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Investigating the New Conservation Laws of Hunter-Saxton Equation Via Lie Symmetries

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Abstract. In this research, using the multiplier method and the 2-dimensional homotopy operator, higher order conservation laws for the Hunter-Saxton equation are computed. Also, in order to construct new conservation laws, the invariance properties of the multipliers are studied using Lie classical symmetries.

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1. Introduction

In the analysis of PDEs, conservation laws have a considerable function in examining the existence and uniqueness of solutions and their stability and extension [4, 19]. Generally, the principal laws in physics, which determine discrete quantities of an isolated system stay stable over time, called conservation laws. These laws are achieved in various ways, including: Noether's method, the multiplier method and the scaling method, etc [1, 4, 18, 19]. To obtain these local laws for the studied PDE, there exist some restrictions in using Noether's theorem. At first, this

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method limited to variational systems. On the other hand, to use this method, the system's Lagrangian must be self-adjoint.

The scaling method was first introduced by Hereman and Poole [19]. This method is based on scaling symmetries. Also, densities are linear combinations of these symmetries and systems that lack these symmetries, lack this kind of conservation laws [14]. Another method is the multiplier method. This method is compatible with the tools of variable calculus and linear algebra [2, 19]. In this technique, by finding a collection of non-singular local multipliers, new and higher order conservation laws are provided. In fact, due to a homotopy integral formula, a one-to-one correspondence arises among the multipliers and these laws. That is, each multiplier leads to the conserved flow using a homotopy formula. We pursue our aim as follows. By applying Lie symmetry generators on the multipliers, new multipliers are constructed for some PDEs, and eventually these multipliers induce other conservation laws.

In this paper, we deal with finding the conservation laws for the Hunter-Saxton(HS) equation. HS equation is a substantial nonlinear hyperbolic PDE in practical sciences. Undoubtedly, the history of this equation goes back to Hunter and Saxton for the theoretical modeling of nematic liquid crystals [8]. If the molecules in the fluid crystal are at first all arranged in a line, and some of them are then a little shaken, this perturbation in orientation will spread along the crystal, and the Hunter-Saxton equation explains some aspects of such orientation waves. The HS equation is

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2. \tag{1}$$

In recent years, many studies have been done on exact solutions, conservation laws and numerical methods to find the solutions of this equation. For example, the conservation laws of HS equation are obtained by using multipliers [22]. Also, it is proven that the HS equation has a notable property [9], namely it can be derived from two additional variational principals and one of them is the high-frequency limit of the variational principle for the Camassa-Holm equation, which is completely integrable [6]. On the other hand, it has infinitely many conserved quantities and a Lax pair [10]. The bi-variational property indicates that the HS equation is a completely integrable bi-Hamiltonian system and may be embedded in the Harry Dym hierarchy [9, 13]. Some explicit representations of the algebro-geometric solutions for the HS equation are obtained in [7] and continuous semigroup of weak and dissipative solutions are stated in [5]. Moreover, studies on symmetries and conservation laws of this equation have been done in [15, 20]. The novelty of this work is due to the fact that by using Lie symmetries, we have obtained new multipliers for HS equation, which has led to the discovery of new conservation laws for this equation.

This article is set as follows. Section 2, is dedicated to recalling the principal definitions and theorems that are helpful in the after sections. In Section 3, the conversation laws of the HS equation are calculated using the multiplier method. Section 4 deals with the application of Lie symmetries to generate new multipliers of the conversation laws.

2. Preliminaries

Assume $\mathbf{R}\{x, u^{(n)}\}$ represents a system of partial differential equations in which $u = (u^1, \cdots, u^q)$ is denoted the dependent variable and $x = (x^1, \cdots, x^p)$ is independent variable. Also, $u^{(n)}$ indicating n'th order partial derivative. Then, a conversation law is a divergence expression as

$$\mathrm{Div}P = \sum_{i=1}^{p} D_i P_i, \tag{2}$$

which is satisfied for all solutions of PDE. Here $P_i(x, u^{(r)})$ implied the fluxes for the conservation law of equation with the r-order derivative. In dynamical systems, for each $u = u(x,t)$ as the solution of the assumed system, (2) is equal to zero, namely P_i's must be unchanged for all solutions of the given PDE. Furthermore, the characteristic form of this notion is determined as

$$\mathrm{Div}P = Q.\ \mathbf{R}(x, t, u^{(n)}). \tag{3}$$

Here Q is called characteristic of conservation laws. The characteristics are unique (up to equivalence). In dynamic issues, the time variable t can be separated from other spatial variables $x = (x^1, \dots, x^p)$. Therefore, (3) is rewritten as

$$D_t \mathrm{T} + \mathrm{Div}\mathrm{X} = 0. \tag{4}$$

Here, T and X denote the conserved density and the corresponding flux, respectively. Also, D_t and Div are the total time derivative and the total divergence, respectively. Generally, if the equation can be written in evolutionary form, we will be allowed to use (4).

Definition 2.1 The total derivative operator is determined as

$$D_t = \frac{\partial}{\partial t} + \sum_{\alpha=1}^{q} \sum_{J} u^{\alpha}_{J,t} \frac{\partial}{\partial u^{\alpha}_{J}}, \qquad J = (j_1, ..., j_s), \quad 0 \leqslant j_s \leqslant p, \quad s \geqslant 0, \tag{5}$$

Where,

$$u^{\alpha}_{J,t} = \frac{\partial u^{\alpha}_{J}}{\partial t} = \frac{\partial^{s+1} u^{\alpha}}{\partial t \partial x^{j_1} \cdots \partial x^{j_s}}.$$

Definition 2.2 The zeroth-Euler operator for u^{α} is specified as

$$\mathbf{E}_{u^{\alpha}} = \partial / \partial u_{\alpha} + \sum_{s \geqslant 1} (-1)^s D_{j_1} \cdots D_{j_s} \partial / \partial u^{\alpha}_{j_1 \cdots j_s}, \qquad \alpha = 1, ..., q. \tag{6}$$

Definition 2.3 Let $\mathbf{x} = (x, t)$, also $g(\mathbf{x}, u^{(N)}(\mathbf{x}))$ and $G(\mathbf{x}, u^{(N-1)}(\mathbf{x}))$ are scalar and vector differentiable functions, respectively. If equality $g = \mathrm{Div}G$ holds, then g is called an exact (or divergence) function.

Theorem 2.1 The exactness of $g = g(\mathbf{x}, u^{(M)}(\mathbf{x}))$ must satisfy $\mathbf{E}_{u(x,t)} g = 0$.

3. Computing conservation laws of HS equation

In the first step, we assume a collection of non-singular local multipliers of a specific rank. Then we multiply these multipliers in the PDE. By applying the Euler operator on the equality of the previous step, and solving this system, we obtain local multipliers. Then, T and X of equation (4) are found using the inverse divergence operator. In fact, By applying the homotopy operator, inverting the divergence is reduced to a one-dimensional integration.

Theorem 3.1 Each of the local non-singular multipliers $\lambda(x, u^{(r)})$ represents a conservation law for system $\mathbf{R}(x, u^{(n)})$ whenever

$$E_{u^\alpha}(\lambda(x, u^{(r)}) \cdot \mathbf{R}(x, u^{(n)})) \equiv 0, \tag{7}$$

holds identically. In order to find all the multipliers in the form $\lambda(x, t, u, u_t, u_x, u_{tt}, u_{xx})$ of the conservation laws for PDEs system (1), the determining equations (7) are equal to

$$E_{u^\alpha}[\lambda(x, t, u, u_t, u_x, u_{tt}, u_{xx}) \cdot (u_{xt} + \frac{1}{2}u_x^2 + uu_{xx})] \equiv 0. \tag{8}$$

Solving (8) provides a collection of local multipliers for the nontrivial conservation laws of equation (1). we get,

$$\lambda = \frac{1}{2}\frac{1}{\big(\exp(-c_1/u_x)\big)^2}\big(2c_5c_6(-c_1+u_x)\exp(-c_1t) + 2x(c_1t+c_2)u_x + (c_1t^2+2c_4 \\ +2(c_2+2c_3)t)u_t + (2c_3u - 2xc_1)\exp(-c_1/u_x) + 2\exp(-c_1t)c_5c_7(c_1+u_x)\big).$$

By substituting c_i's for $1 \leqslant i \leqslant 7$ as the arbitrary constants, the local multipliers are obtained as follows,

$$\lambda_1 = \frac{1}{2}t^2u_t + xtu_x - x, \quad \lambda_2 = tu_t + xu_x, \quad \lambda_3 = tu_t + u, \quad \lambda_4 = u_t. \tag{9}$$

Each of λ_i induces a specified conservation law for HS equation. As a result, the characteristic form is equivalent to

$$D_t\mathrm{T} + D_x\mathrm{X} \equiv \lambda_i \cdot (u_{xt} + \frac{1}{2}u_x^2 + uu_{xx}).$$

To computing conserved quantities T and X, It sufficient to use an integration of a statement in multidimensional containing several functions and it's derivatives. That is cumbersome work. Instead, this problem can be decided using the tools of the homotopy operator (explicit formula)[11]. This operator was introduced as a powerful algorithmic tool in Volterra's works.

Definition 3.1 Vector $(\mathcal{H}^{(x)}_{u(x,t)}g, \mathcal{H}^{(t)}_{u(x,t)}g)$ is called a 2-dimensional homotopy operator and defined by

$$\mathcal{H}^{(z)}_{u(x,t)}g = \int_0^1 (\sum_{\alpha=1}^{q} \mathcal{I}^{(z)}_{u^\alpha(x,t)}g)[\lambda u]\frac{d\lambda}{\lambda}. \tag{10}$$

Here $z = x, t$ and $\mathcal{I}^{(x)}_{u^j(x,t)}g$ and $\mathcal{I}^{(t)}_{u^j(x,t)}g$ are respectively x-integrand and t-integrand that are determined by

$$\mathcal{I}^{(x)}_{u^j(x,t)}g=\sum_{k_1=1}^{M_1^j}\sum_{k_2=0}^{M_2^j}\left(\sum_{i_1=0}^{k_1-1}\sum_{i_2=0}^{k_2}(B^{(x)}u^j_{i_1xi_2t}(-D_x)^{k_1-i_1-1}(-D_t)^{k_2-i_2}\right)\frac{\partial g}{\partial u^j_{k_1xk_2t}}, \quad (11)$$

$$\mathcal{I}^{(t)}_{u^j(x,t)}g=\sum_{k_1=0}^{M_1^j}\sum_{k_2=1}^{M_2^j}\left(\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2-1}(B^{(t)}u^j_{i_1xi_2t}(-D_x)^{k_1-i_1}(-D_t)^{k_2-i_2-1}\right)\frac{\partial g}{\partial u^j_{k_1xk_2t}}.$$

Also M_1^j, M_2^j are respectively the order of g in u^j with respect to x and t, and combinatorial coefficient $B^{(x)}$ demonstrated as

$$B^{(x)}=B(i_1,i_2,k_1,k_2)=\frac{\binom{i_1+i_2}{i_1}\binom{k_1+k_2-i_1-i_2-1}{k_1-i_1-1}}{\binom{k_1+k_2}{k_1}}.$$

Similarly, $B^{(t)}=B(i_2,i_1,k_2,k_1)$ is defined in terms of cyclic permutations.

Theorem 3.2 Suppose g be an exact function, i.e., $g=\text{Div }G$ where $G=G(\mathbf{x},u^{(M-1)}(\mathbf{x}))$. Then, $G=\text{Div}^{-1}g=(\mathcal{H}^{(x)}_{u(x,t)}g,\mathcal{H}^{(t)}_{u(x,t)}g)$.

The results of applying (11) for (1) are stated in the following theorem.

Theorem 3.3 Conservation laws of the HS equation obtained as follows.
case1: $\lambda_1=\frac{1}{2}t^2u_t+xtu_x-x$
Therefore, the conserved vector results as:

$$\begin{aligned}\mathrm{T}=&\frac{1}{4}t^2uu_{xt}+\frac{1}{12}t^2u_x^2u+\frac{1}{6}t^2u^2u_{2x}+\frac{1}{8}t^2u_tu_x+\frac{1}{4}xtu_x^2-\frac{1}{2}xu_x-\frac{1}{8}t^2uu_{tx}\\&-\frac{1}{4}tuu_x-\frac{1}{4}xtuu_{2x}+\frac{1}{2}u,\\\mathrm{X}=&\frac{1}{3}t^2uu_xu_t+\frac{1}{2}xtuu_{xt}+\frac{1}{2}xtuu_x^2-\frac{5}{4}xuu_x+\frac{1}{8}t^2u_t^2+\frac{1}{4}xtu_xu_t-\frac{1}{2}xu_t-\frac{1}{4}tuu_t\\&-\frac{1}{8}t^2uu_{2t}-\frac{1}{3}tu^2u_x-\frac{1}{6}t^2uu_tu_x+\frac{1}{2}xuu_x+\frac{1}{2}u^2.\end{aligned}$$

case2: $\lambda_2=tu_t+xu_x$
Therefore, the following conserved vector is obtained:

$$\begin{aligned}\mathrm{T}=&\frac{1}{2}tuu_{xt}+\frac{1}{6}tuu_x^2+\frac{1}{3}tu^2u_{2x}+\frac{1}{4}tu_tu_x+\frac{1}{4}xu_x^2-\frac{1}{4}tuu_{tx}-\frac{1}{4}uu_x-\frac{1}{4}xuu_{2x},\\\mathrm{X}=&\frac{1}{2}xuu_x^2+\frac{1}{2}xuu_{xt}+\frac{1}{3}xu^2u_{2x}+\frac{1}{4}tu_t^2+\frac{1}{4}xu_tu_x-\frac{1}{4}uu_t-\frac{1}{4}tuu_{2t}-\frac{1}{3}tu^2u_{tx}\\&-\frac{1}{3}u^2u_x.\end{aligned}$$

case3: $\lambda_3=tu_t+u$
The conserved quantities are computed:

$$\begin{aligned}\mathrm{T}=&\frac{1}{4}tuu_{xt}+\frac{1}{6}tuu_x^2+\frac{1}{3}tu^2u_{2x}+\frac{1}{4}tu_tu_x+\frac{1}{4}uu_x-\frac{1}{4}uu_x,\\\mathrm{X}=&\frac{1}{3}tuu_xu_t+\frac{2}{3}u^2u_x-\frac{1}{2}uu_t-\frac{1}{4}tuu_{2t}+\frac{1}{4}tu_t^2+\frac{1}{4}uu_t-\frac{1}{3}tu^2u_{tx}-\frac{2}{3}u^2u_x.\end{aligned}$$

case4: $\lambda_4 = u_t$
Hence, we get,

$$\mathrm{T} = \frac{1}{4}uu_{xt} + \frac{1}{6}uu_x^2 + \frac{1}{3}u^2u_{2x} + \frac{1}{4}u_xu_t,$$
$$\mathrm{X} = -\frac{1}{4}uu_{tt} + \frac{1}{4}u_t^2 - \frac{1}{3}u^2u_{tx} + \frac{1}{3}uu_xu_t.$$

The conservation laws obtained in the above theorem are completely new. One reason for this is that the method of obtaining them is the multipliers method and so far this method has not been used for HS equation. The second reason is that these conservation laws are not equivalent to the conservation laws obtained so far for HS equation. For example, the difference of each of the above conservation laws with the conservation laws obtained in [15, 22] is not divergences, which indicates that they are not equivalent.

4. Application of lie symmetries to generate new multiplier

Lie symmetries are widely used in investigating solutions, invariant solutions, similarity reductions and also conservation laws of a PDE [3, 12, 16, 17, 21]. When a reversible transformation convert any PDE system to another PDE system, each conservation law of the first system transforms to new law of the other system. If each transformation is a symmetry of PDE system, then the associated conservation law is a conservation law of itself. Bluman, Temurchaolo, and Anco in [4] proposed two formulas. In continuation, instead of calculating the conservation laws, new multipliers are obtained with a special technique. In some systems, by applying Lie symmetries to the assumed multipliers, the another multipliers are obtained. Eventually these new multipliers induce other conservation laws.

Theorem 4.1 Assume a conservation law of PDE system can be written as $D_iP_i[u] = 0$. There exist functions $\{\Psi_i[W]\}_{i=1}^n$ with respect to point symmetries if these functions satisfy

$$J[W]D_iP_i[u] = D_i\Psi_i[W].$$

Explicitly, by substituting the i-th column of the Jacobian determinant

$$J[W] = \frac{D(x^1, \cdots, x^n)}{D(z^1, \cdots, z^n)},$$

with column$\{P_1[u], \cdots, P_n[u]\}$, $\Psi_i[W]$ will be obtained [4].

Corollary 4.1 Such that any reversible point transformation is a symmetry of the PDE system, then conservation law $D_iP_i[u] = 0$ provides another conservation law $D_i\Psi_i[u] = 0$.

Proposition 4.2 A collection of multipliers $\{\bar{\lambda}_\nu(x, u^{(r)})\}_{\nu=1}^k$ provides a set of new conservation law for the PDE system $\mathbf{R}_\nu(x, u^{(n)})$, if and only if this collection on all solutions $u(x)$ of PDE system is independent of $\{\lambda_\nu(x, u^{(r)})\}_{\nu=1}^k$.

Table 1. Symmetric analysis of multipliers with respect to generators $\mathbf{X}_1, \cdots, \mathbf{X}_7$.

$\mathbf{X}_i, \lambda_j$	λ_1	λ_2	λ_3	λ_4
$\mathbf{X}_1$	$tu_x - 1$	u_x	0	0
$\mathbf{X}_2$	λ_2	λ_4	λ_4	0
$\mathbf{X}_3$	xtu_x	xu_x	u	0
$\mathbf{X}_4$	$t^2\lambda_4$	$t\lambda_4$	$t\lambda_4 - u$	0
$\mathbf{X}_5$	$\frac{1}{2}t^3\lambda_4 + t^2xu_x - tx$	$txu_x + \frac{1}{2}t^2\lambda_4$	$\frac{1}{2}t^2\lambda_4 + x$	0
$\mathbf{X}_6$	$t^2u_x - t$	tu_x	1	0
$\mathbf{X}_7$	$\frac{1}{2}t^2(tu_x - 1)$	$\frac{1}{2}t^2u_x$	t	0

5. New conservation laws for the HS equation

The Lie point symmetries of equation (1) are calculated as follows [15],

$$
\begin{aligned}
&\mathbf{X}_1 = \partial_x, && \mathbf{X}_2 = \partial_t, && \mathbf{X}_3 = x\partial_x + u\partial_t,\\
&\mathbf{X}_4 = t\partial_t - u\partial_u, && \mathbf{X}_5 = tx\partial_x + \frac{1}{2}t^2\partial_t + x\partial_u, && \mathbf{X}_6 = t\partial_x + \partial_u,\\
&\mathbf{X}_7 = \frac{1}{2}t^2\partial_x + t\partial_u.
\end{aligned}
\tag{12}
$$

Symmetric analysis of multipliers with respect to generators $\mathbf{X}_1, \cdots, \mathbf{X}_7$ are shown in Table 1. According to the proposition (4.2) the action of the generator $\mathbf{X}_1, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5, \mathbf{X}_6$ and $\mathbf{X}_7$ generate new multipliers as

$$
\begin{aligned}
&Q_1 = tu_x - 1, && Q_2 = u_x, && Q_3 = xtu_x,\\
&Q_4 = xu_x, && Q_5 = u, && Q_6 = t^2\lambda_4,\\
&Q_7 = t\lambda_4, && Q_8 = t\lambda_4 - u, && Q_9 = \frac{1}{2}t^3\lambda_4 + t^2xu_x - tx,\\
&Q_{10} = txu_x + \frac{1}{2}t^2\lambda_4, && Q_{11} = tu_x, && Q_{12} = \frac{1}{2}t^2\lambda_4 + x,\\
&Q_{13} = t^2u_x - t, && Q_{14} = \frac{1}{2}t^2(tu_x - 1).
\end{aligned}
\tag{13}
$$

As we know, there are two types of trivial conservation laws. The first type is a law that is equal to zero on the equation, and the second type is a law that has total divergence. If the difference of two conservation laws is divergence, we call them equivalent [14]. Regarding the above conservation laws, it is clear that none of these multipliers are trivial and their two-by-two differences are also not total divergences. Therefore, non-trivial and non-equivalent conservation laws have been obtained As a result, these multipliers would induce new conservation laws for (1).

6. Conclusions

In this paper, the Hunter-Saxton equation, which is one of the most important equations in fluid mechanics, was investigated. This equation is a completely integrable equation and has infinitely many conservation laws. There are several methods to derive conservation laws. We used the multipliers method to obtain the density coefficients. By utilizing the Euler operator, determining equations for multipliers were computed. Also, by applying the 2-dimensional Homotopy formula, higher-order conservation laws of the Hunter-Saxton equation were obtained.

In continuation, by analyzing the action of point symmetries on the multipliers, new conservation laws were constructed for the Hunter-Saxton equation. These conservation laws are not trivial, nor are they equivalent. By comparing these laws with the previous conservation laws, it can be seen that the effect of Euler operator on the difference of these laws is non-zero. So they are not total divergence, and as a result, the obtained conservation laws are not equivalent and are therefore new.

References

[1] R. Abedian, The comparison of two high-order semi-discrete central schemes for solving hyperbolic conservation laws, International Journal of Mathematical Modelling & Computations, **7 (1)** (2017) 39–54.

[2] S. C. Anco and G. W. Bluman, Direct construction method for conservation laws of partial differential equations, Part II: General treatment, Eur. J. Appl. Math., **13** (2002) 567–585.

[3] Y. AryaNejad and R. Mirzavand, Conservation laws and invariant solutions of time-dependent Calogero-Bogoyavlenskii-Schiff equation, Journal of Linear and Topological Algebra, **11 (04)** (2022) 243–252.

[4] G. W. Bluman, New conservation laws obtained directly from symmetry action on a known conservation law, J. Math. Anal. Appl., **322** (2006) 233–250.

[5] A. Bressan and A. Constantin, Global solutions of the Hunter-Saxton equation, SIAM J. Math. Anal., **37 (3)** (2005) 996–1026.

[6] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett., **71 (11)** (1993) 1661–1664.

[7] Y. Hou, E. G. Fan and P. Zhao, Algebro-geometric solutions for the Hunter-Saxton hierarchy, Zeitschrift fur angewandte Mathematik und Physik, **65 (3)** (2014) 487–520.

[8] J. K. Hunter and R. Saxton, Dynamics of director fields, SIAM J. Appl. Math, **51** (1991) 1498–1521.

[9] J. K. Hunter and Y. X. Zheng, On a completely integrable nonlinear hyperbolic variational equation, Phys. D., **79 (2-4)** (1994) 361–386.

[10] R. I. Ivanov, Algebraic discretization of the Camassa-Holm and Hunter-Saxton equations, J. Nonlinear Math. Phys., **15 (2)** (2008) 1–12.

[11] M. Jafari, S. Mahdion, A. Akgül and S. M. Eldin, New conservation laws of the Boussinesq and generalized Kadomtsev-Petviashvili equations via homotopy operator, Results in Physics, **47** (2023) 106369.

[12] M. Jafari and A. Tanhaeivash, Symmetry group analysis and similarity reductions of the thin film equation, Journal of Linear and Topological Algebra, **10 (04)** (2021) 287–293.

[13] M. Jafari, A. Zaeim and S. Mahdion, Scaling symmetry and a new conservation law of the Harry Dym equation, Mathematics Interdisciplinary Research, **6 (2)** (2021) 151–158.

[14] M. Jafari, A. Zaeim and A. Tanhaeivash, Symmetry group analysis and conservation laws of the potential modified KdV equation using the scaling method, International Journal of Geometric Methods in Modern Physics, **19 (07)** (2022) 2250098.

[15] M. Nadjafikhah and F. Ahangari, Symmetry analysis and conservation laws for the Hunter-Saxton equation, Communications in Theoretical Physics, **59 (3)** (2013) 335.

[16] M. Nadjafikhah, M. Jafari, Computation of partially invariant solutions for the Einstein Walker manifolds identifying equations, Communications in Nonlinear Science and Numerical Simulation, **18 (12)** (2013) 3317–3324.

[17] M. Nadjafikhah and M. Jafari, Symmetry reduction of the twodimensional Ricci flow equation, Geometry, **2013 (1)** (2013) 373701.

[18] P. J. Olver, Applications of Lie Groups to Differential Equations, In: Graduate Texts in Mathematics, **107**, Springer, New York, (1993).

[19] D. Poole and W. Hereman, The homotopy operator method for symbolic integration by parts and inversion of divergences with applications, Appl. Anal., **87** (2010) 433–455.

[20] K. Tian and Q. P. Liu, Conservation laws and symmetries of Hunter-Saxton equation: revisited, Nonlinearity, **29 (3)** (2016) 737.

[21] H. Yazdani, M. Nadjafikhah and M. Toomanian, Apply new optimized MRA & invariant solutions on the generalized-FKPP equation, International Journal of Mathematical Modelling & Computations, **11 (4)** (2021) 1–12.

[22] Z. Zhao, Conservation laws and nonlocally related systems of the Hunter-Saxton equation for liquid crystal, Anal. Math. Phys., **9** (2019) 2311–2327.