medium can be described as [5]:

$$F = \frac{9\mu\nu}{2R^2}V + \frac{1}{2}\rho_f \nu \frac{dV}{dt} + \frac{9\mu\nu}{2R}\sqrt{\frac{\nu}{\pi}}\int_0^t \frac{dV}{d\tau}\frac{d\tau}{\sqrt{t-\tau}}, \tag{1}$$

where μ and ν are the fluid dynamic and kinetic viscosity, respectively. ρ_f is the particle mass density. R is the particle radius and V is the particle velocity. The last term which represents the effect of particle acceleration in the past tense on its current motion is known as the history force. In a more general condition, this term would be expressed by [2]:

$$F_h = \int_0^t h''(\tau)K(t-\tau, h'(\tau))d\tau \tag{2}$$

Assuming that the particle Reynolds number is sufficiently small (Creeping flow), Basset kernel, $K(t-\tau, h'(\tau)) = \frac{1}{\sqrt{t-\tau}}$, is used for calculation of the history force.

Based on Newton's second low, the general form for equation of particle motion in the creeping flow can be written as:

$$h''(t) = f(t, h(t), h'(t)) + C_h \overbrace{\int_0^t \frac{1}{\sqrt{t-\tau}} h''(\tau) d\tau}^{I_1(t)}, \qquad 0 \leqslant t \leqslant \tag{3}$$

where, $f(t, h(t), h'(t))$ stands for the particle acceleration produced by a general force field (including Eq. (1) without the last term) and C_h is a numerical factor associated with the history force. Different forms of this equation have been studied widely from various points of view (See [20, 21]).

3. Numerical Solution

Consider Eq. (3) with the initial conditions $h(0) = h_0$ and $h'(0) = v_0$ as the initial position and velocity of the particle, respectively.

By integrating both sides of (3) over [0,t], and changing the order of integration, we have

$$\begin{aligned} h'(t) &= v_0 + \int_0^t f(s, h(s), h'(s)) ds + C_h \int_0^t \int_0^s \frac{1}{\sqrt{s-\tau}} h''(\tau) d\tau ds \\ &= v_0 + \underbrace{\int_0^t f(s, h(s), h'(s)) ds}_{I_2(t)} + 2C_h \underbrace{\int_0^t \sqrt{t-\tau} h''(\tau) d\tau}_{I_3(t)} \end{aligned} \tag{4}$$

and in a similar way,

$$h(t) = h_0 + v_0 t + \overbrace{\int_0^t (t-s) f(s, h(s), h'(s)) ds}^{I_4(t)} + \frac{4}{3} C_h \overbrace{\int_0^t \sqrt{(t-\tau)^3} h''(\tau) d\tau}^{I_5(t)}. \tag{5}$$

We solve system (3)-(4)-(5) to obtain h, h' and h''. We present a quadrature based method for numerical solution of this system. There are five integrals $I_1(t), \ldots, I_5(t)$. We use HQR to compute $I_2(t)$ and $I_4(t)$ and GQR for the rest integrals. In the following, we explain HQR and, GQR will be explained in the subsections 3.1 and 3.3.

Notation guidance

For enhancing the readability of the paper, it is worthy to note that $\phi(t_j)$ and ϕ_j stand for the exact and approximate value of an arbitrary function ϕ in the point t_j, respectively. For example, h_j, h'_j and h''_j denote approximate value of $h(t_j)$, $h'(t_j)$ and $h''(t_j)$, respectively.

Hybrid Quadrature Rule (HQR)

Let $Q_N(f) = \sum_{j=0}^{i} \omega_j f(t_j)$ be a Newton-Cotes quadrature rule to compute $\int_0^{t_i} f(x)dx$. There can be either an explicit or an implicit algebraic system for (h_i, h_i', h_i'') depending on whether $\omega_i = 0$ or not. The first case may occur if left-side rectangular quadrature rule is used and the second case may occur if trapezoidal rule is used. It is evident that trapezoidal rule is more accurate than left-side rectangular one. However, in the case of having nonlinear f, the trapezoidal rule results in implicit scheme. Fortunately, there is still a way to benefit from the accuracy of trapezoidal quadrature rule with explicit scheme. It is sufficient to divide the interval $[0, t_i]$ into the subintervals $[0, t_{i-1}]$ and $[t_{i-1}, t_i]$ for $i = 2, ..., n$. We use the composite trapezoidal rule for the first subinterval and the left-side rectangular for the second one. This quadrature rule is called a hybrid quadrature rule [26] and is denoted by $H(.)$. By applying HQR to $\int_0^{t_i} g(s)ds$ we have

$$\int_0^{t_i} g(s)ds \approx \sum_{j=0}^{i-1} \omega_j^h g(t_j) := H\Big(\int_0^{t_i} g(s)ds\Big),$$

where

$$\omega_j^h = \begin{cases} \frac{\Delta t}{2}, & j = 0, \\ \Delta t, & j = 1, \ldots, i-2, \\ \frac{3\Delta t}{2}, & j = i-1. \end{cases} \tag{6}$$

Theorem 3.1 *(See [7]) If $g \in C^2[0, T]$, then*

$$\left| \int_0^{t_i} g(s)ds - H\Big(\int_0^{t_i} g(s)ds\Big) \right| \leqslant \Big(\frac{T}{12} M_2 + M_1\Big)\Delta t^2,$$

where $M_2 = \max_{t\in[0,T]} |g''(t)|$ and $M_1 = \max_{t\in[0,T]} |g'(t)|$.

We use this kind of quadrature rule to compute $I_2(t_i)$ and $I_4(t_i)$ and denote them by $H(I_2(t_i))$ and $H(I_4(t_i))$, respectively, i.e.

$$H\big(I_2(t_i)\big) := \sum_{j=0}^{i-1} \omega_j^h f\big(t_j, h(t_j), h'(t_j)\big) \tag{7}$$

$$H\big(I_4(t_i)\big) := \sum_{j=0}^{i-1} \omega_j^h (t_i - t_j) f\big(t_j, h(t_j), h'(t_j)\big) \tag{8}$$

From theorem 3.1, if f has continuous second derivative, then

$$|I_2(t_i) - H\big(I_2(t_i)\big)| \leqslant K_1 \Delta t^2 \tag{9}$$

$$|I_4(t_i) - H\big(I_4(t_i)\big)| \leqslant K_2 \Delta t^2 \tag{10}$$

where constants $K_1, K_2 \in \mathbb{R}$.

Please note that, in step ith, we doesn't have the exact value of $h(t_j)$, $h'(t_j)$ and $h''(t_j)$ for $j = 0, 1, \ldots, i-1$. But we have approximate value of them. Therefore, we will used

$$\hat{H}\big(I_2(t_i)\big) := \sum_{j=0}^{i-1} \omega_j^h f\big(t_j, h_j, h'_j\big) \tag{11}$$

$$\hat{H}\big(I_4(t_i)\big) := \sum_{j=0}^{i-1} \omega_j^h (t_i - t_j) f\big(t_j, h_j, h'_j\big) \tag{12}$$

instead of $H\big(I_2(t_i)\big)$ and $H\big(I_4(t_i)\big)$.

The problem understudy in this paper contains singularity in term $I_1(t)$ in (3). In the literature of integral equations, there are many approaches for overcomeing singularities. To remove the singularity appeared in $I_1(t)$, we use GQR introduced in [19]. In this way, we approximate $h''(\tau)$ with a constant (Scheme I) and with a one degree interpolation (Scheme II). This quadrature rule, in addition to removing singularity, has high accuracy.

3.1 *Scheme I*

Generalized Quadrature Rule (GQR)

Consider coinciding equidistant meshes on t and τ:

$$0 = t_0 = \tau_0 < t_1 = \tau_1 < \ldots \leqslant t_n = \tau_n = T$$

where $t_i = i\Delta t$ and $\Delta t = \frac{T}{n}$

Lemma 3.2 *Let* $h \in C^3[0,T]$ *and* $M_3 = \max\limits_{x\in[0,T]} |h'''(x)|$. *We have*

$$\int_{t_j}^{t_{j+1}} \sqrt{(t_i-\tau)^3}\, h''(\tau) d\tau = h''(t_j) A_{ij} + E_1(t_j) \tag{13}$$

$$\int_{t_j}^{t_{j+1}} \sqrt{t_i-\tau}\, h''(\tau) d\tau = h''(t_j) \bar{A}_{ij} + E_2(t_j) \tag{14}$$

$$\int_{t_j}^{t_{j+1}} \frac{1}{\sqrt{t_i-\tau}} h''(\tau) d\tau = h''(t_j) \bar{\bar{A}}_{ij} + E_3(t_j) \tag{15}$$

where

$$A_{ij} = \frac{2}{5}\Delta t^{5/2}\big(\sqrt{(i-j)^5} - \sqrt{(i-j-1)^5}\big) \tag{16}$$

$$\bar{A}_{ij} = \frac{2}{3}\Delta t^{3/2}\big(\sqrt{(i-j)^3} - \sqrt{(i-j-1)^3}\big) \tag{17}$$

$$\bar{\bar{A}}_{ij} = 2\Delta t^{1/2}\big(\sqrt{i-j} - \sqrt{i-j-1}\big) \tag{18}$$

$\big|E_1(t_j)\big| \leqslant M_3 T^{3/2}\Delta t^2$, $\big|E_2(t_j)\big| \leqslant M_3 T^{1/2}\Delta t^2$ *and* $\big|E_3(t_j)\big| \leqslant M_3 T^{-1/2}\Delta t^2$.

Proof We prove (13). The two other inequalities are proved in similar way. Using Taylor's remainder theorem, we have

$$h''(\tau) = h''(t_j) + (\tau - t_j) h'''(\xi_j)$$

where $\xi_j \in [t_j, \tau]$. Thus

$$\begin{aligned}\int_{t_j}^{t_{j+1}} \sqrt{(t_i-\tau)^3}h''(\tau)d\tau &= \frac{2}{5}h''(t_j)\big(\sqrt{(t_i-t_j)^5}-\sqrt{(t_i-t_{j+1})^5}\big)+\int_{t_j}^{t_{j+1}}\sqrt{(t_i-\tau)^3}(\tau-t_j)h'''(\xi_j)d\tau\\ &= \frac{2}{5}h''(t_j)\big(\sqrt{(i\Delta t-j\Delta t)^5}-\sqrt{(i\Delta t-(j+1)\Delta t)^5}\big)+E_1(t_j)\\ &= h''(t_j)A_{ij}+E_1(t_j)\end{aligned}$$

where

$$\begin{aligned}\big|E_1(t_j)\big| &= \Big|\int_{t_j}^{t_{j+1}}\sqrt{(t_i-\tau)^3}(\tau-t_j)h'''(\xi_j)d\tau\Big|\\ &\leqslant \int_{t_j}^{t_{j+1}}\sqrt{(t_i-\tau)^3}(\tau-t_j)|h'''(\xi_j)|d\tau\\ &\leqslant M_3\int_{t_j}^{t_{j+1}}\sqrt{(t_i-t_j)^3}(t_{j+1}-t_j)d\tau\\ &\leqslant M_3\sqrt{(t_i-t_j)^3}(t_{j+1}-t_j)^2\leqslant M_3T^{3/2}\Delta t^2.\end{aligned}$$

■

By using Lemma 3.2, the integrals $I_1(t_i)$, $I_3(t_i)$ and $I_5(t_i)$ can be computed as follows

$$I_1(t_i)=\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}\frac{1}{\sqrt{t_i-\tau}}h''(\tau)d\tau\approx\sum_{j=0}^{i-1}\bar{\bar{A}}_{ij}h''(t_j):=G_0\big(I_1(t_i)\big)$$

$$I_3(t_i)=\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}\sqrt{t_i-\tau}h''(\tau)d\tau\approx\sum_{j=0}^{i-1}\bar{A}_{ij}h''(t_j):=G_0\big(I_3(t_i)\big)$$

$$I_5(t_i)=\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}\sqrt{(t_i-\tau)^3}h''(\tau)d\tau\approx\sum_{j=0}^{i-1}A_{ij}h''(t_j):=G_0\big(I_5(t_i)\big)$$

These quadrature rules are called generalized quadrature rule of degree zero (GQR_0) and is denoted by $G_0(.)$. Note that the index zero in GQR_0 stand for approximation of $h''(\tau)$ by a constant. In the case of using h''_j (as an approximate value of $h''(t_j)$) instead of $h''(t_j)$, we will have

$$\hat{G}_0\big(I_1(t_i)\big):=\sum_{j=0}^{i-1}\bar{\bar{A}}_{ij}h''_j \tag{19}$$

$$\hat{G}_0\big(I_3(t_i)\big):=\sum_{j=0}^{i-1}\bar{A}_{ij}h''_j \tag{20}$$

$$\hat{G}_0\big(I_5(t_i)\big):=\sum_{j=0}^{i-1}A_{ij}h''_j \tag{21}$$

Finally, in scheme I, we solve following explicit system.

$$\begin{cases} h_i'' = f(t_i, h_i, h_i') + C_h \hat{G}_0\big(I_1(t_i)\big), \\ h_i' = v_0 + \hat{H}\big(I_2(t_i)\big) + 2C_h \hat{G}_0\big(I_3(t_i)\big), \\ h_i = h_0 + v_0 t_i + \hat{H}\big(I_4(t_i)\big) + \frac{4}{3} C_h \hat{G}_0\big(I_5(t_i)\big), \end{cases} \tag{22}$$

for $i \geqslant 1$.

Since in (22), in every time step, five quadrature rule should be calculated, for high values t, this method is too time-consuming. Thus, we try to save the calculations of some previous steps to use in the current time step. Let

$$V_i := v_0 + \hat{H}\big(I_2(t_i)\big)$$
$$S_i := \sum_{j=0}^{i-1} \omega_j^h t_j f(t_j, h_j, h_j')$$
$$M_i := h_0 + v_0 t_i + \hat{H}\big(I_4(t_i)\big).$$

By a simple calculation we have

$$M_i = h_0 + t_i V_i - S_i$$

and consequently,

$$V_{i+1} := V_i + \frac{3\Delta t}{2} f(t_i, h_i, h_i') - \frac{\Delta t}{2} f(t_{i-1}, h_{i-1}, h_{i-1}'),$$
$$S_{i+1} := S_i + \frac{3\Delta t}{2} t_i f(t_i, h_i, h_i') - \frac{\Delta t}{2} t_{i-1} f(t_{i-1}, h_{i-1}, h_{i-1}'),$$
$$M_{i+1} := M_i + \Delta t V_i - \frac{\Delta t^2}{2} f(t_{i-1}, h_{i-1}, h_{i-1}').$$

In summary, we have the following algorithm.

3.2 *Algorithm of scheme I*

The algorithm for scheme I is:

1.

$$\begin{cases} h_0 = h_0 \\ h_0' = v_0 \\ h_0'' = f(0, h_0, v_0) \end{cases}$$

2.

$$\begin{cases} h_1 = h_0 + v_0 t_1 + \Delta t^2 f(t_0, h_0, h_0') + \frac{4}{3} C_h \hat{G}_0\big(I_5(t_1)\big), \\ h_1' = v_0 + \Delta t f(t_0, h_0, h_0') + 2C_h \hat{G}_0\big(I_3(t_1)\big), \\ h_1'' = f(t_1, h_1, h_1') + C_h \hat{G}_0\big(I_1(t_1)\big). \end{cases}$$

3.

$$\begin{cases} M_2 := h_0 + v_0 t_2 + \Delta t^2 f(t_0, h_0, h_0') + \frac{3\Delta t^2}{2} f(t_1, h_1, h_1') \\ V_2 := v_0 + \frac{\Delta t}{2} f(t_0, h_0, h_0') + \frac{3\Delta t}{2} f(t_1, h_1, h_0') \end{cases}$$

4. For $i = 2, 3, ...$

$$\begin{cases} h_i := M_i + \frac{4}{3} C_h \hat{G}_0\big(I_5(t_i)\big), \\ h_i' := V_i + 2C_h \hat{G}_0\big(I_3(t_i)\big), \\ h_i'' := f(t_i, h_i, h_i') + C_h \hat{G}_0\big(I_1(t_i)\big), \\ V_{i+1} := V_i + \frac{3\Delta t}{2} f(t_i, h_i, h_i') - \frac{\Delta t}{2} f(t_{i-1}, h_{i-1}, h_{i-1}'), \\ M_{i+1} := M_i + \Delta t V_i - \frac{\Delta t^2}{2} f(t_{i-1}, h_{i-1}, h_{i-1}'). \end{cases}$$

Note. It is worthy to note that in the second step of this algorithm, we use left side rectangular quadrature rule to comput $I_2(t_1)$ and $I_4(t_1)$.

3.3 *scheme II*

The difference between scheme I and scheme II is in computing the integrals $I_1(t_i)$, $I_3(t_i)$ and $I_5(t_i)$.

Lemma 3.3 *Let $h \in C^3[0, T]$ and $M_3 = \max_{\tau \in [t_j, t_{j+1}]} |h'''(\tau)|$. We have*

$$\begin{aligned} \int_{t_j}^{t_{j+1}} \sqrt{(t_i - \tau)^3} h''(\tau) d\tau &= \Delta t^{5/2} \Big[h''(t_{j+1}) B_{ij} + h''(t_j) C_{ij} \Big] + E_1(t_i), \\ \int_{t_j}^{t_{j+1}} \sqrt{t_i - \tau} h''(\tau) d\tau &= \Delta t^{3/2} \Big[h''(t_{j+1}) \bar{B}_{ij} + h''(t_j) \bar{C}_{ij} \Big] + E_2(t_i) \\ \int_{t_j}^{t_{j+1}} \frac{1}{\sqrt{t_i - \tau}} h''(\tau) d\tau &= \Delta t^{1/2} \Big[h''(t_{j+1}) \bar{\bar{B}}_{ij} + h''(t_j) \bar{\bar{C}}_{ij} \Big] + E_3(t_i) \end{aligned}$$

for $j = 0, \dots, i-2,$ *where*

$$B_{ij} := \frac{1}{35}\Big[4(i-j)^{7/2} + 10(i-j-1)^{7/2} - 14(i-j)(i-j-1)^{5/2}\Big]$$
$$C_{ij} := \frac{1}{35}\Big[4(i-j-1)^{7/2} + 10(i-j)^{7/2} - 14(i-j-1)(i-j)^{5/2}\Big]$$
$$\bar{B}_{ij} := \frac{1}{15}\Big[4(i-j)^{5/2} + 6(i-j-1)^{5/2} - 10(i-j)(i-j-1)^{3/2}\Big]$$
$$\bar{C}_{ij} := \frac{1}{15}\Big[4(i-j-1)^{5/2} + 6(i-j)^{5/2} - 10(i-j-1)(i-j)^{3/2}\Big]$$
$$\bar{\bar{B}}_{ij} := \frac{1}{3}\Big[4(i-j)^{3/2} + 2(i-j-1)^{3/2} - 6(i-j)(i-j-1)^{1/2}\Big]$$
$$\bar{\bar{C}}_{ij} := \frac{1}{3}\Big[4(i-j-1)^{3/2} + 2(i-j)^{3/2} - 6(i-j-1)(i-j)^{1/2}\Big]$$

and

$$|E_1(t_i)| \leqslant \frac{\Delta t^3}{8} M_3 T^{3/2}, \tag{23}$$
$$|E_2(t_i)| \leqslant \frac{\Delta t^3}{8} M_3 T^{1/2}, \tag{24}$$
$$|E_3(t_i)| \leqslant \frac{\Delta t^{5/2}}{8} M_3. \tag{25}$$

Proof Let

$$P_h(t) = \frac{t - t_j}{\Delta t} h''(t_{j+1}) + \frac{t_{j+1} - t}{\Delta t} h''(t_j)$$

be linear interpolation of h'' for the interpolation points $\{(t_j, h''_j), (t_{j+1}, h''_{j+1})\}$. Then

$$h''(\tau) = P_h(\tau) + \frac{(\tau - t_j)(\tau - t_{j+1})}{2} h'''(\xi).$$

Assuming $M_3 = \max_{\tau \in [t_j, t_{j+1}]} |h'''(\tau)|$, then since $\max_{\tau \in [t_j, t_{j+1}]} |\frac{(\tau - t_j)(\tau - t_{j+1})}{2}| = \frac{\Delta t^2}{8}$, we have

$$|h''(\tau) - P_h(\tau)| \leqslant \frac{\Delta t^2}{8} M_3, \quad \forall \tau \in [t_j, t_{j+1}]. \tag{26}$$

Thus, $h''(\tau) \approx P_h(\tau)$ for $\tau \in [t_j, t_{j+1}]$. Therefore

$$\int_{t_j}^{t_{j+1}} \sqrt{(t_i - \tau)^3} h''(\tau) d\tau = \int_{t_j}^{t_{j+1}} \sqrt{(t_i - \tau)^3} \Big(\frac{\tau - t_j}{\Delta t} h''(t_{j+1}) + \frac{t_{j+1} - \tau}{\Delta t} h''(t_j) \Big) d\tau + E_1(t_i) \tag{27}$$
$$\int_{t_j}^{t_{j+1}} \sqrt{t_i - \tau} h''(\tau) d\tau = \int_{t_j}^{t_{j+1}} \sqrt{t_i - \tau} \Big(\frac{\tau - t_j}{\Delta t} h''(t_{j+1}) + \frac{t_{j+1} - \tau}{\Delta t} h''(t_j) \Big) d\tau + E_2(t_i)$$
$$\int_{t_j}^{t_{j+1}} \frac{1}{\sqrt{t_i - \tau}} h''(\tau) d\tau = \int_{t_j}^{t_{j+1}} \frac{1}{\sqrt{t_i - \tau}} \Big(\frac{\tau - t_j}{\Delta t} h''(t_{j+1}) + \frac{t_{j+1} - \tau}{\Delta t} h''(t_j) \Big) d\tau + E_3(t_i).$$

It is a matter of simple algebraic manipulations to show that

$$\begin{aligned}\int_{t_j}^{t_{j+1}} \sqrt{(t_i-\tau)^3}h''(\tau)d\tau \approx \frac{h''(t_{j+1})\Delta t^{5/2}}{35}(4(i-j)^{7/2}+10(i-j-1)^{7/2}-14(i-j)(i-j-1)^{5/2})\\ +\frac{h''(t_j)\Delta t^{5/2}}{35}(4(i-j-1)^{7/2}+10(i-j)^{7/2}-14(i-j-1)(i-j)^{5/2})\\ = h''(t_{j+1})B_{ij}+h''(t_j)C_{ij}\end{aligned}$$

$$\begin{aligned}\int_{t_j}^{t_{j+1}} \sqrt{t_i-\tau}h''(\tau)d\tau \approx \frac{h''_{j+1}\Delta t^{3/2}}{15}(4(i-j)^{5/2}+6(i-j-1)^{5/2}-10(i-j)(i-j-1)^{3/2})\\ +\frac{h''(t_j)\Delta t^{3/2}}{15}(4(i-j-1)^{5/2}+6(i-j)^{5/2}-10(i-j-1)(i-j)^{3/2})\\ = h''(t_{j+1})\bar{B}_{ij}+h''(t_j)\bar{C}_{ij}\end{aligned}$$

$$\begin{aligned}\int_{t_j}^{t_{j+1}} \frac{1}{\sqrt{t_i-\tau}}h''(\tau)d\tau \approx \frac{h''_{j+1}\Delta t^{1/2}}{3}(4(i-j)^{3/2}+2(i-j-1)^{3/2}-6(i-j)(i-j-1)^{1/2})\\ +\frac{h''(t_j)\Delta t^{1/2}}{3}(4(i-j-1)^{3/2}+2(i-j)^{3/2}-6(i-j-1)(i-j)^{1/2})\\ = h''(t_{j+1})\bar{\bar{B}}_{ij}+h''(t_j)\bar{\bar{C}}_{ij}\end{aligned}$$

From (26) and (27), we have

$$\begin{aligned}|E_1(t_i)| &= \Big|\int_{t_j}^{t_{j+1}} \sqrt{(t_i-\tau)^3}\big(h''(\tau)-P_h(\tau)\big)d\tau\\ &\leqslant \int_{t_j}^{t_{j+1}} \sqrt{(t_i-\tau)^3}\big|h''(\tau)-P_h(\tau)\big|d\tau\\ &\leqslant \frac{\Delta t^2}{8}M_3T^{3/2}\int_{t_j}^{t_{j+1}} d\tau = \frac{\Delta t^3}{8}M_3T^{3/2}.\end{aligned}$$

which proves (23). In a similar way, (24) can be proven and

$$\begin{aligned}|E_3(t_i)| &= \Big|\int_{t_j}^{t_{j+1}} \frac{1}{\sqrt{(t_i-\tau)}}\big(h''(\tau)-P_h(\tau)\big)d\tau\\ &\leqslant \int_{t_j}^{t_{j+1}} \frac{1}{\sqrt{(t_i-\tau)}}\big|h''(\tau)-P_h(\tau)\big|d\tau\\ &\leqslant \frac{\Delta t^2}{8}M_3\frac{1}{\sqrt{\Delta t}}\int_{t_j}^{t_{j+1}} d\tau = \frac{\Delta t^{5/2}}{8}M_3.\end{aligned}$$

■

By using this Lemma, we can define new type of generalized quadrature rules of degree one (GQR$_1$) to compute $I_1(t_i)$, $I_3(t_i)$ and $I_5(t_i)$:

$$\begin{aligned} I_1(t_i) &= \sum_{j=0}^{i-2}\int_{t_j}^{t_{j+1}} \frac{1}{\sqrt{t_i-\tau}}h''(\tau)d\tau + \int_{t_{i-1}}^{t_i}\frac{1}{\sqrt{t_i-\tau}}h''(\tau)d\tau \\ &\approx \Delta t^{1/2}\sum_{j=0}^{i-2}\Big(h''(t_{j+1})\bar{\bar{B}}_{ij} + h''(t_j)\bar{\bar{C}}_{ij}\Big) + \int_{t_{i-1}}^{t_i}\frac{1}{\sqrt{t_i-\tau}}h''(t_{i-1})d\tau \\ &= \Delta t^{1/2}\left[\sum_{j=0}^{i-2}\Big(h''(t_{j+1})\bar{\bar{B}}_{ij} + h''(t_j)\bar{\bar{C}}_{ij}\Big) + 2h''(t_{i-1})\right] = Q_1^h\big(I_1(t_i)\big) \qquad (28) \end{aligned}$$

$$\begin{aligned} I_3(t_i) &= \sum_{j=0}^{i-2}\int_{t_j}^{t_{j+1}} \sqrt{t_i-\tau}\,h''(\tau)d\tau + \int_{t_{i-1}}^{t_i}\sqrt{t_i-\tau}\,h''(\tau)d\tau \\ &\approx \Delta t^{3/2}\sum_{j=0}^{i-2}\Big(h''(t_{j+1})\bar{B}_{ij} + h''(t_j)\bar{C}_{ij}\Big) + \int_{t_{i-1}}^{t_i}\sqrt{t_i-\tau}\,h''(t_{i-1})d\tau \\ &= \Delta t^{3/2}\left[\sum_{j=0}^{i-2}\Big(h''(t_{j+1})\bar{B}_{ij} + h''(t_j)\bar{C}_{ij}\Big) + \frac{2}{3}h''(t_{i-1})\right] = Q_1^h\big(I_3(t_i)\big) \qquad (29) \end{aligned}$$

and

$$\begin{aligned} I_5(t_i) &= \sum_{j=0}^{i-2}\int_{t_j}^{t_{j+1}} \sqrt{(t_i-\tau)^3}\,h''(\tau)d\tau + \int_{t_{i-1}}^{t_i}\sqrt{(t_i-\tau)^3}\,h''(\tau)d\tau \\ &\approx \Delta t^{5/2}\left[\sum_{j=0}^{i-2}\Big(h''(t_{j+1})B_{ij} + h''(t_j)C_{ij}\Big) + \frac{2}{5}h''(t_{i-1})\right] = Q_1^h\big(I_5(t_i)\big). \qquad (30) \end{aligned}$$

Since in application, we have h''_j instead of $h''(t_j)$, similar to the (19), (20) and (21), we can define $\hat{Q}_1^h\big(I_1(t_i)\big)$, $\hat{Q}_1^h\big(I_3(t_i)\big)$ and $\hat{Q}_1^h\big(I_5(t_i)\big)$.

3.4 *Algorithm*

1.

$$\begin{cases} h_0 := h_0 \\ h'_0 := v_0 \\ h''_0 := f(0, h_0, v_0) \end{cases}$$

2.

$$\begin{cases} h_1 = h_0 + v_0 t_1 + \Delta t^2 f(t_0, h_0, h'_0) + \frac{8}{15}C_h h''_0\sqrt{\Delta t^5}, \\ h'_1 := v_0 + \Delta t f(t_0, h_0, h'_0) + \frac{4}{3}C_h h''_0\sqrt{\Delta t^3}, \\ h''_1 := f(t_1, h_1, v_1) + 2C_h h''_0\sqrt{\Delta t}. \end{cases}$$

3.

$$\begin{cases} M_2 := h_0 + v_0 t_2 + \Delta t^2 f(t_0, h_0, h'_0) + \frac{3\Delta t^2}{2} f(t_1, h_1, h'_1) \\ V_2 := v_0 + \frac{\Delta t}{2} f(t_0, h_0, h'_0) + \frac{3\Delta t}{2} f(t_1, h_1, h'_0) \end{cases}$$

4. For $i = 2, 3, ...$

$$\begin{cases} h_i := M_i + \frac{4}{3} C_h \hat{Q}_1^h \big(I_5(t_i)\big), \\ h'_i := V_i + 2 C_h \hat{Q}_1^h \big(I_3(t_i)\big), \\ h''_i := f(t_i, h_i, h'_i) + C_h \hat{Q}_1^h \big(I_1(t_i)\big). \\ V_{i+1} := V_i + \frac{3\Delta t}{2} f(t_i, h_i, h'_i) - \frac{\Delta t}{2} f(t_{i-1}, h_{i-1}, h'_{i-1}), \\ M_{i+1} := M_i + \Delta t V_i - \frac{\Delta t^2}{2} f(t_{i-1}, h_{i-1}, h'_{i-1}). \end{cases}$$

4. Examples

In this section, we have given two examples to confirm the theoretical results. The numerical results are given in Table **??** in which absolute errors are reported. As seen, the second scheme results in more accurate solution.

Example 4.1 Consider the nonlinear second order implicit integro-differential equation

$$h''(t) = t - \frac{4}{3}\sqrt{6h(t) - 6} + \int_0^t \frac{h''(\tau)}{\sqrt{t - \tau}} d\tau$$

with initial condition $h(0) = 1$ and $h'(0) = 0$ and exact solution $h(t) = \frac{1}{6}t^3 + 1$.

Example 4.2 Consider the nonlinear second order implicit integro-differential equation

$$h''(t) = 6h(t) - \frac{3}{10}t^5 + t - \frac{32}{35}t^{7/2} - \frac{4}{3}t^{3/2} + \int_0^t \frac{h''(\tau)}{\sqrt{t - \tau}} d\tau$$

with initial condition $h(0) = 0$ and $h'(0) = 0$ and exact solution $h(t) = \frac{1}{20}t^5 + \frac{1}{6}t^3$.

5. Conclusion

An implicit second order integro-differential equation governing a particle dynamics in a fluid medium was solved numerically. It was shown that by using hybrid and generalized quadrature rules, it is possible to tackle the drawbacks pertaining to the numerical solution of this equation. Two schemes were proposed while the second scheme is more accurate than the first one. Even more accuracy can be reached by using the same procedure that was used to derive the scheme II from I. In order to reduce the calculation cost including the demanded time, a recursive plan was implemented. Numerical examples show the effectiveness of the method for both linear and nonlinear forms of the equation understudy.

Table 1. Comparison of the numerical results obtained for Examples 4.1 and 4.2.

Errors for Δ t=0.1				
	Examle 4.1		Examle 4.2	
t	Scheme1	Scheme2	Scheme1	Scheme2
t=1	$9.31e^{-2}$	$1.11e^{-2}$	$4.65e^{-2}$	$8.83e^{-3}$
t=3	$2.54e^{-1}$	$1.01e^{-1}$	$4.91e^{-2}$	$1.11e^{-2}$
t=5	$4.43e^{-1}$	$1.12e^{-1}$	$6.51e^{-2}$	$5.13e^{-2}$
t=7	$5.51e^{-1}$	$1.87e^{-1}$	$8.81e^{-2}$	$6.89e^{-2}$
t=9	$7.06e^{-1}$	$2.21e^{-1}$	$1.61e^{-1}$	$9.45e^{-2}$
Errors for Δ t=0.01				
	Examle 4.1		Examle 4.2	
t	Scheme1	Scheme2	Scheme1	Scheme2
t=1	$1.22e^{-2}$	$2.89e^{-2}$	$4.51e^{-3}$	$1.89e^{-4}$
t=3	$1.53e^{-2}$	$2.91e^{-2}$	$5.16e^{-3}$	$5.53e^{-4}$
t=5	$1.68e^{-2}$	$3.11e^{-2}$	$6.34e^{-3}$	$7.88e^{-4}$
t=7	$2.02e^{-2}$	$3.29e^{-2}$	$7.78e^{-3}$	$1.23e^{-3}$
t=9	$2.19e^{-2}$	$4.01e^{-2}$	$8.21e^{-3}$	$2.76e^{-3}$

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