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Lie symmetry analysis of 3D unsteady diffusion and Reaction-Diffusion with singularities

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Abstract. In this study, we apply the basic Lie symmetry method to investigate of transient three dimensional (3D) reaction-diffusion equation with singularities. We obtain the classical Lie symmetries for the equation under consideration. Therefore, we respond to the question of classification of the equation symmetries and, as a result, derived the infinitesimal symmetries and thirteen basic combinations of vector fields which are used to reduce the order of the given equation. We create the optimal system of Lie subalgebras and the symmetry reductions of the considered equation.

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1. Introduction

A symmetry classification of partial differential equation modeling a transient diffusive reactive three-dimensional phenomenon is introduced in this section. The study is conducted in the $\Theta = E \otimes \Omega \subset R^3, E \subset R, \Omega \subset R^3$ domain, where E and Ω are limited and closed domains.

$$\psi \frac{\partial T}{\partial t} + k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + k_z \frac{\partial^2 T}{\partial z^2} + BT = 0. \tag{1}$$

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In this equation, it is assumed that k_x, k_y, k_z are constants and not equal to zero, and involves the functions $T = T(x, y, z, t), B = B(x, y, z)$, and $\psi = \psi(x, y, z)$, where $x, y, z, t \in R$. The equation also includes bounded conditions of the first and second type, as well as an initial condition. Partial differential equations givern or represent the vast majority of physical problems. Mathematical methods, such as those used for heat transfer, can produce analytical solutions for a limited range of problems [2, 11], these methods are insufficient for solving most real-world problems. It is essential to keep the symmetry for solving the equations, as it is the key for solving non-linear differntial equations. The classical and non-classical methods generate some exact arbitrary function and thus exhibit various solutions. The symmetry group of equations is known as the most fundamental transformation local group, acting on the dependent and independent variables in the system [3, 6, 7, 10, 19]. Indeed, studies in nonlinear equations are growing exponentially because such equations depict the modes and characteristics of nonlinear phenomena. These equations expand the view of scientists in terms of physical aspects, and in this regard, they find more usages in engineering and other sciences. The establishment of a completely integrable model, which describes the true characteristics of the scientific and engineering fields, is in progress, and a wide range of useful findings are being obtained. Several properties of the integrable equations include the presence of a Lax pair, which can be solved by the IST technique, satisfying the Painlev criterion, having infinite symmetry and Hamiltonian and Bi-Hamiltonian formulas, and other criteria [8, 16]. Through the Lie symmetry group process, the problem of symmetry categorization is extensively taken into account for various equations in different spaces [1, 4, 8]. Indeed, the Lie approach (symmetry group approach), as a computational, algorithmic technique to obtain constant group solutions, is widely utilized to solve differential equations. During the mentioned process, suitable solutions are obtained through known solutions. Also, its other applications are checking fixed solutions and reducing the order of ODEs [9, 12, 17]. Studies in this area are in progress since such equations depict the states and properties of nonlinear phenomena, broaden vision in terms of physical aspects, and then become more practical in engineering and other sciences. So, the search for accurate solutions is important in non-linear equations in several ways, like plasma laser radiation [5, 13].

The paper is presented in several chapters as follows. The infinitesimal generators of the symmetry algebra of Equation (1) are specified along with some other results obtained in Section 2. In Section 3, we make the optimal ideal subalgebras of Eq.(1). Following the third section, we discover the similarity solutions, Lie invariants, and similarity reduction based on the infinitesimal symmetries of Eq.(1). In Section 4, we show reductions for differential equations as well as for definite solutions.

2. Symmetry classification of Equation (1)

The symmetry group of a set of differential equations is defined as the largest local group of transformations that acts on the independent and dependent variables of the system, such that solutions of the system are transformed into other solutions. More formally, let G be a local group of transformations acting on a manifold M, and let $\mathcal{L}$Cm be a subset of M. If $\mathcal{L}$Cm is invariant under $\mathcal{G}$ then G G is said to be a symmetrizing group of $\mathcal{L}$Cm. For a system of differential equations R, a symmetry group of the system $\mathcal{L}$ is a local group of transformations G that acts on an open subsetμ of the space of independent and dependent variables, such that solutions of the system are invariant under G. To ensure the integrability of the system, we aim to use the Painlev analysis to check the compatibility of each equation's coefficients. Consider a system of PDEs of order p: $\Delta_\alpha\left(n, u^{(\rho)}\right)=0, \alpha=1, \ldots, N$. , where $x=\left(x^1, \ldots, x^m\right)$ and $T=\left(T^1, \ldots, T^n\right)$ are the m independent and n dependent variables, Respectively and $T^{(i)}$ is the i-order derivative of T with Z respect to x , $0<i<4$. Infinitesimal transformations of a Lie group act on both x, y as follows:

$$
\begin{aligned}
& \tilde{x}^i=x^i+\varepsilon \xi^i(x, T)+0\left(\varepsilon^2\right) \quad 1 \leqslant i \leqslant m, \\
& \tilde{T}^j=T^j+\varepsilon \phi_j(x, T)+0\left(\varepsilon^2\right) \quad 1 \leqslant j \leqslant n,
\end{aligned}
$$

here ξ^i, ϕ_j represent the infinitesimal transformations for $\left\{x^1, \ldots, x^\rho\right\}$ and $\left\{T^1, \ldots, T^q\right\}$. An arbitrary infinitesimal generator corresponding to the groups of transformation is:

$$
v=\sum_{i=1}^p \xi^i(x, T) \partial_x^i+\sum_{j=1}^q \phi^j(x, T) \partial_T^j
$$

We apply x, y,z and t instead of x^1, x^2, x^3 and x^4 respectively, and for simplicity

$$
\begin{gathered}
\xi^j:=\xi^j(x, y, t, T), \qquad j=1, \cdots, 3, \\
\phi:=\phi(x, y, t, T).
\end{gathered}
$$

Here, an infinitesimal transformations one-parameter Lie group is taken to apply the process for Eq.(1) as:

$$
\begin{aligned}
& \tilde{x}=x+\varepsilon \xi^1(x, y, z, t, T)+0\left(\varepsilon^2\right), \\
& \tilde{y}=y+\varepsilon \xi^2(x, y, z, t, T)+0\left(\varepsilon^2\right), \\
& \tilde{z}=z+\varepsilon \xi^3(x, j, z, t, T)+0\left(\varepsilon^2\right), \\
& \tilde{t}=t+\varepsilon \xi^4(x, y, z, t, T)+0\left(\varepsilon^2\right), \\
& \tilde{T}=T+\varepsilon \phi,(n, y, z, t, T)+0\left(\varepsilon^2\right),
\end{aligned}
$$

The corresponding symmetry generator is:

$$
\begin{aligned}
v= & \xi^1(x, y, z, t, T) \partial_x+\xi^2(x, y, z, t, T) \partial_y+\xi^3(x, y, z, t, T) \partial_z \\
& +\xi^t(x, y, z, t, T) \partial_t+\phi_1(x, y, z, t, T) \partial_T .
\end{aligned}
$$

The condition of being invariance corresponds to equations:

$$
p r^{(2)} v\left[\tau \frac{\partial T}{\partial t}+k_x \frac{\partial^2 T}{\partial x^2}+k_y \frac{\partial^2 T}{\partial y^2}+k_z \frac{\partial^2 T}{\partial z^2}+B T\right]=0
$$

whenever $\tau \frac{\partial T}{\partial t}+k_x \frac{\partial^2 T}{\partial x^2}+k_y \frac{\partial^2 T}{\partial y^2}+k_z \frac{\partial^2 T}{\partial z^2}+B T=0$ since $\xi^1, \xi^2, \xi^3, \xi^4, \phi_1$, only dependent on x, y, z, t and T. Setting the coefficients equal to zero, the number of generated equations is 52 . We express the answer to the above set of equations in the form of the following theorem:

Theorem 2.1 *The point symmetries Lie group of equation* (1) *contain Lie algebra generated by Equation. The obtained coefficients are the infinetesimals in the following forms:*

$$
\begin{aligned}
&\phi_1 = \frac{1}{8}\Big(-4Te^{\frac{tc_1k_1}{\psi_1}}e^{\sqrt{-c_3}z}e^{\sqrt{-c_2}y}e^{\frac{tB}{\psi}}((\tfrac{3}{2}c_1t \\
&-2c_{13})\,\psi + tB\,(c_1t+2c_2)\,k_3 - z\left(\frac{1}{4}c_1z+c_{11}\right)\psi^2\Big)\,k_2 - \frac{1}{4} \\
&\psi^2k_3y\,(c_1y+4c_8))\,k_1 - \frac{1}{4}\psi^2x\,(c_1x+4c_7)k_2k_3)\,e^{\sqrt{c_1}x}e^{\frac{tk_3c_3}{\psi}} \\
&e^{\frac{tk_2c_2}{\psi}} + 8\psi_{c_{20}}k_1k_2k_3\left(c_{19}+\left(e^{\sqrt{c_3}z}\right)^2c_{18}\right)\left(\left(e^{\sqrt{c_2}y}\right)^2c_{16}+c_{17}\right) \\
&\left(c_{15}+\left(e^{\sqrt{c_1x})^2}c_{14}\right)\right)\Big/\Big(k_1k_2k_3e^{\frac{tc_1k_1}{\psi}}e^{\frac{tk_2c_2}{\psi}}e^{\frac{t_{13}}{\psi}}e^{\frac{tk_3c_3}{\psi}} \\
&e^{\sqrt{c_3}z}e^{\sqrt{c_2}y}e^{\sqrt{c_1x}}\Big), \\
&\xi_1 = \frac{1}{2}\,(c_1x+2c_7)\,t + c_5z + \frac{1}{2}xc_2 + c_4y + c_6, \\
&\xi_2 = \frac{1}{2}\,(c_1t+c_2)\,y - \frac{k_2c_4x}{k_1} + c_{10}z + c_8t + c_9, \\
&\xi_3 = \frac{1}{2}\left(c_1t + c_2z - \frac{k_3c_5x}{k_1} - \frac{k_3c_{10}y}{k_2} + c_{11}t + c_{12}\right), \\
&\xi_4 = \frac{1}{2}c_1t^2 + c_2t + c_3,
\end{aligned}
$$

where $c_i \in R, i = 1, ..., T$ *and* $a(T)$ *is a function satisfying Eq.* (1).

Corollary 2.2 *All one parameter Lie groups of point symmetrices for Equation* *have the following infinitesimal generators:*

$$\begin{aligned}
&\vartheta_{13} = \frac{1}{2}xt\partial_x + \frac{1}{2}ty\partial_y + \frac{1}{2}tz\partial_z + \frac{1}{2}t^2\partial_t \\
&-\frac{1}{8}\frac{1}{k_1k_2k_3\psi}\left(T\left(6k_1k_2k_3t\psi + 4k_1k_2k_3t^rB - k_1k_2z^2\psi^2\right.\right. \\
&\left.\left.-k_1\psi^2k_3y^2 - \psi^2x^2k_2k_3\partial_T\right)\right), \\
&\vartheta_{12} = \frac{1}{2}x\partial_x + \frac{1}{2}y\partial y + \frac{1}{2}z\partial_z + t\partial_T - \frac{Tt_{13}\partial T}{\psi}, \\
&\vartheta_{11} = \partial_t, \\
&\vartheta_{10} = y\partial_x - \frac{k_2x\partial_y}{k_1}, \\
&\vartheta_9 = z\partial_x - \frac{k_3x\partial_z}{k_1}, \\
&\vartheta_8 = \partial_x, \\
&\vartheta_7 = t\partial_x + \frac{1}{2}\frac{T\psi_x\partial_T}{k_1}, \\
&\vartheta_6 = t\partial_y + \frac{1}{2}\frac{T\psi_y\partial_T}{k_2}, \\
&\vartheta_5 = \partial y, \\
&\vartheta_4 = z\partial y - \frac{k_3y\partial_z}{k_2}, \\
&\vartheta_3 = t\partial_z + \frac{1}{2}\frac{T + 2\partial_T\psi_z\partial_T}{k_3}, \\
&\vartheta_2 = \partial_z, \\
&\vartheta_1 = T\partial_T.
\end{aligned}$$

Table 1. Lie algebra for Eq. (1).

	ϑ_1	ϑ_2	ϑ_3	ϑ_4	ϑ_5	ϑ_6	ϑ_7	ϑ_8	ϑ_9	ϑ_{10}	ϑ_{11}	ϑ_{12}	ϑ_{13}
ϑ_1	0	0	0	0	0	0	0	0	0	0	0	0	0
ϑ_2	0	0	$\frac{1}{2}\frac{\psi\vartheta_1}{k_3}$	ϑ_5	0	0	0	0	ϑ_8	0	0	$\frac{1}{2}\vartheta_2$	$\frac{1}{2}\vartheta_3$
ϑ_3	0	$-\frac{1}{2}\frac{\psi\vartheta_1}{k_3}$	0	ϑ_6	0	0	0	0	ϑ_7	0	$-\vartheta_2$	$-\frac{1}{2}\vartheta_3$	0
ϑ_4	0	$-\vartheta_5$	$-\vartheta_6$	0	$\frac{k_3\vartheta_3}{k_2}$	$k_3\vartheta_3$	0	0	$-\frac{k_3\vartheta_{10}}{k_2}$	ϑ_9	0	0	0
ϑ_5	0	0	0	$\frac{k_3\vartheta_2}{k_2}$	0	$\frac{1}{2}\frac{\psi\vartheta_1}{k_2}$	0	0	0	ϑ_8	0	$\frac{1}{2}\vartheta_5$	$\frac{1}{2}\vartheta_6$
ϑ_6	0	0	0	$\frac{-k_3\vartheta_2}{k_2}$	0	$\frac{-1}{2}\frac{\psi\vartheta_1}{k_2}$	0	0	0	ϑ_7	$-\vartheta_5$	$\frac{1}{2}\vartheta_6$	0
ϑ_7	0	0	0	0	0	0	0	$\frac{-1}{2}\frac{\psi\vartheta_1}{k_1}$	$\frac{k_3\vartheta_2}{k_1}$	$\frac{k_2\vartheta_6}{k_1}$	$-\vartheta_8$	$\frac{-1}{2}\vartheta_7$	0
ϑ_8	0	0	0	0	0	0	$\frac{1}{2}\frac{\psi\vartheta_1}{k_1}$	0	$\frac{-k_3\vartheta_2}{k_1}$	$\frac{-k_2\vartheta_5}{k_1}$	0	$\frac{1}{2}\vartheta_8$	$\frac{1}{2}\vartheta_7$
ϑ_9	0	$-\vartheta_8$	$-\vartheta_7$	$\frac{-k_3\vartheta_{10}}{k_2}$	0	0	$\frac{k_3\vartheta_3}{k_1}$	$\frac{k_3\vartheta_2}{k_1}$	0	$\frac{-k_2\vartheta_4}{k_1}$	0	0	0
ϑ_{11}	0	0	ϑ_2	0	0	ϑ_5	ϑ_8	0	0	0	0	$-\frac{B\vartheta_1}{\psi} + \vartheta_{11}$	$-\frac{3\vartheta_1}{4} + \vartheta_{12}$
ϑ_{12}	0	$\frac{-1}{2}\vartheta_2$	$\frac{1}{2}\vartheta_3$	0	$\frac{-1}{2}\vartheta_5$	$\frac{1}{2}\vartheta_6$	$\frac{1}{2}\vartheta_7$	$\frac{-1}{2}\vartheta_8$	0	0	$\frac{B\vartheta_4}{\psi} - \vartheta_{11}$	0	ϑ_{13}
ϑ_{13}	0	$\frac{-1}{2}\vartheta_3$	0	0	$\frac{-1}{2}\vartheta_6$	0	0	$\frac{-1}{2}\vartheta_7$	0	0	$\frac{3\vartheta_1}{4} - \vartheta_{12}$	$-\vartheta_{13}$	0

Table 2. Adjoint representation of the Lie algebra.

ϑ_1	ϑ_1	ϑ_2	ϑ_3	ϑ_4	ϑ_5
ϑ_2	$\vartheta_1+\frac{s_2\psi}{2k_3}\vartheta_2+\frac{s_2^2\psi}{8k_3}\vartheta_{13}$	$\vartheta_2+\frac{s_2}{2}\vartheta_{12}$	$\vartheta_2+\frac{s_2}{2}\vartheta_{13}$	ϑ_4	
ϑ_3	$\vartheta_1-\frac{s_3\psi}{2k_3}\vartheta_3+\frac{s_2^2\psi}{4k_3}\vartheta_{11}$	$\vartheta_2-s_3\vartheta_{11}$	$\vartheta_3-\frac{s_3}{2}\vartheta_{12}$	ϑ_4	ϑ_5
ϑ_4	ϑ_1	$a_1+a_2\vartheta_2-a_3\vartheta_5$	$a_1+a_2\vartheta_3-a_3\vartheta_6$	ϑ_4	$a_1+a_2\vartheta_5-a_3\vartheta_2$
ϑ_5	ϑ_1	$a_1+a_2\vartheta_2-a_3\vartheta_5$	$a_1+a_2\vartheta_3-a_3\vartheta_6$	ϑ_4	$a_1+a_2\vartheta_5-a_3\vartheta_2$
ϑ_6	$\vartheta_1-\frac{s_6\psi}{2k_2}\vartheta_5+\frac{s_2^2\psi}{4k_2}\vartheta_{11}$	ϑ_2	$\vartheta_3-\frac{s_6k_3}{k_2}\vartheta_4$	ϑ_4	$\vartheta_5-s_6\vartheta_{11}$
ϑ_7	$\vartheta_1-\frac{s_7\psi}{2k_1}\vartheta_8+\frac{s_2^2\psi}{4k_1}\vartheta_{11}$	ϑ_2	$\vartheta_3-\frac{s_7k_3}{k_1}\vartheta_9$	ϑ_4	ϑ_5
ϑ_8	$\vartheta_1+\frac{s_8\psi}{2k_1}\vartheta_7+\frac{s_8^2\psi}{8k_1}\vartheta_{12}$	$\vartheta_2-\frac{s_8k_3}{k_1}\vartheta_9$	ϑ_3	ϑ_4	$\vartheta_5-\frac{s_8k_3}{k_1}\vartheta_{10}$
ϑ_9	ϑ_1	$a_4+a_5\vartheta_2-a_6\vartheta_8$	$a_4+a_5\vartheta_3-a_6\vartheta_7$	$a_4+a_5\vartheta_4+a_6\vartheta_{10}$	ϑ_5
ϑ_{10}	ϑ_1	ϑ_2	ϑ_3	$a_7+a_8\vartheta_4-a_9\vartheta_9$	$a_7+a_8\vartheta_5-a_9\vartheta_8$
ϑ_{11}	$\vartheta_1-\frac{s_{11}B}{\psi}-\frac{s_{11}(2s_{11}B+3\psi)}{4\psi}\vartheta_{13}$	$\vartheta_2+s_{11}\vartheta_3$	ϑ_3	ϑ_4	$\vartheta_5+s_{11}\vartheta_6$
ϑ_{12}	$\vartheta_1-\frac{B(-1+e^{-s_{12}})}{\psi}\vartheta_{11}$	$e^{-\frac{s_{12}}{2}}\vartheta_2$	$e^{\frac{s_{12}}{2}}\vartheta_3$	ϑ_4	$e^{-\frac{s_{12}}{2}}\vartheta_5$
ϑ_{13}	$\vartheta_1+\frac{3s_{13}}{4}\vartheta_{11}$	ϑ_2	$\vartheta_3-\frac{s_3}{2}\vartheta_{13}$	ϑ_4	ϑ_5

ϑ_1	ϑ_6	ϑ_7	ϑ_8	ϑ_9	ϑ_{10}
ϑ_2	$s_2\vartheta_4+\vartheta_5$	ϑ_6	ϑ_7	$\vartheta_8+s_2\vartheta_9$	ϑ_9
ϑ_3	$s_3\vartheta_4+\vartheta_6$	$\vartheta_7+s_3\vartheta_9$	ϑ_8	ϑ_9	ϑ_{10}
ϑ_4	$a_1+a_2\vartheta_6-a_3\vartheta_3$	ϑ_7	ϑ_8	$a_1+a_2\vartheta_9-a_3\vartheta_{10}$	$a_1+a_2\vartheta_9-a_3\vartheta_{10}$
ϑ_5	$a_1+a_2\vartheta_6-a_3\vartheta_3$	ϑ_7	ϑ_8	$a_1+a_2\vartheta_9-a_3\vartheta_{10}$	$a_1+a_2\vartheta_9-a_3\vartheta_{10}$
ϑ_6	$\vartheta_6-\frac{s_6}{2}\vartheta_{12}$	$\vartheta_7+s_6\vartheta_{10}$	ϑ_8	ϑ_9	ϑ_{10}
ϑ_7	$\vartheta_6-\frac{s_7k_3}{k_1}\vartheta_{10}$	$\vartheta_7-\frac{s_7}{2}\vartheta_{12}$	$\vartheta_8-s_7\vartheta_{11}$	ϑ_9	ϑ_{10}
ϑ_8	ϑ_6	$\vartheta_7+\frac{s_8}{2}\vartheta_{13}$	$\vartheta_8+\frac{s_8}{2}\vartheta_{12}$	ϑ_9	ϑ_{10}
ϑ_9	ϑ_6	$a_4+a_5\vartheta_7+a_6\vartheta_3$	$a_4+a_5\vartheta_8+a_6\vartheta_2$	ϑ_9	$a_4+a_5\vartheta_{10}-a_6\vartheta_4$
ϑ_{10}	$a_7+a_8\vartheta_7+a_9\vartheta_6$	$a_7+a_8\vartheta_8+a_9\vartheta_5$	$a_7+a_8\vartheta_9+a_9\vartheta_4$	ϑ_9	ϑ_{10}
ϑ_{11}	ϑ_6	ϑ_7	$\vartheta_8+s_{11}\vartheta_7$	ϑ_9	ϑ_{10}
ϑ_{12}	$e^{\frac{s_{12}}{2}}\vartheta_6$	$e^{\frac{s_{12}}{2}}\vartheta_7$	$e^{\frac{s_{12}}{2}}\vartheta_8$	ϑ_9	ϑ_{10}
ϑ_{13}	$\vartheta_6-\frac{s_{13}}{2}\vartheta_5$	$\vartheta_7-\frac{s_{13}}{2}\vartheta_8$	ϑ_8	ϑ_9	ϑ_{10}

ϑ_1	ϑ_{11}	ϑ_{12}	ϑ_{13}
ϑ_2	ϑ_{11}	ϑ_{12}	ϑ_{13}
ϑ_3	ϑ_{11}	ϑ_{12}	ϑ_{13}
ϑ_4	ϑ_{11}	ϑ_{12}	ϑ_{13}
ϑ_5	ϑ_{11}	ϑ_{12}	ϑ_{13}
ϑ_6	ϑ_{11}	ϑ_{12}	ϑ_{13}
ϑ_7	ϑ_{11}	ϑ_{12}	ϑ_{13}
ϑ_8	ϑ_{11}	ϑ_{12}	ϑ_{13}
ϑ_9	ϑ_{11}	ϑ_{12}	ϑ_{13}
ϑ_{10}	ϑ_{11}	ϑ_{12}	ϑ_{13}
ϑ_{11}	$\vartheta_{11}+s_{11}\vartheta_{12}+\frac{s_{11}^2}{2}\vartheta_{13}$	$\vartheta_{12}+s_{11}\vartheta_{13}$	ϑ_{13}
ϑ_{12}	$e^{-s_{12}}\vartheta_{11}$	ϑ_{12}	$e^{s_{12}}\vartheta_{13}$
ϑ_{13}	ϑ_{11}	$\vartheta_{12}-s_{13}\vartheta_{11}$	$\vartheta_{13}-s_{13}\vartheta_{12}+\frac{s_{13}^2}{2}$

where

$$
\begin{aligned}
a_1 &= \frac{1}{2}e^{\frac{\sqrt{-k_2k_3}s_4}{k_2}}, a_2 = \frac{1}{2}e^{-\frac{\sqrt{-k_2k_3}s_4}{k_2}}, a_3 = \frac{k_3(e^{-\frac{\sqrt{-k_2k_3}s_4}{k_2}}-e^{\frac{\sqrt{-k_2k_3}s_4}{k_2}})}{2\sqrt{-k_2k_3}}\\
a_4 &= \frac{1}{2}e^{\frac{\sqrt{-k_1k_3}s_9}{k_1}}, a_5 = \frac{1}{2}e^{-\frac{\sqrt{-k_1k_3}s_9}{k_1}}, a_6 = \frac{k_3(e^{-\frac{\sqrt{-k_1k_3}s_9}{k_1}}-e^{\frac{\sqrt{-k_1k_3}s_9}{k_1}})}{2\sqrt{-k_1k_3}}\\
a_7 &= \frac{1}{2}e^{\frac{\sqrt{-k_1k_2}s_{10}}{k_1}}, a_8 = \frac{1}{2}e^{-\frac{\sqrt{-k_1k_2}s_{10}}{k_1}}, a_9 = \frac{k_3(e^{-\frac{\sqrt{-k_1k_2}s_{10}}{k_1}}-e^{\frac{\sqrt{-k_1k_2}s_{10}}{k_1}})}{2\sqrt{-k_1k_2}}.
\end{aligned}
$$

G_i groups containing one parameter produced by ϑ_i are described in the following expressions:

$$exp(\varepsilon\vartheta_i)(x,t,u) = (\tilde{x},\tilde{t},\tilde{u})$$
$$G_1 = (x,y,z,t,Te^{\varepsilon}),$$
$$G_2 = (x,y,\varepsilon+z,t,T),$$
$$G_3 = \left(x,y,t\varepsilon+z,t,Te\int_0^{\varepsilon}\frac{1}{2}\frac{\psi\,(z_{19}+z)}{k_3}dz_{19}\right),$$
$$G_4 = \left(x,\frac{z\sqrt{k_2}sin(\frac{\sqrt{k_3}\varepsilon}{\sqrt{k_2}})}{\sqrt{k_3}}+ycos(\frac{\sqrt{k_3}\varepsilon}{\sqrt{k_2}}),-\frac{\sqrt{k_3}(-\frac{-z\sqrt{k_2}cos(\frac{\sqrt{k_3}\varepsilon}{\sqrt{k_2}})}{\sqrt{k_3}})+ysin(\frac{\sqrt{k_3}\varepsilon}{\sqrt{k_2}})}{\sqrt{k_2}},t,T\right),$$
$$G_5 = (x,\varepsilon+y,z,t,T),$$
$$G_6 = \left(x,t_{\varepsilon+y},z,t,Te\int_0^{e}\frac{1}{2}\frac{\left(t_{\,z_{19}+y)\psi}\right.}{k_2}dz_{19}\right),$$
$$G_7 = \left(t\varepsilon+x,y,z,t,Te^{\int_0^e\frac{1}{2}\frac{\psi(z_{19}+y)}{k_1}dz_{19}}\right),$$
$$G_8 = (\varepsilon{+}x,y,z,t,T),$$
$$G_9 = \left(\frac{z\sqrt{k_1}\sin\left(\frac{\sqrt{k_3}}{\sqrt{k_1}}\varepsilon\right)}{\sqrt{k_3}}+x\cos\left(\frac{\sqrt{k_3}\varepsilon}{\sqrt{k_1}}\right),y,-\frac{\sqrt{k_2}\left(-z\frac{\sqrt{k_1}\cos\left(\frac{\sqrt{k_3}\varepsilon}{\sqrt{k_1}}\right)}{\sqrt{k_3}}+x\sin(\right.}{\sqrt{k_1}}\right.$$
$$\left.\frac{\sqrt{k_3}}{\sqrt{k_1}}\varepsilon),t,T\right),$$
$$G_{10} = \left(\frac{y\sqrt{k_1}\sin\left(\frac{\sqrt{k_2}}{\sqrt{k_1}}\varepsilon\right)}{\sqrt{k_2}}+x\cos\left(\frac{\sqrt{k_2}\varepsilon}{\sqrt{k_1}}\right),-\frac{\sqrt{k_2}\left(-y\frac{\sqrt{k_1}\cos\left(\frac{\sqrt{k_2}\varepsilon}{\sqrt{k_1}}\right)}{\sqrt{k_2}}+x\sin(\right.}{\sqrt{k_1}}\right.$$
$$\left.\frac{\sqrt{k_2}}{\sqrt{k_1}}\varepsilon),z,t,T\right),$$
$$G_{11} = (x,y,z,\varepsilon+t,T),$$
$$G_{12} = \left(xe^{\frac{1}{2}\varepsilon},ye^{\frac{1}{2}\varepsilon},ze^{\frac{1}{2}\varepsilon},te^{\varepsilon},Te^{\frac{tB}{\psi}}e^{-\frac{te^{\varepsilon}B}{\psi}}\right),$$
$$G_{13} = \left(\frac{2x}{t\left(-\varepsilon+\frac{2}{t}\right)},\frac{2y}{t\left(-\varepsilon+\frac{2}{t}\right)},\frac{2z}{t\left(-\varepsilon+\frac{2}{t}\right)},\frac{2}{-\varepsilon+\frac{2}{t}},\right.$$
$$\left.\frac{Te^{-\frac{1}{2}\frac{\psi x^2}{k_1\left(\varepsilon-\frac{2}{t}\right)t^2}-\frac{1}{2}\frac{\psi z^2}{k_3\left(\varepsilon-\frac{2}{t}\right)t^2}+\frac{2B}{\psi\left(\varepsilon-\frac{2}{t}\right)}-\frac{1}{2}\frac{\psi y^2}{k_2\left(\varepsilon-\frac{2}{t}\right)t^2}+\frac{3}{2}\ln\left(\frac{\varepsilon-2}{t}\right)}}{e^{-\frac{1}{4}\frac{\psi^2x^2k_2k_3-k_1k_2z^2t^2+4k_1k_2k_3t^2B-k_1\psi^2k_3y^2-6ln(\frac{-2}{3})k_1k_2\psi k_1t}{tk_1k_3\psi k_2}}}\right).$$

3. Classification of 1D subalgebras

Using the developed symmetry group, we can now determine the optimal system with one parameter group of equations. It is important to obtain the subgroups of the system since they offer various types of solutions for the equation. Therefore, finding solutions that remain unaffected is essential. To achieve this, our proposed approach provides an expression for an optimal group of subalgebras. The classification procedure of subalgebras is similar to the classification for representation orbits that are adjointed. By assigning one representation from every subalgebra group within the system, we can obtain a solution for a problem with an optimal group of subalgebras. Assuming g to be the Lie algebra determined by Corollary 2.2, we can obtain the adjoint action for the equation in table 2. We can then set up an optimal system of the subalgebras for the equation.

$$\text{Ad }\left(\exp\left(s\cdot\vartheta_t\right)\cdot\vartheta_r\right)=\vartheta_r-s\cdot\left[\vartheta_t,\vartheta_r\right]+\frac{s^2}{2}\cdot\left[\vartheta_t,\left[\vartheta_t,\vartheta_r\right],\ldots\right].$$

In this equation, s is a variable and $[\vartheta_t, \vartheta_r]$ is defined in Table 1 for $t, r = 1, \ldots, 13$, where s is a variable in this representation. Assuming g to be the Lie algebra as determined by Corollary 2.2, we can derive the adjoint action for the equation, as shown in Table 2. Using this information, we can establish an optimal system of subalgebras for the Equation. Specifically, a one-dimensional optimal system of Equation is provided below:

Theorem 3.1 *An example of a one-dimensional optimal system of the equation is given by:*

1) $\vartheta_{13}+c_1\vartheta_1+c_2\vartheta_2+c_3\vartheta_4+c_4\vartheta_5+c_5\vartheta_6+c_6\vartheta_8+c_7\vartheta_9+c_8\vartheta_{10}+c_9\vartheta_{11}$,

2) $\vartheta_{12}+c_1\vartheta_1+c_2\vartheta_4+c_3\vartheta_5+c_4\vartheta_9+c_5\vartheta_{10}+c_6\vartheta_{11}$,

3) $\vartheta_{11}+c_1\vartheta_3+c_2\vartheta_4+c_3\vartheta_6+c_4\vartheta_7+c_5\vartheta_9+c_6\vartheta_{10}$,

4) $\vartheta_{10}+c_1\vartheta_1+c_2\vartheta_2+c_3\vartheta_3+c_4\vartheta_4+c_5\vartheta_5+c_6\vartheta_8+c_7\vartheta_9$,

5) $\vartheta_9+c_1\vartheta_1+c_2\vartheta_4+c_3\vartheta_5+c_4\vartheta_6+c_5\vartheta_8$,

6) $\vartheta_8+c_1\vartheta_1+c_2\vartheta_2+c_3\vartheta_3+c_4\vartheta_4+c_5\vartheta_5+c_6\vartheta_6$,

7) $\vartheta_7+c_1\vartheta_1+c_2\vartheta_2+c_3\vartheta_3+c_4\vartheta_4+c_5\vartheta_6$,

8) $\vartheta_6+c_1\vartheta_1+c_2\vartheta_2+c_3\vartheta_3+c_4\vartheta_4$,

9) $\vartheta_5+c_1\vartheta_1+c_2\vartheta_2+c_3\vartheta_3+c_4\vartheta_4$,

10) $\vartheta_4+c_1\vartheta_1+c_2\vartheta_2+c_3\vartheta_3$,

11) ϑ_3,

12) $\vartheta_2+c_1\vartheta_1$,

13) ϑ_1.

Proof Looking at Table 1, it is enough to determine the subalgebras of

$$\langle\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5,\vartheta_6,\vartheta_7,\vartheta_8,\vartheta_9,\vartheta_{10},\vartheta_{11},\vartheta_{12},\vartheta_{13}\rangle$$

function $F_t^s : g \to g$ defined by $\vartheta \to Ad(Exp(s\vartheta_i)\vartheta)$ is a linear map, for $i = 1, \ldots, 13$. For example, two matrices $A10$ and $A12$ of $F_i^s, i = 1, \ldots, 13$, with respect to basis, are the following:

$$A10 = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&a_7+a_8&0&0&0&0&-\frac{k_2}{k_3}a_9&0&0&0&0\\
0&0&0&0&a_7+a_8&0&0&-\frac{k_2}{k_3}a_9&0&0&0&0&0\\
0&0&0&0&0&a_7+a_8&-\frac{k_2}{k_3}a_9&0&0&0&0&0&0\\
0&0&0&0&0&-\frac{k_1}{k_3}a_9&a_7+a_8&0&0&0&0&0&0\\
0&0&0&0&-\frac{k_1}{k_3}a_9&0&0&a_7+a_8&0&0&0&0&0\\
0&0&0&-\frac{k_1}{k_3}a_9&0&0&0&0&a_7+a_8&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}$$

$$A12 := \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&-\frac{B(-1+e^{-s}12)}{\psi}&0&0\\
0&e^{-\frac{1}{2}s}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&e^{\frac{1}{2}s}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&e^{-\frac{1}{2}s}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&e^{\frac{1}{2}s}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&e^{\frac{1}{2}s}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&e^{-\frac{1}{2}s}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&e^{-s}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&e^{s^{12}}
\end{bmatrix}$$

Alternatively by acting these matrices on $x = \sum_{i=1}^{13} c_i\vartheta_i$. x is a vector field, x can by simplified as follows:
By taking $c_{13} \neq 0$, the coefficient of ϑ_3, ϑ_7 and ϑ_{12} can be disappeard by setting $s_2 = -2\dfrac{c_2}{c_{13}}$ $s_8 = -\dfrac{-2c_7}{c_{13}} and s_{11} = -\dfrac{c_{12}}{c_{13}}$
respectively. Scalling x, we assume $c_{13} = 1$, thus, x can be to the case (1). For $c_{13} = 0$ and $c_{12} \neq 0$, the coefficients of ϑ_2, ϑ_3, ϑ_6, ϑ_7 and ϑ_8 can be disappered by setting $s_2 = \dfrac{-2c_2}{c_{12}}$, $s_3 = \dfrac{2c_3}{12}$, $s_6 = \dfrac{2c_6}{c_{12}}$ and $s_8 = -\dfrac{2c_8}{c_{12}}$ respectively. Scalling x, we assume $c_{12} = 1$. Thus x can be to the case (2).
For $c_{13} = c_{12} = 0$ and $c_{11} \neq 0$, the coefficients of ϑ_1, ϑ_2, ϑ_5 and ϑ_8 can be disappered by setting $s_2 = \dfrac{-4c_1}{c_{11}}$, $s_3 = \dfrac{c_2}{c_{11}}$, $s_6 = \dfrac{c_5}{c_{11}}$, $s_7 = \dfrac{c_8}{c_{11}}$ respectively. Scalling x, we assume $c_{11} = 1$. Thus x can be to the case (3).
For $c_{13} = c_{12} = c_{11} = 0$ and $c_{10} \neq 0$, the coefficents of ϑ_6 and ϑ_7 can be disappered by setting $s_7 = \dfrac{k_1c_6}{k_2c_{10}}$ and $s_6 = \dfrac{-c_7}{c_{10}}$ respectively. Scalling x, we assume $c_{10} = 1$. Thus x can be to the case (4).
For $c_{13} = c_{12} = c_{11} = c_{10} = 0$ and $c_9 \neq 0$, the coefficents of ϑ_2, ϑ_3 and ϑ_7 can be

disappered by setting $s_8 = \dfrac{k_1 c_2}{k_3 c_9}$, $s_7 = \dfrac{k_1 c_3}{k_3 c_9}$ and $s_2 = -\dfrac{c_7}{c_9}$ respectively. Scalling x, we assume $c_9 = 1$. Thus x can be to the case (5).
For $c_{13} = c_{12} = c_{11} = c_{10} = c_9 = 0$ and $c_8 \neq 0$, the coefficients of ϑ_7 and ϑ_8 can be disappered by setting $s_{13} = \dfrac{2c_7}{c_8}$ and $s_8 = -\dfrac{2c_{12}}{c_8}$.
respectively. Scalling x, we assume $c_8 = 1$. Thus x can be to the case (6).
For $c_{13} = c_{12} = c_{11} = c_{10} = c_9 = c_8 = c_7 = 0$ and $c_6 \neq 0$, the coefficients of ϑ_5 can be disappered by setting $s_{11} = -\dfrac{c_5}{c_6}$ respectively. Scalling x, we assume $c_6 = 1$. Thus x can be to the case (7).
For $c_{13} = c_{12} = c_{11} = c_{10} = c_9 = c_8 = c_7 = c_6 = c_5 = 0$ and $c_4 \neq 0$, the coefficents of ϑ_3 can be disappered by setting $s_6 = \dfrac{2k_2 c_2}{k_3 c_4}$ respectively. Scalling x, we assume $c_4 = 1$. Thus x can be to the case (8). For $c_{13} = c_{12} = c_{11} = c_{10} = c_9 = c_8 = c_7 = c_6 = c_5 = c_4 = 0$, and $c_3 \neq 0$, the coefficents of ϑ_2 can be disappered bysetting $s_6 = -\dfrac{c_2}{c_3}$ respectively. Scalling x, we assume $c_3 = 1$. Thus x can be to the case (9). ■

4. Reduction of Eq. (1)

First, symmetry reduction of Eq.(1) is classified, taking into account the subalgebras of Theorem 3. It is essential to look for a new form of Eq. (1) in special coordinates. In these new coordinates, reduction occurs. Independent variables p, q and r must be found for the infinitesimal generator to create these coordinates. Hence, the equation is expressed in novel coordinates through the chain rule reducing the system. Table 3 shows the similarity variables ξ_i, eta_i, w_i and h_i for 1D subalgebras in Theorem 3. Using each similarity variable, the reduced PDE of Eq.(1) is found (Table 4).

Table 3. The parameters of Lie invariants and the similarity solutions

H_i	ξ_i	η_i	w_i	u_i
ϑ_2	t	x	y	$h(\xi,\eta)$
ϑ_3	t	x	y	$e^{\frac{1}{4}\frac{\psi z^2}{t k_3}} h(\xi,\eta)$
ϑ_4	t	x	$\frac{k_3 y^2 + k_2 z^2}{k_2}$	$h(\xi,\eta)$
ϑ_4	t	x	z	$h(\xi,\eta)$
ϑ_5	t	x	z	$e^{-\frac{1}{4}\frac{\psi y^2}{k_2 t}} h(\xi,\eta)$
ϑ_5	t	y	z	$e^{-\frac{1}{4}\frac{\psi y^2}{k_2 t}} h(\xi,\eta)$
ϑ_5	t	y	z	$h(\xi,\eta)$
ϑ_5	t	y	$\frac{k_3 x^2 + k_1 z^2}{k_1}$	$h(\xi,\eta)$
ϑ_5	t	z	$\frac{k_2 x^2 + k_1 y^2}{k_1}$	$h(\xi,\eta)$
ϑ_5	x	y	z	$h(\xi,\eta)$
ϑ_{11}	$\frac{t}{x^2}$	$\frac{y}{x}$	$\frac{z}{x}$	$h(\xi,\eta)$
$\vartheta_1 + \vartheta_2$	t	x	y	$h(\xi,\eta)$

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Table 4. Reduced equation

1	$\psi h_\xi + k_1 h_{\eta\eta} + k_2 h_w + Bh = 0,$
2	$2\psi\xi h_\xi + 2k_1\xi h_{\eta\eta} + 2k_2\xi h_{ww} + \psi h + 2Bh = 0,$
3	$k_2\psi\xi h_\xi + k_1 h_{\eta\eta} + 4k_3 w h_{ww} + 4k_3 h w + Bh = 0,$
4	$\psi h_\xi + k_1 h_{\eta\eta} + k_3 h_{ww} + Bh = 0,$
5	$2\psi h_\xi + 2k_1\xi h_{\eta\eta} + \psi h + 2k_3\xi h_{ww} + 2B\xi h = 0,$
6	$2\psi\xi h_\xi + \psi h + 2k_2\xi h_{\eta\eta} + 2k_3\xi h_{ww} + 2B\xi h = 0,$
7	$\psi h_\xi + k_2 h_{\eta\eta} + k_3 h_{ww} + Bh = 0,$
8	$k_1\psi h_\xi + 4k_3 w h_{ww} + 4k_3 h_w + k_2 h_{\eta\eta} + Bh = 0,$
10	$k_1 h_{\xi\xi} + k_2 h_{\eta\eta} + k_3 h_{ww} + Bh = 0,$
11	$\psi h_\xi + k_1\eta\eta h_{\eta\eta} + 2k_1\eta w h_{\eta w} + 4k_1\eta\xi h_{\xi\eta} + 2k_1\eta h_\eta + k_1 w w h_{ww} + 4k_1 w h_{w\xi} + 2k_1 w h_w +$ $4k_1\xi\xi h_{\xi\xi} + 4k_1\xi h_\xi + k_2 h_{\eta\eta} + k_3 h_w = 0,$
12	$\psi h + k_1 h_{\eta\eta} + k_2 h_{ww} + k_3 h + Bh = 0.$

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