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Estimation of multi-component reliability parameter in a non-identical-component strengths system under dependency of stress and strength components

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Abstract. Generating more realistic stress-strength model is main attempt, in this paper. For this aim, inference on stress-strength parameter was considered in a multi-component system with the non-identical-component strengths, based on the Kumaraswamy generalized distribution, when the stress and strength variables are dependent. The dependency assumption is studied by Copula theory, one of the most important concept in dependent variables. The maximum likelihood estimation (MLE), bootstrap confidence interval, Bayesian approximation and highest posterior density (HPD) interval are obtained, for the multi-component stress-strength parameter. Employing Monte Carlo simulations, the performance of different estimations are compared together. Finally, one real data set is analyzed for illustrative purposes.

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1. Introduction

In lifetime data analysis, the generalized distributions play very important role in considering of real data. In this paper, we use two generalized distributions. One of them is Kumaraswamy generalized (KuG) distribution. KuG was born from the idea compilation of [12] and [4]. This distribution is a class of beta generalized

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distributions and a distribution for double-bounded random processes. In fact, for the flexibility of this model, in many applicable situations, KuG can be employed to analyze real data, so that since its introduction in 2011 until now, the KuG distribution has received a considerable amount of attention from the statistical community, with over 926 citations. Recently, [15] show the flexibility of this model in studying of stress-strength reliability in a non-identical-component strengths system based on upper record values. KuG distribution, for a basic cumulative distribution function (cdf) $G(x)$ with probability density function (pdf) $g(x)$, has the pdf and cdf respectively, as follows:

$$f(x) = \alpha\beta g(x)G^{\alpha-1}(x)\big(1 - G^{\alpha}(x)\big)^{\beta-1}, \quad \alpha, \beta > 0,$$
$$F(x) = 1 - \big(1 - G^{\alpha}(x)\big)^{\beta}, \quad \alpha, \beta > 0.$$

One multi-component system is a system with more than component. Recently, the reliability of such system has attracted enormous interest. In this system, one common stress component and k strength components, which are independent and identical, work together. So, as long as at least s from k strength components exceed its stress, the system is reliable. The multi-component reliability parameter, in this system can be derived as

$$R_{s,k} = P\big(\text{at least } s \text{ of } (U_1, \cdots, U_k) > Y\big), \tag{1}$$

where $(U_1, \cdots, U_k)$ are the strengths have the same pdf $f_U(\cdot)$ and the random variable Y known as stress has the pdf $f_Y(\cdot)$. Some authors have been considered this model, when the strength and stress variables are independent, such as [9] and [10].

In following, we study the system which includes two components $\mathbf{k} = (k_1, k_2)$. Let us put $f_U(\cdot)$ and $f_V(\cdot)$ be the pdfs of the first and second components, respectively. Also, a common stress Y with the pdf $f_Y(\cdot)$ impressions all components. So, as long as at least $\mathbf{s} = (s_1, s_2)$ of $\mathbf{k} = (k_1, k_2)$ strength components exceed its stress, the system is reliable. This parameter can be defind as follows:

$$R_{\mathbf{s},\mathbf{k}} = P\big(\text{at least } s_1 \text{ of } (U_1, \cdots, U_{k_1}) \text{ and at least } s_2 \text{ of } (V_1, \cdots, V_{k_2}) > Y\big). \tag{2}$$

We assume that k_1 components' strengths $U_1, \cdots, U_{k_1}$ are independent and k_2 components' strengths $V_1, \cdots, V_{k_2}$ are independent, but the strength components and stress are dependent variables. This model is complete general, so that if $\mathbf{k} = (1, 0)$ the stress-strength parameter can be obtained from it. Also, when $\mathbf{k} = (k, 0)$, we can derived the multi-component reliability parameter in (1). Recently, [11] have considered this parameter when the strength and stress are independent variables. In this paper, the researchers can be found some examples, in nature, which can be modeled by this system.

In this paper, we obtain the estimation of $R_{\mathbf{s},\mathbf{k}}$ in Equation (2), assuming that the strength components are dependent random variables and they are dependent with the stress variable. One of the most powerful tools to demonstrate the dependence between two variables is copula function. Recently, copula theory has been extended in some fields such as financial and reliability. In reliability theory, some authors used the copula theory for describing the dependence of stress and strength variables, for example, [5] provided a copula-based approach to account for dependence in stress-strength models. [7] studied a dependent stress-strength interference model based on mixed copula function. [1] considered the reliability estimation of multi-component stress-strength model based on copula function under progressively hybrid censoring. Very recently, [16], considered the multi-component reliability estimation, when the strength and stress component are dependent. In fact, we obtain this parameter based on order statistics.

The rest of this paper is as follows. In Section 2, we consider some basic concepts such as copula theory and multi-component reliability model, when the stress and strength components are dependent. In Section 3, we obtain the classical estimation of $R_{\mathbf{s},\mathbf{k}}$ such as MLE and Bootstrap confidence interval. Bayesian inference on $R_{\mathbf{s},\mathbf{k}}$ is consider in Section 4. In Section 5, we derive the simulation results and study one real data analysis. Finally, we conclude the paper in Section 6.

2. Basic concepts

2.1 *Copula theory*

Copulas are the most important tools to describe dependence of two or more variables and in field of reliability and finance, they have many applicable. In following, we study some basic concepts of copula theory.

Suppose that $\mathbf{X} = (X_1, \cdots, X_p)$ is a continuous random vector with the joint distribution function $H(x_1, \cdots, x_p)$ and marginal $F_{X_1}(x_1) = F(x_1, +\infty, \cdots, +\infty)$, $F_{X_2}(x_2) = F(+\infty, x_2, +\infty, \cdots, +\infty)$, $\cdots$, $F_{X_p}(x_p) = F(+\infty, \cdots, +\infty, x_p)$. In this condition, based on Sklar's theorem, then there exists a unique p-dimensional copula function $C : [0, 1]^n \to [0, 1]$ satisfying

$$H(x_1, \cdots, x_p) = C(F_{X_1}(x_1), \cdots, F_{X_p}(x_p)).$$

So, if let $h(x_1, \cdots, x_p)$ be the joint probability density function of $\mathbf{X} = (X_1, \cdots, X_p)$, then we can write

$$h(x_1, \cdots, x_p) = c(F_{X_1}(x_1), \cdots, F_{X_n}(x_p)) \prod_{i=1}^{p} f_{X_i}(x_i),$$

where

$$c(u_1, \cdots, u_p) = \frac{\partial^p C(u_1, \cdots, u_p)}{\partial u_1 \cdots \partial u_p}.$$

Many authors have considered different copulas and investigated their properties, for more details see [14]. In following, we consider the Archimedean copula which introduced by [8]. The general form of this copula can be written as follows:

$$C(u_1, \cdots, u_p) = \psi(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \cdots + \psi^{1}(u_p)).$$

Considering $\psi(x) = (1+\gamma x)^{-\frac{1}{\gamma}}$, $\gamma > 0$, the Archimedean copula convert to a famous copula called Clayton copula. So, easily, we can write p-dimensional Clayton copula $(p \geqslant 2)$ as follows:

$$C(u_1, u_2, \cdots, u_p) = (u_1^{-\gamma} + u_2^{-\gamma} + \cdots + u_p^{-\gamma} - p + 1)^{-\frac{1}{\gamma}}. \quad (3)$$

Also, by some calculations, the $c(u_1, u_2, \cdots, u_p)$ can be derived, as follows:

$$c(u_1, \cdots, u_p) = \frac{\partial^p C(u_1, \cdots, u_p)}{\partial u_1 \cdots \partial u_p} = \Big(\prod_{j=1}^{p-1} (\gamma + j)\Big) \times \Big(\sum_{j=1}^{p} u_j^{-\gamma} - p + 1\Big)^{-\frac{1}{\gamma}-p} \times \Big(\prod_{j=1}^{p} u_j^{-\gamma-1}\Big)$$

2.2 *Multi-component reliability model when stress and strengths are dependent*

The reliability in a multi-component stress-strength model, with two non-identical-component strengths, is defined by (2) as

$$R_{\mathbf{s},\mathbf{k}} = \sum_{p_1=s_1}^{k_1} \sum_{p_2=s_2}^{k_2} \binom{k_1}{p_1}\binom{k_2}{p_2} \int_0^\infty \underbrace{\int_0^y \cdots \int_0^y}_{k_1-p_1+k_2-p_2} \underbrace{\int_y^\infty \cdots \int_y^\infty}_{p_1+p_2} f_{U_1,\cdots,U_{k_1},V_1,\cdots,V_{k_2},Y}(u_1, \cdots, u_{k_1}, v_1, \cdots, v_{k_2}, y) du_1 \cdots du_{k_1} dv_1 \cdots dv_{k_2} dy. \quad (4)$$

Let $U_1, \cdots, U_{k_1}$, $V_1, \cdots, V_{k_2}$ be the strength components and follow the pdfs $f_U(\cdot)$ and $f_V(\cdot)$, respectively. Now, assuming that Y, stress component, is a random variable with pdf $f_Y(\cdot)$, we can write the $k_1 + k_2$ inner integrals in (4) as follows:

$$\underbrace{\int_0^y \cdots \int_0^y}_{k_1-p_1+k_2-p_2} \underbrace{\int_y^\infty \cdots \int_y^\infty}_{p_1+p_2} \frac{\partial^{k_1+k_2+1} C\big(F_U(u_1), \cdots, F_U(u_{k_1}), F_V(v_1), \cdots, F_V(v_{k_2}), F_Y(y)\big)}{\partial F_U(u_1) \cdots \partial F_U(u_{k_1}) \partial F_V(v_1) \cdots \partial F_V(v_{k_2}) \partial F_Y(y)} \times f_U(u_1) \cdots f_U(u_{k_1}) \times f_V(v_1) \cdots f_V(v_{k_2}) f_Y(y) du_1 \cdots du_{k_1} dv_1 \cdots dv_{k_2}.$$

By some calculations, we solve these integrals and obtain the following:

$$= \sum_{\substack{w_1 \\ w_i=1,F_U(y)}} \cdots \sum_{w_{p_1}} \sum_{\substack{z_1 \\ z_i=1,F_V(y)}} \cdots \sum_{z_{p_2}} (-1)^{a_1+a_2} \times \frac{\partial C\big(w_1, \cdots, w_{p_1}, z_1, \cdots, z_{p_2}, \overbrace{F_U(y), \cdots, F_U(y)}^{k_1-p_1}, \overbrace{F_V(y), \cdots, F_V(y)}^{k_2-p_2}, F_Y(y)\big)}{\partial F_Y(y)},$$

where a_1 and a_2 are the numbers of $F_U(y)$ and $F_V(y)$, respectively, in each sentences of the summation. So, we can obtain $R_{\mathbf{s},\mathbf{k}}$ in equation (4) as follows:

$$R_{\mathbf{s},\mathbf{k}} = \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\int_0^\infty \underset{w_i=1,F_U(y)}{\sum_{w_1}\cdots\sum_{w_{p_1}}}\underset{z_i=1,F_V(y)}{\sum_{z_1}\cdots\sum_{z_{p_2}}}(-1)^{a_+a_2}$$
$$\times \frac{\partial C\big(w_1,\cdots,w_{p_1},,z_1,\cdots,z_{p_2}\overbrace{F_U(y),\cdots,F_U(y)}^{k_1-p_1}\overbrace{F_V(y),\cdots,F_V(y)}^{k_2-p_2},F_Y(y)\big)}{\partial F_Y(y)}$$
$$\times f_Y(y)dy. \tag{5}$$

Corollary 2.1 When the variables $v_1,\cdots,v_p$ are independent, we have the following copula:

$$C(v_1,\cdots,v_p)=\prod_{i=1}^{p} v_i.$$

Now, assuming the independence of strengths and stress variables, from the equation (5), we can obtain the $R_{\mathbf{s},\mathbf{k}}$ which is considered in [11], in the following form:

$$R_{\mathbf{s},\mathbf{k}} = \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\int_0^\infty (1-F_U(y))^{p_1}\big(F_U(y)\big)^{k_1-p_1}\big(1-F_V(y)\big)^{p_2}$$
$$\times \big(F_V(y)\big)^{k_2-p_2}f_Y(y)dy. \ \square$$

Now, using the p-dimensional Clayton copula, which is given in (3), we can write the integral part of the equation (5), as follows:

$$\int_0^\infty \underset{w_i=1,F_U(y)}{\sum_{w_1}\cdots\sum_{w_{p_1}}}\underset{z_i=1,F_V(y)}{\sum_{z_1}\cdots\sum_{z_{p_2}}}(-1)^{a_+a_2}$$
$$\times \frac{\partial C\big(w_1,\cdots,w_{p_1},,z_1,\cdots,z_{p_2}\overbrace{F_U(y),\cdots,F_U(y)}^{k_1-p_1}\overbrace{F_V(y),\cdots,F_V(y)}^{k_2-p_2},F_Y(y)\big)}{\partial F_Y(y)}$$
$$\times f_Y(y)dy = \underset{w_i=1,F_U(y)}{\sum_{w_1}\cdots\sum_{w_{p_1}}}\underset{z_i=1,F_V(y)}{\sum_{z_1}\cdots\sum_{z_{p_2}}}(-1)^{a_+a_2}$$
$$\times \int_0^\infty \frac{\Big(\sum_{i=1}^{p_1} w_i^{-\gamma}+\sum_{j=1}^{p_2} z_j^{-\gamma}+(k_1-p_1)F_U^{-\gamma}(y)+(k_2-p_2)F_V^{-\gamma}(y)+F_Y^{-\gamma}(y)\Big)^{-\frac{1}{\gamma}}}{\partial F_Y(y)}$$
$$\times f_Y(y)dy,$$

where

$$F_U(u)=1-\big(1-G^{\alpha_1}(u)\big)^{\beta_1},\ F_V(v)=1-\big(1-G^{\alpha_2}(v)\big)^{\beta_2},$$
$$F_Y(y)=1-\big(1-G^{\alpha_3}(v)\big)^{\beta_3}.$$

So, $R_{\mathbf{s},\mathbf{k}}$ in equation (5) can be obtained as follows:

$$R_{\mathbf{s},\mathbf{k}} = \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\sum_{\substack{w_1 \\ w_i=1,F_U(y)}}\cdots\sum_{w_{p_1}}\sum_{\substack{z_1 \\ z_i=1,F_V(y)}}\cdots\sum_{z_{p_2}}(-1)^{a_+a_2}$$
$$\times \int_0^{\infty} \frac{\left(\sum_{i=1}^{p_1} w_i^{-\gamma} + \sum_{j=1}^{p_2} z_j^{-\gamma} + (k_1-p_1)F_U^{-\gamma}(y) + (k_2-p_2)F_V^{-\gamma}(y) + F_Y^{-\gamma}(y)\right)^{-\frac{1}{\gamma}}}{\partial F_Y(y)}$$
$$\times f_Y(y)dy. \tag{6}$$

3. Classical estimation

In this section, we obtain some classical estimations of $R_{\mathbf{s},\mathbf{k}}$, such as MLE and bootstrap confidence interval.

3.1 *MLE of $R_{\mathbf{s},\mathbf{k}}$*

Let $X_1 \sim KuG(\alpha_1, \beta_1)$, $X_2 \sim KuG(\alpha_2, \beta_2)$ and $Y \sim KuG(\alpha_3, \beta_3)$ is three dependent random variables. Now, we construct the likelihood function, with n systems on the lifetime experiment. So, the samples are as follows:

$$Y = \overset{\text{Observed stress variables}}{\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}} \quad \text{and} \quad \overset{\text{Observed strength variables}}{X_1 = \begin{bmatrix} U_{11} & \dots & U_{1k_1} \\ \vdots & \ddots & \vdots \\ U_{n1} & \dots & U_{nk_1} \end{bmatrix}, X_2 = \begin{bmatrix} V_{11} & \dots & V_{1k_2} \\ \vdots & \ddots & \vdots \\ V_{n1} & \dots & V_{nk_2} \end{bmatrix}}.$$

Therefore, assuming that $\{U_{i1}, \dots, U_{ik_1}\}$, $\{V_{i1}, \dots, V_{ik_2}\}$, $i = 1, \dots, n$, and $\{Y_1, \dots, Y_n\}$ are three samples from $KuG(\alpha_1, \beta_1)$, $KuG(\alpha_2, \beta_2)$ and $KuG(\alpha_3, \beta_3)$, respectively, the likelihood function of the unknown parameters are as:

$$L(\alpha_1,\alpha_2,\beta_1,\beta_2,\alpha_3,\beta_3,\gamma|\text{data}) = \prod_{i=1}^{n} h(u_{i1},\cdots,u_{ik_1},v_{i1},\cdots,v_{ik_2},y_i)$$

$$= \prod_{i=1}^{n}\Big(c\big(F_{X_1}(u_{i1}),\cdots,F_{X_1}(u_{ik_1}),F_{X_2}(u_{i1}),\cdots,F_{X_2}(v_{ik_2}),F_Y(y_i)\big)$$

$$\times f_{X_1}(u_{i1})\cdots f_{X_1}(u_{ik_1})f_{X_2}(v_{i1})\cdots f_{X_2}(v_{ik_2})f_Y(y_i)\Big)$$

$$= \Big(\prod_{i=1}^{n} c\big(F_{X_1}(u_{i1}),\cdots,F_{X_1}(u_{ik_1}),F_{X_2}(u_{i1}),\cdots,F_{X_2}(v_{ik_2}),F_Y(y_i)\big)\Big)$$

$$\times\Big(\prod_{i=1}^{n}\prod_{j_1=1}^{k_1} f_{X_1}(u_{ij})\Big)\times\Big(\prod_{i=1}^{n}\prod_{j_2=1}^{k_2} f_{X_2}(v_{ij})\Big)\times\Big(\prod_{i=1}^{n} f_Y(y_i)\Big)$$

$$= \Big(\prod_{i=1}^{n} c\big(F_{X_1}(u_{i1}),\cdots,F_{X_1}(u_{ik_1}),F_{X_2}(u_{i1}),\cdots,F_{X_2}(v_{ik_2}),F_Y(y_i)\big)\Big)$$

$$\times\Big(\prod_{i=1}^{n}\prod_{j_1=1}^{k_1}\alpha_1\beta_1 g(u_{ij_1})G^{\alpha_1-1}(u_{ij_1})\big(1-G^{\alpha_1}(u_{ij_1})\big)^{\beta_1-1}\Big)$$

$$\times\Big(\prod_{i=1}^{n}\prod_{j_2=1}^{k_2}\alpha_2\beta_2 g(v_{ij_2})G^{\alpha_2-1}(v_{ij_2})\big(1-G^{\alpha_2}(v_{ij_2})\big)^{\beta_2-1}\Big)$$

$$\times\Big(\prod_{i=1}^{n}\alpha_3\beta_3 g(y_i)G^{\alpha_3-1}(y_i)\big(1-G^{\alpha_3}(y_i)\big)^{\beta_3-1}\Big)$$

$$= \Big(\prod_{j=1}^{k_1+k_2}(\gamma+j)\Big)^n\times\Big(\prod_{i=1}^{n}\big(\sum_{j_1=1}^{k_1}F_{X_1}^{-\gamma}(u_{ij_1})+\sum_{j_2=1}^{k_2}F_{X_2}^{-\gamma}(v_{ij_2})+F_Y^{-\gamma}(y_i)$$

$$-k_1-k_2\big)^{-\frac{1}{\gamma}-k_1-k_2-1}\Big)\times\Big(\prod_{i=1}^{n}F_Y^{-\gamma-1}(y_i)\Big)\times\Big(\prod_{i=1}^{n}\prod_{j_1=1}^{k_1}F_{X_1}^{-\gamma-1}(u_{ij_1})\Big)$$

$$\times\Big(\prod_{i=1}^{n}\prod_{j_2=1}^{k_2}F_{X_2}^{-\gamma-1}(v_{ij_2})\Big)$$

$$\times \alpha_1^{nk_1}\beta_1^{nk_1} \times \Big(\prod_{i=1}^{n}\prod_{j_1=1}^{k_1} g(u_{ij_1})\Big) \times \Big(\prod_{i=1}^{n}\prod_{j_1=1}^{k_1} G^{\alpha_1-1}(u_{ij_1})\Big)$$

$$\times \Big(\prod_{i=1}^{n}\prod_{j_1=1}^{k_1} \big(1-G^{\alpha_1}(u_{ij_1})\big)^{\beta_1-1}\Big)$$

$$\times \alpha_2^{nk_2}\beta_2^{nk_2} \times \Big(\prod_{i=1}^{n}\prod_{j_2=1}^{k_2} g(v_{ij_2})\Big) \times \Big(\prod_{i=1}^{n}\prod_{j_2=1}^{k_2} G^{\alpha_2-1}(v_{ij_2})\Big)$$

$$\times \Big(\prod_{i=1}^{n}\prod_{j_2=1}^{k_2} \big(1-G^{\alpha_2}(v_{ij_2})\big)^{\beta_2-1}\Big)$$

$$\times \alpha_3^{n}\beta_3^{n} \times \Big(\prod_{i=1}^{n} g(y_i)\Big) \times \Big(\prod_{i=1}^{n} G^{\alpha_3-1}(y_i)\Big) \times \Big(\prod_{i=1}^{n} \big(1-G^{\alpha_3}(y_i)\big)^{\beta_3-1}\Big).$$

The log-likelihood function, by replacing $\mathfrak{R}(t,\alpha_c) = 1 - G^{\alpha_c}(t),\ c = 1,2,3$, based on the observed samples, can be obtained by:

$$\ell(\alpha_1,\alpha_2,\beta_1,\beta_2,\alpha_3,\beta_3,\gamma|\text{data}) = n\sum_{j=1}^{k_1+k_2}\log(\gamma+j) - (\frac{1}{\gamma}+k_1+k_2+1)$$

$$\times \sum_{i=1}^{n}\log\Big(\sum_{j_1=1}^{k_1}\big(1-\mathfrak{R}^{\beta_1}(u_{ij_1},\alpha_1)\big)^{-\gamma} + \sum_{j_2=1}^{k_2}\big(1-\mathfrak{R}^{\beta_2}(v_{ij_2},\alpha_2)\big)^{-\gamma}$$

$$+ \big(1-\mathfrak{R}^{\beta_3}(y_i,\alpha_3)\big)^{-\gamma} - k_1 - k_2\Big) - (\gamma+1)\sum_{i=1}^{n}\log\big(1-\mathfrak{R}^{\beta_3}(y_i,\alpha_3)\big)$$

$$-(\gamma+1)\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\log\big(1-\mathfrak{R}^{\beta_1}(u_{ij_1},\alpha_1)\big) - (\gamma+1)\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\log\big(1-\mathfrak{R}^{\beta_2}(v_{ij_2},\alpha_2)\big)$$

$$+ nk_1\log(\alpha_1) + nk_1\log(\beta_1) + \sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\log\big(g(u_{ij_1})\big) + (\alpha_1-1)\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\log\big(G(u_{ij_1})\big)$$

$$+ (\beta_1-1)\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\log\big(\mathfrak{R}^{\beta_1}(u_{ij_1},\alpha_1)\big) + nk_2\log(\alpha_2) + nk_2\log(\beta_2)$$

$$+ \sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\log\big(g(v_{ij_2})\big) + (\alpha_2-1)\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\log\big(G(v_{ij_2})\big) + (\beta_2-1)$$

$$\times \sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\log\big(\mathfrak{R}^{\beta_2}(v_{ij_2},\alpha_2)\big) + n\log(\alpha_3) + n\log(\beta_3) + \sum_{i=1}^{n}\log\big(g(y_i)\big)$$

$$+ (\alpha_3-1)\sum_{i=1}^{n}\log\big(G(y_i)\big) + (\beta_3-1)\sum_{i=1}^{n}\log\big(\mathfrak{R}^{\beta_3}(y_i,\alpha_3)\big).$$

Now, by replacing $\mathfrak{T}\big(A,B\big) = A^B\log(A)$ and $\mathfrak{Q}(t,\alpha_c,\beta_c) = 1-\mathfrak{R}^{\beta_c}(t,\alpha_c),\ c = 1,2,3$, the values of $\widehat{\alpha}_1,\ \widehat{\alpha}_2,\ \widehat{\alpha}_3,\ \widehat{\beta}_1,\ \widehat{\beta}_2,\ \widehat{\beta}_3$ and $\widehat{\gamma}$, MLEs of $\alpha_1,\ \alpha_2,\ \alpha_3\ \beta_1,\ \beta_2,\ \beta_3$

and γ, respectively, should be dived by solving the following equations:

$$\frac{\partial \ell}{\partial \alpha_1} = \beta_1\big(1+(k_1+k_2+1)\gamma\big)$$
$$\times \sum_{i=1}^{n} \frac{\sum_{j_1=1}^{k_1}\Big(\mathfrak{T}\big(G(u_{ij_1}),\alpha_1\big)\mathfrak{R}^{\beta_1-1}(u_{ij_1},\alpha_1)\mathfrak{Q}^{-\gamma-1}(u_{ij_1},\alpha_1,\beta_1)\Big)}{\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)}$$
$$+\beta_1(\gamma+1)\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\frac{\mathfrak{T}\big(G(u_{ij_1}),\alpha_1\big)\mathfrak{R}^{\beta_1-1}(u_{ij_1},\alpha_1)}{\mathfrak{Q}(u_{ij_1},\alpha_1,\beta_1)}+\frac{nk_1}{\alpha_1}$$
$$+\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\log\big(G(u_{ij_1})\big)-(\beta_1-1)\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\frac{\mathfrak{T}\big(G(u_{ij_1}),\alpha_1\big)}{\mathfrak{R}(u_{ij_1},\alpha_1)},$$

$$\frac{\partial \ell}{\partial \alpha_2} = \beta_2\big(1+(k_1+k_2+1)\gamma\big)$$
$$\times \sum_{i=1}^{n} \frac{\sum_{j_2=1}^{k_2}\Big(\mathfrak{T}\big(G(v_{ij_2}),\alpha_2\big)\mathfrak{R}^{\beta_2-1}(v_{ij_2},\alpha_2)\mathfrak{Q}^{-\gamma-1}(v_{ij_2},\alpha_2,\beta_2)\Big)}{\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)}$$
$$+\beta_2(\gamma+1)\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\frac{\mathfrak{T}\big(G(v_{ij_2}),\alpha_2\big)\mathfrak{R}^{\beta_2-1}(v_{ij_2},\alpha_2)}{\mathfrak{Q}(v_{ij_2},\alpha_2,\beta_2)}+\frac{nk_2}{\alpha_2}$$
$$+\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\log\big(G(v_{ij_2})\big)-(\beta_2-1)\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\frac{\mathfrak{T}\big(G(v_{ij_2}),\alpha_2\big)}{\mathfrak{R}(v_{ij_2},\alpha_2)},$$

$$\frac{\partial \ell}{\partial \alpha_3} = \beta_3\big(1+(k_1+k_2+1)\gamma\big)\sum_{i=1}^{n}\frac{\mathfrak{T}\big(G(y_i),\alpha_3\big)\mathfrak{R}^{\beta_3-1}(y_i,\alpha_3)\mathfrak{Q}^{-\gamma-1}(y_i,\alpha_3,\beta_3)}{\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)}$$
$$+\beta_3(\gamma+1)\sum_{i=1}^{n}\frac{\mathfrak{T}\big(G(y_i),\alpha_3\big)\mathfrak{R}^{\beta_3-1}(y_i,\alpha_3)}{\mathfrak{Q}(y_i,\alpha_3,\beta_3)}+\frac{n}{\alpha_3}$$
$$+\sum_{i=1}^{n}\log\big(G(y_i)\big)-(\beta_3-1)\sum_{i=1}^{n}\frac{\mathfrak{T}\big(G(y_i),\alpha_3\big)}{\mathfrak{R}(y_i,\alpha_3)},$$

$$\frac{\partial \ell}{\partial \beta_1} = -\big(1+(k_1+k_2+1)\gamma\big)\sum_{i=1}^{n}\frac{\sum_{j_1=1}^{k_1}\Big(\mathfrak{T}\big(\mathfrak{R}(u_{ij_1},\alpha_1),\beta_1\big)\mathfrak{Q}^{-\gamma-1}(u_{ij_1},\alpha_1,\beta_1)\Big)}{\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)}$$
$$+(\gamma+1)\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\frac{\mathfrak{T}\big(\mathfrak{R}(u_{ij_1},\alpha_1),\beta_1\big)}{\mathfrak{Q}(u_{ij_1},\alpha_1,\beta_1)}+\frac{nk_1}{\beta_1}+\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\log\big(\mathfrak{R}(u_{ij_1},\alpha_1)\big),$$

$$\frac{\partial \ell}{\partial \beta_2} = -\big(1+(k_1+k_2+1)\gamma\big)\sum_{i=1}^{n}\frac{\sum_{j_2=1}^{k_2}\Big(\mathfrak{T}\big(\mathfrak{R}(v_{ij_2},\alpha_2),\beta_2\big)\mathfrak{Q}^{-\gamma-1}(v_{ij_2},\alpha_2,\beta_2)\Big)}{\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)}$$
$$+(\gamma+1)\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\frac{\mathfrak{T}\big(\mathfrak{R}(v_{ij_2},\alpha_2),\beta_2\big)}{\mathfrak{Q}(v_{ij_2},\alpha_2,\beta_2)}+\frac{nk_2}{\beta_2}+\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\log\big(\mathfrak{R}(v_{ij_2},\alpha_2)\big),$$

$$\frac{\partial \ell}{\partial \beta_3} = -\big(1+(k_1+k_2+1)\gamma\big)\sum_{i=1}^{n}\frac{\mathfrak{T}\big(\mathfrak{R}(y_i,\alpha_3),\beta_3\big)\mathfrak{Q}^{-\gamma-1}(y_i,\alpha_3,\beta_3)}{\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)}$$

$$+(\gamma+1)\sum_{i=1}^{n}\frac{\mathfrak{T}\big(\mathfrak{R}(y_i,\alpha_3),\beta_3\big)}{\mathfrak{Q}(y_i,\alpha_3,\beta_3)}+\frac{n}{\beta_3}+\sum_{i=1}^{n}\log\big(\mathfrak{R}(y_i,\alpha_3)\big),$$

$$\frac{\partial \ell}{\partial \gamma} = n\sum_{j=1}^{k_1+k_2}\frac{1}{\gamma+j}+\frac{1}{\gamma^2}\sum_{i=1}^{n}\log\big(\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)\big)+(\frac{1}{\gamma}+k_1+k_2+1)$$

$$\times\sum_{i=1}^{n}\frac{\sum_{j_1=1}^{k_1}\mathfrak{T}\big(\mathfrak{Q}(u_{ij_1},\alpha_1,\beta_1),-\gamma\big)+\sum_{j_2=1}^{k_2}\mathfrak{T}\big(\mathfrak{Q}(v_{ij_2},\alpha_2,\beta_2),-\gamma\big)+\mathfrak{T}\big(\mathfrak{Q}(y_i,\alpha_3,\beta_3),-\gamma\big)}{\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)}$$

$$-\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\log\big(\mathfrak{Q}(u_{ij_1},\alpha_1,\beta_1)\big)-\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\log\big(\mathfrak{Q}(v_{ij_2},\alpha_2,\beta_2)\big)-\sum_{i=1}^{n}\log\big(\mathfrak{Q}(y_i,\alpha_3,\beta_3)\big),$$

where

$$\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)=\sum_{j_1=1}^{k_1}\mathfrak{Q}^{-\gamma}(u_{ij_1},\alpha_1,\beta_1)$$
$$+\sum_{j_2=1}^{k_2}\mathfrak{Q}^{-\gamma}(v_{ij_2},\alpha_2,\beta_2)+\mathfrak{Q}^{-\gamma}(y_i,\alpha_3,\beta_3)-k_1-k_2.$$

Using one numerical method, the above seven equations are solved together and then utilizing the invariance property, $\widehat{R}^{MLE}_{\mathbf{s},\mathbf{k}}$, the MLE value of $R_{\mathbf{s},\mathbf{k}}$, can be obtained.

3.2 *Bootstrap confidence interval*

As the form of $R_{\mathbf{s},\mathbf{k}}$ is so complicated, the asymptotic confidence interval cannot be obtained, directly. Therefore, we use one parametric bootstrap method which introduced by [6] and known as percentile bootstrap method (Boot-p) to obtain the confidence interval. For this aim, we consider the following steps:

(1) Put $\{U_{i1},\ldots,U_{ik_1}\}$, $\{V_{i1},\ldots,V_{ik_2}\}$, $i=1,\cdots,n$ and $\{Y_1,\ldots,Y_n\}$ be three bootstrap samples from $KuG(\alpha_1,\beta_1)$, $KuG(\alpha_2,\beta_2)$ and $KuG(\alpha_3,\beta_3)$, respectively and obtain MLEs of the unknown parameters, $(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\beta}_1,\widehat{\beta}_2,\widehat{\lambda},\widehat{\gamma})$.
(2) Put $\{U^*_{i1},\ldots,U^*_{ik_1}\}$, $\{V^*_{i1},\ldots,V^*_{ik_2}\}$, $i=1,\ldots,n$, and $\{Y^*_1,\ldots,Y^*_n\}$ be three bootstrap samples from $KuG(\widehat{\alpha}_1,\widehat{\beta}_1)$, $KuG(\widehat{\alpha}_2,\widehat{\beta}_2)$ and $KuG(\widehat{\alpha}_3,\widehat{\beta}_3)$, respectively and obtain the bootstrap estimate $\widehat{R}^{MLE*}_{\mathbf{s},\mathbf{k}}$.
(3) Repeat NBOOT times Step 2.
(4) Put $G^*(x)=P(\widehat{R}^{MLE*}_{\mathbf{s},\mathbf{k}}\leqslant x)$ be cdf of $\widehat{R}^{MLE*}_{\mathbf{s},\mathbf{k}}$ and put $\widehat{R}^{Bp}_{\mathbf{s},\mathbf{k}}(x)=G^{*-1}(x)$, for a given x. A $100(1-\eta)\%$ Boot-p confidence interval of $R_{\mathbf{s},\mathbf{k}}$ can be constructed as follows:

$$(\widehat{R}^{Bp}_{\mathbf{s},\mathbf{k}}(\frac{\eta}{2}),\widehat{R}^{Bp}_{\mathbf{s},\mathbf{k}}(1-\frac{\eta}{2})).$$

4. Bayes estimation

In this section, under the squared error loss function, assuming that independence of prior distributions, we obtain Bayes estimation and associated credible interval of $R_{\mathbf{s},\mathbf{k}}$. For this aim, gamma distribution is considered as the prior distribution of α_1, α_2, α_3, β_1, β_2 and β_3. Also, we assume that the copula parameter, γ, follows as a uniform distribution. So, we have

$$\pi_1(\alpha_1) \propto \alpha_1^{a_1-1}e^{-b_1\alpha_1}, \quad \pi_2(\alpha_2) \propto \alpha_2^{a_2-1}e^{-b_2\alpha_2}, \quad \pi_3(\alpha_3) \propto \alpha_3^{a_3-1}e^{-b_3\alpha_3},$$
$$\pi_4(\beta_1) \propto \beta_1^{c_1-1}e^{-d_1\beta_1}, \quad \pi_5(\beta_2) \propto \beta_2^{c_2-1}e^{-d_2\beta_2}, \quad \pi_6(\beta_3) \propto \beta_3^{c_3-1}e^{-d_3\beta_3}, \quad \pi_7(\gamma) \propto 1.$$

By this, the joint posterior density function can be derived by:

$$\pi(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma|\text{data}) \propto L(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma|\text{data})\pi_1(\alpha_1)\pi_2(\alpha_2)\pi_3(\alpha_3)$$
$$\times \pi_4(\beta_1)\pi_5(\beta_2)\pi_6(\beta_3)\pi_7(\gamma). \qquad (7)$$

Equation (7) shows that we cannot obtain the estimation of α_1, α_2, α_3, β_1, β_2, β_3, γ and $R_{\mathbf{s},\mathbf{k}}$ parameters in a closed form. So, the Bayes estimation of the parameters and $R_{\mathbf{s},\mathbf{k}}$ is approximated by MCMC method.

4.1 *MCMC method*

From (7), we obtain the posterior pdfs of the α_1, α_2, α_3, β_1, β_2, β_3 and γ parameters by the following:

$$\pi(\alpha_1|\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma,\text{data}) \propto \prod_{i=1}^{n}\left(\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)\right)^{-\frac{1}{\gamma}-k_1-k_2-1}$$
$$\times \prod_{i=1}^{n}\prod_{j_1=1}^{k_1}\mathfrak{Q}^{-\gamma-1}(u_{ij_1},\alpha_1,\beta_1) \times \prod_{i=1}^{n}\prod_{j_1=1}^{k_1}G^{\alpha_1-1}(u_{ij_1})$$
$$\times \prod_{i=1}^{n}\prod_{j_1=1}^{k_1}\mathfrak{R}^{\beta_1-1}(u_{ij_1},\alpha_1) \times \alpha_1^{nk_1+a_1-1} \times e^{-b_1\alpha_1},$$

$$\pi(\alpha_2|\alpha_1,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma,\text{data}) \propto \prod_{i=1}^{n}\left(\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)\right)^{-\frac{1}{\gamma}-k_1-k_2-1}$$
$$\times \prod_{i=1}^{n}\prod_{j_1=1}^{k_2}\mathfrak{Q}^{-\gamma-1}(v_{ij_2},\alpha_2,\beta_2) \times \prod_{i=1}^{n}\prod_{j_2=1}^{k_2}G^{\alpha_2-1}(v_{ij_2})$$
$$\times \prod_{i=1}^{n}\prod_{j_2=1}^{k_2}\mathfrak{R}^{\beta_2-1}(v_{ij_2},\alpha_2) \times \alpha_2^{nk_2+a_2-1} \times e^{-b_2\alpha_2},$$

$$\pi(\alpha_3|\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\gamma,\text{data}) \propto \prod_{i=1}^{n}\left(\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)\right)^{-\frac{1}{\gamma}-k_1-k_2-1}$$
$$\times \prod_{i=1}^{n}\mathfrak{Q}^{-\gamma-1}(y_i,\alpha_3,\beta_3) \times \prod_{i=1}^{n}G^{\alpha_3-1}(y_i)$$

$$\times \prod_{i=1}^{n} \mathfrak{R}^{\beta_3-1}(y_i,\alpha_3) \times \alpha_3^{n+a_3-1} \times e^{-b_3\alpha_3},$$

$$\pi(\beta_1|\alpha_1,\alpha_2,\alpha_3,\beta_2,\beta_3,\gamma,\text{data}) \propto \prod_{i=1}^{n} \left(\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)\right)^{-\frac{1}{\gamma}-k_1-k_2-1}$$

$$\times \prod_{i=1}^{n}\prod_{j_1=1}^{k_1} \mathfrak{Q}^{-\gamma-1}(u_{ij_1},\alpha_1,\beta_1) \times \prod_{i=1}^{n}\prod_{j_1=1}^{k_1} \mathfrak{R}^{\beta_1-1}(u_{ij_1},\alpha_1) \times \beta_1^{nk_1+c_1-1} \times e^{-d_1\beta_1},$$

$$\pi(\beta_2|\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_3,\gamma,\text{data}) \propto \prod_{i=1}^{n} \left(\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)\right)^{-\frac{1}{\gamma}-k_1-k_2-1}$$

$$\times \prod_{i=1}^{n}\prod_{j_2=1}^{k_2} \mathfrak{Q}^{-\gamma-1}(v_{ij_2},\alpha_2,\beta_2) \times \prod_{i=1}^{n}\prod_{j_2=1}^{k_2} \mathfrak{R}^{\beta_2-1}(v_{ij_2},\alpha_2) \times \beta_2^{nk_2+c_2-1} \times e^{-d_2\beta_2},$$

$$\pi(\beta_3|\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\gamma,\text{data}) \propto \prod_{i=1}^{n} \left(\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)\right)^{-\frac{1}{\gamma}-k_1-k_2-1}$$

$$\times \prod_{i=1}^{n} \mathfrak{Q}^{-\gamma-1}(y_i,\alpha_3,\beta_3) \times \prod_{i=1}^{n} \mathfrak{R}^{\beta_3-1}(y_i,\alpha_3) \times \beta_3^{n+c_3-1} \times e^{-d_3\beta_3},$$

$$\pi(\gamma|\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\text{data}) \propto \prod_{i=1}^{n} \left(\mathfrak{A}_i(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)\right)^{-\frac{1}{\gamma}-k_1-k_2-1}$$

$$\times \prod_{j=1}^{k_1+k_2} (\gamma+j)^n \times \prod_{i=1}^{n}\prod_{j_1=1}^{k_1} \mathfrak{Q}^{-\gamma-1}(u_{ij_1},\alpha_1,\beta_1)$$

$$\times \prod_{i=1}^{n}\prod_{j_1=1}^{k_2} \mathfrak{Q}^{-\gamma-1}(v_{ij_2},\alpha_2,\beta_2) \times \prod_{i=1}^{n} \mathfrak{Q}^{-\gamma-1}(y_i,\alpha_3,\beta_3).$$

It is notable that as these posterior pdfs have not well-known distributions, generating random numbers from them is not possible. So, we use the Metropolis-Hastings method and implement the Gibbs sampling algorithm as follows:

(1) Begin with initial values $(\alpha_{1(0)},\ \alpha_{2(0)},\ \alpha_{3(0)},\ \beta_{1(0)},\ \beta_{2(0)},\ \beta_{3(0)},\ \gamma_{(0)})$.
(2) Set $t = 1$.
(3) Generate $\alpha_{1(t)}$ from $\pi(\alpha_1|\alpha_{2(t-1)},\alpha_{3(t-1)},\beta_{1(t-1)},\beta_{2(t-1)},\beta_{3(t-1)},\gamma_{(t-1)},\text{data})$ using Metropolis-Hastings method, with $N(\alpha_{1(t-1)},1)$ proposal distribution as follows:
   a) Generate w_t from $W(\cdot|\alpha_{1(t-1)},1) = N(\alpha_{1(t-1)},1)$ and u from $U(0,1)$.
   b) If $u < \min\{1,\xi\}$, let $\alpha_{1(t)} = w_t$, where

$$\xi = \frac{\pi(w_t|\alpha_{2(t-1)},\alpha_{3(t-1)},\beta_{1(t-1)},\beta_{2(t-1)},\beta_{3(t-1)},\gamma_{(t-1)},\text{data})}{\pi(\alpha_{1(t-1)}|\alpha_{2(t-1)},\alpha_{3(t-1)},\beta_{1(t-1)},\beta_{2(t-1)},\beta_{3(t-1)},\gamma_{(t-1)},\text{data})} \times \frac{W(\alpha_{1(t-1)}|w_t,1)}{W(w_t|\alpha_{1(t-1)},1)},$$

   else, go to Step (a).
(4) Generate $\alpha_{2(t)}$ from $\pi(\alpha_2|\alpha_{1(t-1)},\alpha_{3(t-1)},\beta_{1(t-1)},\beta_{2(t-1)},\beta_{3(t-1)},\gamma_{(t-1)},\text{data})$ by a similar algorithm in Step 3.

(5) Generate $\alpha_{3(t)}$ from $\pi(\alpha_3|\alpha_{1(t-1)}, \alpha_{2(t-1)}, \beta_{1(t-1)}, \beta_{2(t-1)}, \beta_{3(t-1)}, \gamma_{(t-1)}, \text{data})$ by a similar algorithm in Step 3.
(6) Generate $\beta_{1(t)}$ from $\pi(\beta_1|\alpha_{1(t-1)}, \alpha_{2(t-1)}, \alpha_{3(t-1)}, \beta_{2(t-1)}, \beta_{3(t-1)}, \gamma_{(t-1)}, \text{data})$ by a similar algorithm in Step 3.
(7) Generate $\beta_{2(t)}$ from $\pi(\beta_2|\alpha_{1(t-1)}, \alpha_{2(t-1)}, \alpha_{3(t-1)}, \beta_{1(t-1)}, \beta_{3(t-1)}, \gamma_{(t-1)}, \text{data})$ by a similar algorithm in Step 3.
(8) Generate $\beta_{3(t)}$ from $\pi(\beta_3|\alpha_{1(t-1)}, \alpha_{2(t-1)}, \alpha_{3(t-1)}, \beta_{1(t-1)}, \beta_{2(t-1)}, \gamma_{(t-1)}, \text{data})$ by a similar algorithm in Step 3.
(9) Generate $\gamma_{(t)}$ from $\pi(\gamma|\alpha_{1(t-1)}, \alpha_{2(t-1)}, \alpha_{3(t-1)}, \beta_{1(t-1)}, \beta_{2(t-1)}, \beta_{3(t-1)}, \text{data})$ by a similar algorithm in Step 3.
(10) Evaluate $R_{(t)\mathbf{s},\mathbf{k}}$ with $\alpha_{1(t)}$, $\alpha_{2(t)}$, $\alpha_{3(t-1)}$, $\beta_{1(t)}$, $\beta_{2(t)}$, $\beta_{3(t-1)}$ and $\gamma_{(t)}$.
(11) Set $t = t + 1$.
(12) Repeat T times in Steps 3-11.

Now, we obtain the Bayes estimation of $R_{\mathbf{s},\mathbf{k}}$ by

$$\widehat{R}^{MC}_{\mathbf{s},\mathbf{k}} = \frac{1}{T}\sum_{t=1}^{T} R_{(t)\mathbf{s},\mathbf{k}}. \tag{8}$$

Also, we construct the $100(1-\eta)\%$ HPD credible intervals of $R_{\mathbf{s},\mathbf{k}}$, using the idea of [2], as follows. For this aim, we order $R_{(1)\mathbf{s},\mathbf{k}}, \cdots, R_{(T)\mathbf{s},\mathbf{k}}$ as $R^{(1)}_{\mathbf{s},\mathbf{k}} < \cdots < R^{(T)}_{\mathbf{s},\mathbf{k}}$ and construct all the $100(1-\eta)\%$ confidence intervals of $R_{\mathbf{s},\mathbf{k}}$, as

$$(R^{(1)}_{\mathbf{s},\mathbf{k}}, R^{([T(1-\eta)])}_{\mathbf{s},\mathbf{k}}), \cdots, (R^{([T\eta])}_{\mathbf{s},\mathbf{k}}, R^{(T)}_{\mathbf{s},\mathbf{k}}),$$

where $[T]$ symbolizes the largest integer less than or equal to T. The HPD credible interval of $R_{\mathbf{s},\mathbf{k}}$ is the shortest length interval.

5. Numerical simulation and data analysis

In this section, the performance of different methods are compared with together, using the Monte Carlo simulations. Also, one real data set is analyzed to illustrative purposes.

5.1 *Numerical experiments and simulations*

In this section, we compare different estimations utilizing the Monte Carlo simulations. The point estimates are compared together with mean square errors (MSEs) and interval estimates are compared together with average lengths (AL) and coverage percentages (CP). We derive the simulation results based on 2000 repetitions, the re-sampling numbers in bootstrap method is NBOOT = 350 and the number of repetitions in Gibbs sampling algorithm is $T = 3000$. To derive the value of $R_{\mathbf{s},\mathbf{k}}$, from the observed data, we should generate dependent samples $(\mathbf{P}, \mathbf{Q}, \mathbf{S})$. For this aim, considering conditional situations in [3], first, let

$$C_{p,q}(s) = C(s|p,q) = \frac{\partial^2 C(p,q,s)/\big(\partial p \partial q\big)}{\partial^2 C(p,q,1)/\big(\partial p \partial q\big)}, \quad C_p(q) = C(q|p) = \partial C(p,q,1)/\partial p.$$

So, we can employ the following algorithm:

1. Generate n-dimensional independent uniform $(0,1)$ vectors $\mathbf{p}, \mathbf{w}_1, \cdots, \mathbf{w}_{k_1}, \mathbf{r}_1, \cdots, \mathbf{r}_{k_2}$.
2. Compute $\mathbf{q}_{j_1} = C_p^{-1}(\mathbf{w}_{j_1}),\ j_1 = 1, \cdots, k_1$ and $\mathbf{s}_{j_2} = C_{p,q}^{-1}(\mathbf{r}_{j_2}),\ j_2 = 1, \cdots, k_2$.
3. Calculate

$$\mathbf{u}_{j_1} = G^{-1}\big(1 - (1 - \mathbf{q}_{j_1})^{\frac{1}{\beta_1}}\big)^{\frac{1}{\alpha_1}}, \quad j_1 = 1, \cdots, k_1,$$

$$\mathbf{v}_{j_2} = G^{-1}\big(1 - (1 - \mathbf{s}_{j_2})^{\frac{1}{\beta_2}}\big)^{\frac{1}{\alpha_2}}, \quad j_2 = 1, \cdots, k_2,$$

$$\mathbf{y} = G^{-1}\big(1 - (1 - \mathbf{p})^{\frac{1}{\beta_3}}\big)^{\frac{1}{\alpha_3}}.$$

Now, based on the generated samples, we obtain the simulation results from the family of KuG and EPf distributions, in two cases.

Case I: We use the exponential distribution with rate parameter equal 2 as the baseline $G(\cdot)$. The results are obtained based on parameter values as $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma) = (2, 1, 2, 3, 2, 2, 2)$. We also assume three priors to study the Bayesian inference by

$$\text{Prior 1}: a_i = b_i = c_i = d_i = 0, \qquad i = 1, 2, 3,$$

$$\text{Prior 2}: a_i = c_i = 0.2,\ b_i = d_i = 0.3,\ i = 1, 2, 3,$$

$$\text{Prior 3}: a_i = c_i = 2,\ b_i = d_i = 4, \qquad i = 1, 2, 3.$$

In this case, the results are given in Table 1.

Case II: We use the Weibull distribution with scale and shape parameters equal 2 and 3, respectively, as the baseline $G(\cdot)$. The results are obtained based on parameter values as $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma) = (3, 2, 2, 2, 1, 1.5, 2)$. We also assume three priors to study the Bayesian inference by

$$\text{Prior 4}: a_i = b_i = c_i = d_i = 0, \qquad i = 1, 2, 3,$$

$$\text{Prior 5}: a_i = c_i = 0.1,\ b_i = d_i = 0.3,\ i = 1, 2, 3,$$

$$\text{Prior 6}: a_i = c_i = 2,\ b_i = d_i = 3, \qquad i = 1, 2, 3.$$

In this case, the results are given in Table 2.

The simulation results in Tables 1 and 2 show the following procedures:

- In point estimates, Bayes estimations perform better than classical ones and in the Bayes estimates, informative priors are better than non-informative ones, based on MSEs.
- In classical estimates, HPD intervals perform better than bootstrap intervals and in the Bayesian inference, informative priors are better than non-informative ones, based on ALs and CPs.
- For fixed $\mathbf{s}$ and $\mathbf{k}$, with increasing n, MSEs and ALs decrease and CP increase.
- For fixed $\mathbf{s}$ and n, with increasing $\mathbf{k}$, MSEs and ALs decrease and CP increase.

5.2 *Data analysis*

In this section, the monthly water capacity of the Shasta reservoir in California, USA, which are avaliable in link http://cdec.water.ca.gov/cgi-progs/queryMonthly?SHA, is analyzed, for illustrative aims. This data by some authors have been analyzed such as [9], [10] and [16]. The inference of this authors is related to the drought occurrence concept. In following, we assumed that there

Table 1. Simulation results in estimation of $R_{\mathbf{s},\mathbf{k}}$ in Case I.

		Point Estimates				Interval Estimates				
			Bayes					Bayes		
(k_1, k_2, s_1, s_2, n)		MLE	Prior 1	Prior 2	Prior 3		Bootstrap	Prior 1	Prior 2	Prior 3
(5,5,2,2,5)	|Bias|	0.0158	0.0163	0.0110	0.0154	AL	0.5028	0.4682	0.4025	0.3764
	MSE	0.0925	0.0900	0.0815	0.0789	CP	0.905	0.914	0.923	0.930
(5,5,3,3,5)	|Bias|	0.0152	0.0097	0.0093	0.0134	AL	0.5039	0.4528	0.4083	0.3774
	MSE	0.0933	0.0907	0.0810	0.0779	CP	0.907	0.915	0.924	0.931
(5,5,4,4,5)	|Bias|	0.0185	0.0117	0.0144	0.0104	AL	0.5087	0.4609	0.4037	0.3709
	MSE	0.0918	0.0904	0.0819	0.0780	CP	0.906	0.914	0.925	0.930
(5,5,2,2,10)	|Bias|	0.0162	0.0108	0.0135	0.0156	AL	0.4885	0.4133	0.3819	0.3528
	MSE	0.0846	0.0795	0.0729	0.0701	CP	0.912	0.919	0.929	0.935
(5,5,3,310)	|Bias|	0.0177	0.0185	0.0140	0.0095	AL	0.4839	0.4158	0.3859	0.3592
	MSE	0.0859	0.0780	0.0730	0.0700	CP	0.914	0.920	0.931	0.936
(5,5,4,4,10)	|Bias|	0.0096	0.0108	0.0172	0.0107	AL	0.4812	0.4100	0.3838	0.3574
	MSE	0.0830	0.0799	0.0719	0.0681	CP	0.913	0.921	0.930	0.935
(10,10,2,2,5)	|Bias|	0.0169	0.0106	0.0182	0.0118	AL	0.4585	0.3956	0.3358	0.2957
	MSE	0.0812	0.0723	0.0689	0.0649	CP	0.915	0.924	0.931	0.946
(10,10,3,3,5)	|Bias|	0.0101	0.0107	0.0147	0.0132	AL	0.4595	0.3915	0.3315	0.2966
	MSE	0.0809	0.0730	0.0680	0.0638	CP	0.916	0.925	0.933	0.947
(10,10,4,4,5)	|Bias|	0.0118	0.0171	0.0144	0.0140	AL	0.4577	0.3977	0.3399	0.2907
	MSE	0.0815	0.0729	0.0693	0.0630	CP	0.915	0.926	0.931	0.946
(10,10,2,2,10)	|Bias|	0.0180	0.0111	0.0163	0.0162	AL	0.4305	0.3484	0.3118	0.2654
	MSE	0.0652	0.0600	0.0559	0.0515	CP	0.921	0.937	0.944	0.950
(10,10,3,310)	|Bias|	0.0121	0.0142	0.0088	0.0085	AL	0.4338	0.3415	0.3108	0.2626
	MSE	0.0630	0.0598	0.0568	0.0519	CP	0.923	0.938	0.945	0.951
(10,10,4,4,10)	|Bias|	0.0138	0.0165	0.0182	0.0094	AL	0.4381	0.3477	0.3183	0.2693
	MSE	0.0681	0.0604	0.0549	0.0508	CP	0.921	0.937	0.946	0.950

Table 2. Simulation results in estimation of $R_{\mathbf{s},\mathbf{k}}$ in Case II.

		Point Estimates				Interval Estimates				
			Bayes					Bayes		
(k_1, k_2, s_1, s_2, n)		MLE	Prior 4	Prior 5	Prior 6		Bootstrap	Prior 4	Prior 5	Prior 6
(5,5,2,2,5)	|Bias|	0.0169	0.0179	0.0093	0.0180	AL	0.6325	0.5942	0.5732	0.5528
	MSE	0.1254	0.1047	0.0985	0.0920	CP	0.900	0.911	0.917	0.923
(5,5,3,3,5)	|Bias|	0.0149	0.0090	0.0110	0.0140	AL	0.6345	0.5927	0.5781	0.5567
	MSE	0.1220	0.1034	0.0973	0.0910	CP	0.902	0.910	0.919	0.925
(5,5,4,4,5)	|Bias|	0.0185	0.0186	0.0097	0.0170	AL	0.6381	0.5982	0.5754	0.5519
	MSE	0.1295	0.1046	0.0980	0.0917	CP	0.900	0.912	0.918	0.923
(5,5,2,2,10)	|Bias|	0.0185	0.0133	0.0168	0.0095	AL	0.6075	0.5528	0.5217	0.4957
	MSE	0.1023	0.0935	0.0867	0.0810	CP	0.907	0.916	0.922	0.931
(5,5,3,310)	|Bias|	0.0126	0.0180	0.0167	0.0185	AL	0.6027	0.5541	0.5247	0.4967
	MSE	0.1010	0.0942	0.0846	0.0800	CP	0.908	0.919	0.925	0.932
(5,5,4,4,10)	|Bias|	0.0152	0.0083	0.0173	0.0182	AL	0.6074	0.5564	0.5294	0.4918
	MSE	0.1054	0.0937	0.0833	0.0815	CP	0.907	0.915	0.920	0.931
(10,10,2,2,5)	|Bias|	0.0154	0.0163	0.0161	0.0123	AL	0.5748	0.5488	0.5027	0.4544
	MSE	0.1007	0.0820	0.0800	0.0782	CP	0.912	0.922	0.927	0.936
(10,10,3,3,5)	|Bias|	0.0152	0.0098	0.0157	0.0083	AL	0.5727	0.5466	0.5084	0.4567
	MSE	0.1023	0.0815	0.0795	0.0775	CP	0.915	0.925	0.930	0.938
(10,10,4,4,5)	|Bias|	0.0110	0.0085	0.0090	0.0170	AL	0.5757	0.5428	0.5037	0.4564
	MSE	0.1009	0.0827	0.0790	0.0765	CP	0.915	0.923	0.929	0.937
(10,10,2,2,10)	|Bias|	0.0156	0.0114	0.0184	0.0083	AL	0.5328	0.4978	0.4332	0.3854
	MSE	0.0985	0.0795	0.0776	0.0746	CP	0.918	0.935	0.945	0.950
(10,10,3,310)	|Bias|	0.0128	0.0121	0.0164	0.0167	AL	0.5374	0.4967	0.4370	0.3819
	MSE	0.0979	0.0789	0.0768	0.0735	CP	0.920	0.937	0.946	0.949
(10,10,4,4,10)	|Bias|	0.0100	0.0133	0.0129	0.0151	AL	0.5394	0.4917	0.4399	0.3837
	MSE	0.0992	0.0790	0.0770	0.0730	CP	0.920	0.935	0.946	0.950

Table 3. Goodness of fit test results.

Dist.	data set	KS	P-value	AIC	BIC	HQIC
Ku-Weibull	U	0.1180	0.8372	-14.6894	-9.8139	-13.3372
	V	0.1618	0.5295	-17.4871	-12.6116	-16.1348
	Y	0.2076	0.9488	-0.3702	-1.9325	-4.5632
Ku-Lomax	U	0.1281	0.7592	-10.5733	-5.6978	-9.2210
	V	0.1634	0.5159	-12.8984	-8.0229	-11.5462
	Y	0.2042	0.9548	0.5254	-1.0367	-3.6674
Ku-Frechet	U	0.1302	0.7422	-10.0478	-5.1723	-8.6956
	V	0.1667	0.4902	-12.8227	-7.9472	-11.4705
	Y	0.2054	0.9527	0.6449	-0.9172	-3.5479
Ku-LogNormal	U	0.1316	0.7310	-10.4566	-5.5811	-9.1043
	V	0.1656	0.4988	-13.05773	-8.1822	-11.7054
	Y	0.2120	0.9407	0.2539	-1.3082	-3.9386
Ku-Rayleigh	U	0.1282	0.7587	-12.9440	-9.2874	-11.9298
	V	0.1654	0.5002	-15.3339	-11.6772	-14.3197
	Y	0.1968	0.9672	-1.5264	-2.6981	-4.6711
Ku-Exp	U	0.1294	0.7490	-12.6060	-8.9494	-11.5918
	V	0.1653	0.5012	-15.0245	-11.3678	-14.0103
	Y	0.1976	0.9659	-1.4848	-2.6565	-4.6295

is no drought if the water capacity of a reservoir in a region on August and July in at least 2 years out of next 5 years is more than the amount of water achieved on December in the previous year such that, $R_{\mathbf{2,5}}$ can be interpreted as the probability of non-occurrence of drought. So, we put $U_{11}, ..., U_{15}$ and $V_{11}, ..., V_{15}$ as capacities of July and August from 1987 to 1991, $U_{21}, ..., U_{25}$ and $V_{21}, ..., V_{25}$ as capacities of July and August from 1993 to 1997, and so on $U_{51}, ..., U_{55}$ and $V_{51}, ..., V_{55}$ as capacities of July and August from 2011 to 2015. Also, $Y_1, Y_2, \ldots, Y_5$ are capacity of December in 1986 and 1992 up to 2010. For simplifying the calculations, we divide all data points by 4552000 acre-foot, total capacity of reservoir. It is notable that by this work, the statistical inference has not changed.

To check dependency of his data, because the number of stress and strength components data are different, we employ the randomness test. Using this method, the statistic is -2.7206 and the corresponding p-value is 0.0060, so the randomness is rejected and samples from U, V and Y can be considered to be dependent.

Now, we fit separately some of the possible KuG models to the data sets. So, Ku-Lomax, Ku-Weibull, Ku-Frechet, Ku-LogNormal, Ku-Rayleigh and Ku-Exp are fitted and the results are given in Table 3. We obtained this results with R software, using the new package "Newdistns" which introduced by [13]. Form Table 3, it is observed that all the models provided good fits for the data sets, which highlights the flexibility of KuG model. The Kolmogorov-Smirnov statistic and corresponding p-values in this table show that Ku-Weibull is the best fit for U and V, and Ku-Rayleigh is the best fit for Y data set. Also, the values of AIC, BIC and HQIC confirm this results. So, in this six models, we prefer to work with the Ku-Weibull for U and V and Ku-Rayleigh for Y data. So, assuming that $\mathbf{s} = (2, 2)$ and $\mathbf{k} = (5, 5)$ and under the non-informative priors, we obtain $\widehat{R}^{MLE}_{\mathbf{s},\mathbf{k}}$, $\widehat{R}^{MC}_{\mathbf{s},\mathbf{k}}$ and the corresponding 95% HPD credible interval are equal to 0.2985, 0.2907 and $(0.1274, 0.4981)$, respectively.

6. Conclusion

In this study, the statistical inference for the multi-component stress-strength reliability, with two non-identical-component strengths has been considered. The most important highlights in this paper is dependency assuming with the stress and strength components. Copula theory was described this as well. Also, KuG distribution, which contains many flexible distributions, was employed for explanation of the stress and strength components distributions. The classical and Bayesian inference of $R_{\mathbf{s},\mathbf{k}}$ is considered. So, MLE, bootstrap confidence interval, Bayes estimation and HPD credible interval are obtained. The simulation study is compared this methods and the applicability of this proposed method has implemented on the real data.

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