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A numerical solution for 2D-nonlinear Fredholm integral equations based on Hybrid functions basis

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Abstract. This work considers a numerical method based on the 2D-hybrid block-pulse functions and normalized Bernstein polynomials to solve 2D-nonlinear Fredholm integral equations of the second type. These problems are reduced to a system of nonlinear algebraic equations and solved by Newtons iterative method along with the numerical integration and collocation methods. Also, the convergence theorem for this algorithm is proved. Finally, some numerical examples are given to show the effectiveness and simplicity of the proposed method.

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1. Introduction

Linear and nonlinear integral equations appear in most fields of science and engineering and they are of great importance due to their wide applications in different fields(see [1–4]). Many problems in mechanics including electromagnetic, oscillation theory, fluid dynamics, and mathematical physics are modeled by Fredholm integral equations. For more details, see [5–8] and references cited therein. A variety of powerful methods e.g. Chebyshev polynomials method [9], Sinc-collocation

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method [10–12], Legendre wavelet method [13], spectral collocation method [14], Bernoulli matrix method [15], Bernoulli polynomials method [16, 17], operational matrix method [18], Legendre polynomials method [19, 20], Adomian decomposition method [21, 22], Homotopy method [23], and hybrid functions [24–27] have been provided to solve different types of integral equations.
Block-pulse functions as a set of basis set of functions have been studied extensively for signal and function approximations. They have also have certain advantages for solving problems involving integrals and derivatives. In the literature, some efforts have been made in order to develop efficient methods based on these functions for solving integral equations. For instance, Maleknejad and Shahabi have introduced hybrid function operational matrices for solving 2D-nonlinear Volterra integral equations [28]. Mohammadi et al. [29], used the new direct method to solve bivariate nonlinear Fredholm integral equations using operational matrices with 2D-hybrid block-pulse functions and Chebyshev polynomials. The hybrid block-pulse and Legendre polynomial function have been also studied [30]. The use of hybrid functions with Bernstein polynomials in solving some Volterra and Fredholm integral equations was investigated by Behiry [31], Hessameddin and Shahbazi [32], and Mirzaei et al. [33]. Moreover, Bernstein polynomials play an important role in different fields (see [34, 35]). Some fundamental works on various aspects of Bernstein polynomials are done by Maleknejad et al. [36], Ramadan [37], and Mirzaei et al.[38, 39].
The general form of the 2D-nonlinear Fredholm integral equation is as follows

$$u(x,t) = f(x,t) + \int_0^1 \int_0^1 k(x,t,y,s,u(y,s))dyds, \qquad (x,t) \in [0,1) \times [0,1), \qquad (1)$$

where f and k are known analytic functions defined on $[0,1) \times [0,1)$ and $\Omega = [0,1)^4 \times \mathbb{R}$, respectively, and u is an unknown function that will be determined.
The purpose of this paper is to apply a new set of basic functions with the help of the hybrid 2D-block-pulse functions and normalized Bernstein polynomials to solve Eq. (1) approximately in a direct approach. The use of normalized Bernstein polynomials guarantees a high validity which leads to a proper approximate solution to this equation. Using the numerical integration and collocation method, Eq. (1) is transformed into a nonlinear set of algebraic equations which will be solved by Newtons iteration method.

This article is organized as follows:
The hybrid functions, approximation, and some of their properties are introduced and explained in Section 2. In Section 3, these functions will be applied to approximate the solution of 2D-nonlinear Fredholm integral equations. In Section 4, the convergence analysis of this technique is investigated. To demonstrate the accuracy and efficiency of the proposed method, three numerical examples are given. It is noteworthy that all calculations are done by the Mathematica software (Version 10.2).

2. Hybrid Block-pulse functions

In this section, some definitions, symbols and properties of Bernstein polynomials, and block-pulse functions are presented, briefly.

2.1 *Hybrid functions*

[26]

The Bernstein polynomials of degree M are defined as follows [40]

$$B_{j,M}(x) = \binom{M}{j} x^j (1-x)^{M-j}, \qquad x \in [0,1), \tag{2}$$

for $j = 0, 1, 2, \dots, M$.
A class of orthonormal polynomials is obtained from Bernstein polynomials by the Gram-Schmidt orthonormalization process on $B_{j,M}(x)$. These polynomials are called orthonormal Bernstein polynomials of degree M and denoted by $b_{j,M}(x)$, $0 \leqslant j \leqslant M$. For $M = 1$, 3, we have

$$\begin{aligned}
b_{0,1}(x) &= -\sqrt{3}(-1+x), & b_{1,1}(x) &= -1+3x,\\
b_{0,3}(x) &= -\sqrt{7}(-1+x)^3, & b_{1,3}(x) &= \sqrt{5}(-1+x)^2(-1+7x),\\
b_{2,3}(x) &= \sqrt{3}(1-13x+33x^2-21x^3), & b_{3,3}(x) &= -1+15x-45x^2+35x^3.
\end{aligned}$$

According to the above definitions, the hybrid functions $h_{i_1 j_1 i_2 j_2}$ are generally defined in $[0,1) \times [0,1)$ as

$$h_{i_1 j_1 i_2 j_2}(x,t) = \begin{cases} \sqrt{N(M+1)}\, b_{j_1,M}(Nx - i_1 + 1) b_{j_2,M}(Nt - i_2 + 1), & (x,t) \in [\frac{i_1-1}{N}, \frac{i_1}{N}) \times [\frac{i_2-1}{N}, \frac{i_2}{N}),\\ 0, & elsewhere, \end{cases} \tag{3}$$

for any $i_1, i_2 = 1, 2, \dots, N$ and $j_1, j_2 = 0, 1, \dots, M$.
It is clear that Eq. (3) is orthonormal and disjoint in $L^2([0,1) \times [0,1))$ with the following orthogonality condition

$$\int_0^1 \int_0^1 h_{i_1 j_1 i_2 j_2}(x,t) h_{i_3 j_3 i_4 j_4}(x,t) dx dt = \begin{cases} 1, & (i_1, i_2) = (i_3, i_4), (j_1, j_2) = (j_3, j_4),\\ 0, & elsewhere. \end{cases}$$

2.2 *Function approximation*

Let $X = L^2([0,1) \times [0,1))$ and

$$\{h_{1010}(x,t), h_{1011}(x,t), \dots, h_{101M}(x,t), h_{1020}(x,t), \dots, h_{102M}(x,t), \\ \dots, h_{NMN0}(x,t), \dots, h_{NMNM}(x,t)\},$$

be the set of hybrid functions that is defined by

$$X_{N,M+1} = span\{h_{i_1,j_1,i_2,j_2}(x,t) |\ i_1, i_2 = 1, 2, \dots, N,\ and\ j_1, j_2 = 0, 1, \dots, M\}.$$

If $u_{N,M+1} \in X_{N,M+1} \subset X$ be the best approximation of $u \in X$, then

$$u(x,t) \simeq u_{N,M+1}(x,t) = \sum_{i_1=1}^{N} \sum_{j_1=0}^{M} \sum_{i_2=1}^{N} \sum_{j_2=0}^{M} c_{i_1 j_1 i_2 j_2} h_{i_1 j_1 i_2 j_2}(x,t) = \mathbf{C}^T \mathbf{H}(x,t), \tag{4}$$

where

$$\mathbf{C} = [c_{1010}, c_{1011}, \ldots, c_{101M}, c_{1020}, \ldots, c_{102M}, \ldots, c_{NMN0}, \ldots, c_{NMNM}]^T,$$

and

$$\mathbf{H}(x,t) = [h_{1010}, h_{1011}, \ldots, h_{101M}, h_{1020}, \ldots, h_{102M}, \ldots, h_{NMN0}, \ldots, h_{NMNM}]^T.$$

Finally, the unknown hybrid coefficients $c_{i_1 j_1 i_2 j_2}$ are obtained by

$$c_{i_1 j_1 i_2 j_2} = \frac{(u(x,t), h_{i_1 j_1 i_2 j_2}(x,t))}{(h_{i_1 j_1 i_2 j_2}(x,t), h_{i_1 j_1 i_2 j_2}(x,t))}, \quad \begin{array}{l} i_1, i_2 = 1, 2, \ldots, N, \\ j_1, j_2 = 0, 1, \ldots, M. \end{array}$$

3. Outline of the solution method

Using the hybrid functions and also applying the collocation method along with numerical integration, a numerical method is introduced to solve the Eq. (1).
Let $u(x,t)$ be a solution to Eq. (1). Then, it is approximated by the proposed method as follows: Consider 2D-nonlinear Fredholm integral Eq. (1). $u(x,t) \in L^2(\Omega)$ is approximated as

$$u(x,t) \simeq \overline{u}_{N,M+1}(x,t) = \sum_{i_1=1}^{N} \sum_{j_1=0}^{M} \sum_{i_2=1}^{N} \sum_{j_2=0}^{M} u_{i_1 j_1 i_2 j_2} h_{i_1 j_1 i_2 j_2}(x,t) = \mathbf{U}^T \mathbf{H}(x,t), \quad (5)$$

where $\mathbf{U}^T$ is an unknown vector that must be determined.
To find u approximation in Eq. (1), we use Eq. (5). Then, we obtain

$$\bar{u}_{N,M+1}(x,t) = f(x,t) + \int_0^1 \int_0^1 k(x,t,y,s,\bar{u}_{N,M+1}(y,s))dyds, \quad (x,t) \in [0,1) \times [0,1).$$

Now, we define the set of collocation nodes $\{(x_i = ih, t_j = jh), for \quad i,j = 1, 2, \ldots, N(M+1)\}$, where $h = \frac{1}{N(M+1)}$ is a fixed step size for the collocation nodes. Then put

$$f(x_i, t_j) = \bar{u}_{N,M+1}(x_i, t_j) - \int_0^1 \int_0^1 k(x_i, t_j, y, s, \bar{u}_{N,M+1}(y,s))dyds. \quad (6)$$

Clearly, integral operator in Eq. (6) can be approximated by the Gauss-Legendre quadrature. Consider the following linear transformations:

$$y = \frac{1}{2}(\tau + 1), \qquad y \in [0,1],$$
$$s = \frac{1}{2}(\eta + 1), \qquad s \in [0,1].$$

Using these transformations, Eq. (6) will be transformed into the following nonlinear equations system:

$$f(x_i,t_j)=\bar{u}_{N,M+1}(x_i,t_j)$$
$$-\frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}k\left(x_i,t_j,\frac{1}{2}(\tau+1),\frac{1}{2}(\eta+1),\bar{u}_{N,M+1}(\frac{1}{2}(\tau+1),\frac{1}{2}(\eta+1))\right)d\tau d\eta,$$
$$i,j=1,2,\ldots,N(M+1).$$

Now, applying the Legendre-Gauss-Lobatto integration formula leads to the following nonlinear system which includes $(N(M+1))^2$ equations

$$f(x_i,t_j)=\bar{u}_{N,M+1}(x_i,t_j)$$
$$-\frac{1}{4}\sum_{n=1}^{r_1}\sum_{m=1}^{r_2}w_n w_m k\left(x_i,t_j,\frac{1}{2}(\tau_m+1),\frac{1}{2}(\eta_n+1),\bar{u}_{N,M+1}(\frac{1}{2}(\tau_m+1),\frac{1}{2}(\eta_n+1))\right),$$
$$i,j=1,2,\ldots,N(M+1), \tag{7}$$

where τ_m and η_n are the Legendre-Gauss points of degrees r_1 and r_2 in $[-1,1)$ respectively, and w_m and w_n are the weights.
This system can be solved by Newtons iteration method.

Remark 3.1 If k is a polynomial of the unknown function u, then Newton's iteration method is convergent.

4. Convergence analysis

In this section, at first, we assume that Newton's iteration method used for solving nonlinear system of Eq. (7), is convergent. Then, an accurate estimation of the proposed method is obtained.
Let

$$\chi_{i_1,i_2}(x,t)=\begin{cases}1, & (x,t)\in[\frac{i_1-1}{N},\frac{i_1}{N})\times[\frac{i_2-1}{N},\frac{i_2}{N}),\\ 0, & elsewhere,\end{cases}\qquad i_1,i_2=1,2,\ldots,N,$$

and $g_{i_1i_2}\in C^{M+1}([0,1]\times[0,1])$ is a known function. Put

$$g(x,t)=\sum_{i_1,i_2=1}^{N}g_{i_1i_2}(x,t)\chi_{i_1,i_2}(x,t). \tag{8}$$

Also, suppose that

$$Y_{i_1i_2}=span\{h_{i_10i_20}(x,t),h_{i_10i_21}(x,t),\ldots,h_{i_10i_2M}(x,t),h_{i_11i_20}(x,t),\ldots,$$
$$h_{i_11i_2M}(x,t),\ldots,h_{i_1Mi_20}(x,t),\ldots,h_{i_1Mi_2M}(x,t)\},\quad i_1,i_2=1,2,\ldots,N.$$

In the following, the convergence theorem for the approximation of g is presented according to [13].

Lemma 4.1 *[13], suppose that $g(x,t)$ be defined as (8). Also, let $\boldsymbol{C}^T_{i_1i_2}\boldsymbol{H}_{i_1i_2}(x,t)$ is the best approximation of $g_{i_1i_2}$ respect to $Y_{i_1i_2}$, and Eq. (4), where*

$$\boldsymbol{C}_{i_1i_2} = [c_{i_10i_20}, c_{i_10i_21}, \dots, c_{i_10i_2M}, c_{i_11i_20}, \dots, c_{i_11i_2M}, \dots, c_{i_1Mi_20}, \dots, c_{i_1Mi_2M}]^T,$$

and

$$\boldsymbol{H}_{i_1i_2}(x,t) = [h_{i_10i_20}(x,t), h_{i_10i_21}(x,t), \dots, h_{i_10i_2M}(x,t), h_{i_11i_20}(x,t), \dots,$$
$$h_{i_11i_2M}(x,t), \dots, h_{i_1Mi_20}(x,t), \dots, h_{i_1Mi_2M}(x,t)]^T, \qquad i_1, i_2 = 1, 2, \dots, N. \quad (9)$$

Then we have

$$\|g(x,t) - \boldsymbol{C}^T\boldsymbol{H}(x,t)\|_2 \leqslant \frac{2^{M+1}\gamma}{N^{M+1}(M+1)!},$$

where $\gamma = \max_{x,t\in[0,1),0\le k\le M+1}\left|\frac{\partial^{M+1}g(x,t)}{\partial x^k \partial t^{M+1-k}}\right|$.

Proof See [30]. ■

Theorem 4.2 *Let $k \in C^1(\overline{\Omega})$, with $C_0 = \sup_\Omega \left|\frac{\partial k}{\partial \xi}(x,t,y,s,\xi)\right| < \infty$. Also, let $u \in C^M([0,1]\times[0,1])$ for $M > 2$ and $\bar{u}_{N,M+1}(x,t)$ be the exact and approximation solution of Eq. (1), respectively. If the matrix $\boldsymbol{A} = (A_{i_1i_2})_{N\times N}$ is a nonsingular matrix such that $A_{i_1i_2}$ similar to $H_{i_1i_2}$ in Eq. (9) is defined*

$$A_{i_1j_1i_2j_2} = h_{i_1j_1i_2j_2}(x_i,t_j) - \int_0^1\int_0^1 \frac{\partial k}{\partial u}(x_i,t_j,y,s,\xi(y,s))h_{i_1j_1i_2j_2}(y,s)dyds, \quad (10)$$

for any collocating points $(x_i = \frac{i}{N(M+1)}, t_j = \frac{j}{N(M+1)})(i,j = 1,2,\dots,N(M+1))$, and
$\xi \in (\min(u_{N,M+1}, \bar{u}_{N,M+1}), \max(u_{N,M+1}, \bar{u}_{N,M+1}))$. Then

$$\|u(x,t) - \bar{u}_{N,M+1}(x,t)\|_2 \leqslant \frac{1}{(M+1)!}(\frac{2}{N})^{M+1}\left(\gamma + N^{\frac{3}{2}}(M+1)^{\frac{3}{2}}C_1\|\boldsymbol{A}^{-1}\|_2\right).$$

Proof Consider Eq. (5), and let

$$u(x,t) \simeq \bar{u}_{N,M+1}(x,t) = \mathbf{U}^T\mathbf{H}(x,t), \quad (11)$$

be an approximation of $u(x,t)$ where unknown vector $\mathbf{U}^T$ is the solution of non-linear system (7).
Substituting Eq. (11) in Eq. (1), gives

$$f(x,t) = \bar{u}_{N,M+1}(x,t) - \int_0^1\int_0^1 k(x,t,y,s,\bar{u}_{N,M+1}(y,s))dyds. \quad (12)$$

Also, substituting Eq. (4) in Eq. (1), gives

$$\hat{f}(x,t) = u_{N,M+1}(x,t) - \int_0^1\int_0^1 k(x,t,y,s,u_{N,M+1}(y,s))dyds. \quad (13)$$

Now, subtracting (13) from (12) gives

$$\hat{f}(x,t) - f(x,t) = u_{N,M+1}(x,t) - \bar{u}_{N,M+1}(x,t) - \left(\int_0^1 \int_0^1 (k(x,t,y,s,u_{N,M+1}(y,s)) - k(x,t,y,s,\bar{u}_{N,M+1}(y,s)))dyds \right).$$

Which turns into the following equation by the mean value theorem

$$\hat{f}(x,t) - f(x,t) = u_{N,M+1}(x,t) - \bar{u}_{N,M+1}(x,t) - \int_0^1 \int_0^1 \frac{\partial k}{\partial u}(x,t,y,s,\xi)(u_{N,M+1}(y,s) - \bar{u}_{N,M+1}(y,s))dyds,$$

where $\xi \in (\min(u_{N,M+1}, \bar{u}_{N,M+1}), \max(u_{N,M+1}, \bar{u}_{N,M+1}))$. By using $(N(M+1))^2$ collocating points $(x_i = \frac{i}{N(M+1)}, t_j = \frac{j}{N(M+1)})(i,j = 1,2,\ldots,N(M+1))$, gives

$$\hat{f}(x_i,t_j) - f(x_i,t_j) = \sum_{i_1=1}^{N} \sum_{j_1=0}^{M} \sum_{i_2=1}^{N} \sum_{j_2=0}^{M} (c_{i_1j_1i_2j_2} - u_{i_1j_1i_2j_2}) \left(h_{i_1j_1i_2j_2}(x_i,t_j) - \int_0^1 \int_0^1 \frac{\partial k}{\partial u}(x_i,t_j,y,s,\xi) h_{i_1j_1i_2j_2}(y,s)dyds \right). \tag{14}$$

So, we can write

$$\hat{\mathbf{F}} - \mathbf{F} = \mathbf{A}(\mathbf{C} - \mathbf{U}), \tag{15}$$

where $\mathbf{A}$ defined in Eq. (10) and

$$\mathbf{F} = [f(x_i,t_j)]_{i,j=1}^{N(M+1)}, \qquad \hat{\mathbf{F}} = \left[\hat{f}(x_i,t_j)\right]_{i,j=1}^{N(M+1)},$$
$$\mathbf{C} = [c_{1010}, c_{1011}, \ldots, c_{101M}, c_{1020}, \ldots, c_{102M}, \ldots, c_{NMN0}, \ldots, c_{NMNM}]^T,$$
$$\mathbf{U} = [u_{1010}, u_{1011}, \ldots, u_{101M}, u_{1020}, \ldots, u_{102M}, \ldots, u_{NMN0}, \ldots, u_{NMNM}]^T.$$

From (15), the following bound is obtained for $\|\mathbf{C} - \mathbf{U}\|_2$

$$\|\mathbf{C} - \mathbf{U}\|_2 \leqslant \|\mathbf{A}^{-1}\|_2 \|\hat{\mathbf{F}} - \mathbf{F}\|_2. \tag{16}$$

Additionally

$$\|\hat{\mathbf{F}} - \mathbf{F}\|_2^2 = \sum_{i,j=1}^{N(M+1)} |\hat{f}(x_i,t_j) - f(x_i,t_j)|^2 \leqslant (N(M+1))^2 \|\hat{f}(x,t) - f(x,t)\|_2^2. \tag{17}$$

Also, we have

$$\hat{f}(x,t) = f(x,t) + (u_{N,M+1}(x,t) - u(x,t)) - \left(\int_0^1 \int_0^1 (k(x,t,y,s,u_{N,M+1}(y,s))dyds - \int_0^1 \int_0^1 (k(x,t,y,s,u(y,s))dyds \right).$$

Now, using the triangle inequality, Lemma 4.1, also the mean value theorem, leads to the following inequality

$$\|\hat{f}(x,t)-f(x,t)\|_2 \leqslant \|u_{N,M+1}(x,t)-u(x,t)\|_2 + C_0\|u_{N,M+1}(x,t)-u(x,t)\|_2$$
$$\leqslant (1+C_0)\frac{2^{M+1}\gamma}{N^{M+1}(M+1)!},$$

where $C_0 = \sup_\Omega |\frac{\partial k}{\partial \xi}(x,t,y,s,\xi)| < \infty$. Consequently

$$\|\hat{\mathbf{F}}-\mathbf{F}\|_2 \leqslant N(M+1)\|\hat{f}(x,t)-f(x,t)\|_2 = C_1\frac{2^{M+1}}{N^M M!}. \tag{18}$$

The following ralation is obtained if we substituting Eq. (18) into Eq. (16)

$$\|\mathbf{C}-\mathbf{U}\|_2 \leqslant C_1\|\mathbf{A}^{-1}\|_2\frac{2^{M+1}}{N^M M!}. \tag{19}$$

According to triangle inequality, we can write

$$\|u(x,t)-\bar{u}_{N,M+1}(x,t)\|_2 \leqslant \|u(x,t)-u_{N,M+1}(x,t)\|_2+\|u_{N,M+1}(x,t)-\bar{u}_{N,M+1}(x,t)\|_2, \tag{20}$$

for the second phrase we have

$$\begin{aligned}
\|u_{N,M+1}(x,t)-\bar{u}_{N,M+1}(x,t)\|_2^2 &= \int_0^1\int_0^1 |u_{N,M+1}(x,t)-\bar{u}_{N,M+1}(x,t)|^2 dxdt\\
&\leqslant \int_0^1\int_0^1 \left|\sum_{i_1,i_2=1}^{N}\sum_{j_1,j_2=0}^{M}(c_{i_1j_1i_2j_2}-u_{i_1j_1i_2j_2})h_{i_1j_1i_2j_2}(x,t)\right|^2 dxdt\\
&\leqslant \sum_{i_1,i_2=1}^{N}\sum_{j_1,j_2=0}^{M}|c_{i_1j_1i_2j_2}-u_{i_1j_1i_2j_2}|^2\sum_{i_1,i_2=1}^{N}\sum_{j_1,j_2=0}^{M}\int_0^1\int_0^1|h_{i_1j_1i_2j_2}(x,t)|^2dxdt\\
&\leqslant \|\mathbf{C}-\mathbf{U}\|_2^2 N(M+1).
\end{aligned}$$

Now, using Eq. (19) gives

$$\|u_{N,M+1}(x,t)-\bar{u}_{N,M+1}(x,t)\|_2 \leqslant C_1\|\mathbf{A}^{-1}\|_2\frac{2^{M+1}\sqrt{M+1}}{N^{M-\frac{1}{2}}M!}. \tag{21}$$

Finally, we find an upper bound of L^2-Norm error from Lemma 4.1, Eqs. (21) and (20) as below

$$\|u(x,t)-\bar{u}_{N,M+1}(x,t)\|_2 \leqslant \frac{1}{(M+1)!}(\frac{2}{N})^{M+1}\left(\gamma + N^{\frac{3}{2}}(M+1)^{\frac{3}{2}}C_1\|\mathbf{A}^{-1}\|_2\right),$$

and the proof is completed. ■

5. Numerical results

In this section, in order to illustrate the accuracy of the introduced method some numerical examples are presented. Absolute errors are calculated at arbitrary points $(x,t) = (\frac{1}{2^r}, \frac{1}{2^r}), r = 1,2,\ldots,6$. All calculations have been run in Mathematica software. For the analysis of error of this algorithm, the Chebyshev and L^2 norms taken over $[0,1]^2$ are used for the error function $e_{N,M+1}(x,t)$ defined as follows

$$e_{N,M+1}(x,t) = |u(x,t) - \bar{u}_{N,M+1}(x,t)|, \qquad (x,t) \in [0,1)^2, and \quad N, M \in \mathbb{N},$$

where $u(x,t)$ is the exact solution and $\bar{u}_{N,M+1}(x,t)$, is the approximate solution of the Eq. (1).

Example 5.1 Consider the following 2D-nonlinear Fredholm integral equation

$$u(x,t) = f(x,t) + \int_0^1 \int_0^1 \sin(x+t)(u(y,s)+1)dyds, \quad (x,t) \in [0,1) \times [0,1),$$

where

$$f(x,t) = x\cos(t) - \frac{1}{2}\sin(x+t)(2+\sin(1)).$$

The exact solution of this equation is $u(x,t) = x\cos(y)$.
After solving the nonlinear system (7), the Chebyshev and L^2 norms for the error function $e_{N,M+1}(x,t)$ have been computed and shown in Table (1). The comparison

Table 1. The absolute values of the error function $e_{N,M+1}(x,t)$ and its Chebyshev and L^2 norms with $N = 2$ and $M = 1,3,5,7$ for Example (5.1).

$(x,t) = (\frac{1}{2^r}, \frac{1}{2^r})$	$N=2, M=1$	$N=2, M=3$	$N=2, M=5$	$N=2, M=7$
$r=1$	$0.12605E(-1)$	$0.24973E(-4)$	$0.27688E(-7)$	$0.17014E(-10)$
$r=2$	$0.95386E(-4)$	$0.28906E(-5)$	$0.85593E(-9)$	$0.33925E(-13)$
$r=3$	$0.92576E(-3)$	$0.34579E(-6)$	$0.23751E(-9)$	$0.72032E(-13)$
$r=4$	$0.10796E(-2)$	$0.40124E(-6)$	$0.57474E(-9)$	$0.19176E(-12)$
$r=5$	$0.75086E(-3)$	$0.90208E(-6)$	$0.21975E(-9)$	$0.33905E(-14)$
$r=6$	$0.44519E(-3)$	$0.73516E(-6)$	$0.48015E(-9)$	$0.20263E(-12)$
$\|e_{N,M+1}(x,t)\|_2$	$0.62857E(-2)$	$0.97485E(-5)$	$0.72765E(-8)$	$0.38016E(-11)$
$\|e_{N,M+1}(x,t)\|_\infty$	$0.24963E(-1)$	$0.63201E(-4)$	$0.57183E(-7)$	$0.34506E(-10)$

of maximum absolute errors of the proposed method and the method proposed in [30] is given in Table (2).
The results show that an approximate solution with more accuracy is obtained by the current method.

Table 2. Comparison of maximum absolute errors for Example (5.1).

$N = 2$ and $M = 3,5,7$	Hybrid Block-Pulse Legendre Method [30]	Present Method
$N=2, M=3$	5.43×10^{-6}	6.32×10^{-5}
$N=2, M=5$	3.14×10^{-7}	5.72×10^{-8}
$N=2, M=7$	7.22×10^{-10}	3.46×10^{-11}

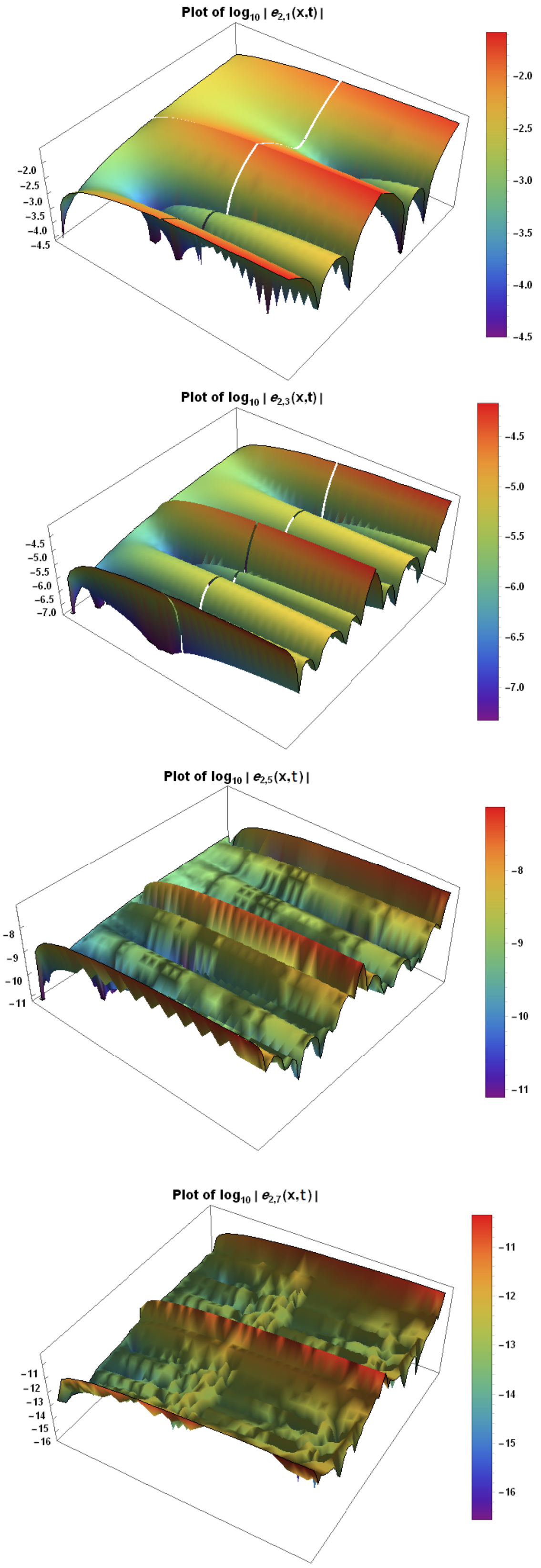

Figure 1. Plots of the Logarithm absolute error function for Example (5.1).

Example 5.2 Consider the following 2D-nonlinear Fredholm integral equation

$$u(x,t) = f(x,t) + \int_0^1 \int_0^1 \frac{x}{1+t}(1+y+s)u^2(y,s)dyds, \quad (x,t) \in [0,1) \times [0,1),$$

where

$$f(x,t) = \frac{1}{(1+x+t)^2} - \frac{x}{6+6t}.$$

The exact solution is $u(x,t) = \frac{1}{(1+x+t)^2}$. Put $N = 2$. Numerical results and the Chebyshev and L^2 norms for the error function $e_{N,M+1}(x,t)$ for various values of $M = 1, 3, 5$, have been computed and presented in Table (3). Also Figure (2) shows the $\bar{u}(x,t)$.

Example 5.3 Let $f(x,t) = xt - \frac{1}{20}(2x+t)$ and

$$u(x,t) = f(x,t) + \int_0^1 \int_0^1 (2xy + st)u^3(y,s)dyds, \quad (x,t) \in [0,1) \times [0,1),$$

with the exact solution $u(x,t) = xt$. Applying the proposed method different values of M and $N = 2$, leads to the following results.

6. Conclusion

In this paper, 2D- nonlinear Fredholm integral equations of the second kind have been solved by the 2D- hybrid block-pulse functions and normalized Bernstein polynomials. These equations are reduced into a nonlinear system of algebraic equations using numerical integration and collocation method. One of the advantages of this technique is less that error than other methods. In addition, the convergence analysis of the proposed scheme has been presented.

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Table 3. The error function $e_{N,M+1}(x,t)$ with $M = 1, 3, 5$ and its Chebyshev and L^2 norms for Example (5.2).

$(x,t) = (\frac{1}{2^r}, \frac{1}{2^r})$	$N = 2, M = 1$	$N = 2, M = 3$	$N = 2, M = 5$
$r = 1$	$0.14126E(-1)$	$0.14394E(-3)$	$0.14466E(-5)$
$r = 2$	$0.18772E(-1)$	$0.13091E(-3)$	$0.60706E(-6)$
$r = 3$	$0.41893E(-3)$	$0.3815E(-3)$	$0.32831E(-7)$
$r = 4$	$0.46417E(-1)$	$0.3076E(-5)$	$0.10684E(-4)$
$r = 5$	$0.86791E(-1)$	$0.16173E(-2)$	$0.16602E(-4)$
$r = 6$	0.11271	$0.32083E(-2)$	$0.68165E(-4)$
$\|e_{N,M+1}(x,t)\|_2$	$0.11949E(-1)$	$0.22414E(-3)$	$0.44646E(-5)$
$\|e_{N,M+1}(x,t)\|_\infty$	0.13292	$0.47293E(-3)$	$0.53282E(-5)$

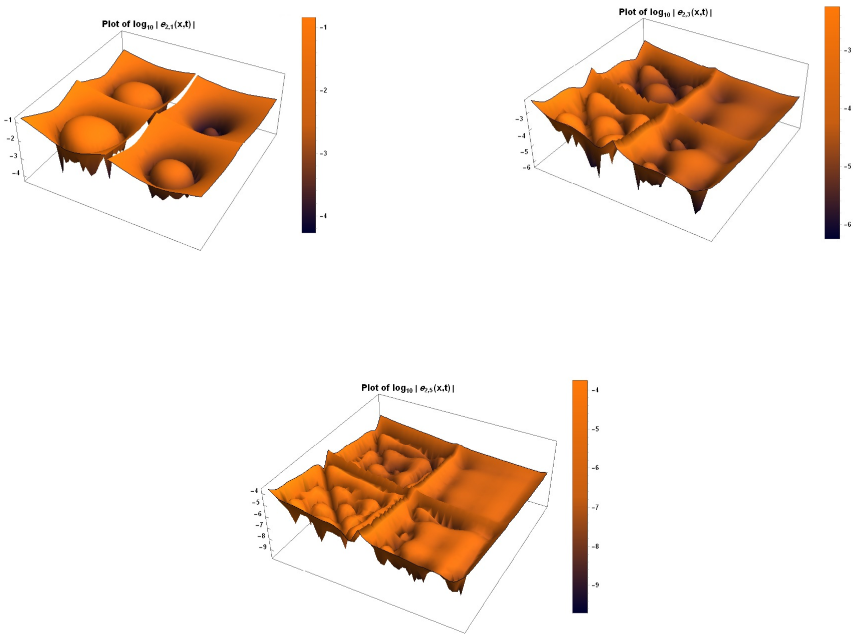

Figure 2. Plots of the Logarithm absolute error function for Example (5.2).

Table 4. Comparison of maximum absolute errors for Example (5.2).

$N = 2$ and $M = 1, 3, 5$	Hybrid Block-Pulse Legendre Method [30]	Present Method
$N = 2, M = 1$	4.58×10^{-2}	1.33×10^{-1}
$N = 2, M = 3$	1.57×10^{-3}	4.73×10^{-4}
$N = 2, M = 5$	5.13×10^{-5}	5.32×10^{-6}

Table 5. Chebyshev and L^2 norms for Example (5.3) for $N = 2$ and $M = 1, 2$.

$(x,t) = (\frac{1}{2^r}, \frac{1}{2^r})$	$N = 2, M = 1$	$N = 2, M = 3$
$r = 1$	$0.15543E(-14)$	$0.66613E(-15)$
$r = 2$	$0.89512E(-15)$	$0.27756E(-16)$
$r = 3$	$0.48225E(-15)$	$0.10408E(-16)$
$r = 4$	$0.22855E(-15)$	$0.69389E(-17)$
$r = 5$	$0.10202E(-15)$	0
$r = 6$	$0.48572E(-16)$	$0.69389E(-17)$
$\|e_{N,M+1}(x,t)\|_2$	$0.26246E(-14)$	$0.11917E(-15)$
$\|e_{N,M+1}(x,t)\|_\infty$	$0.15210E(-13)$	$0.11657E(-14)$

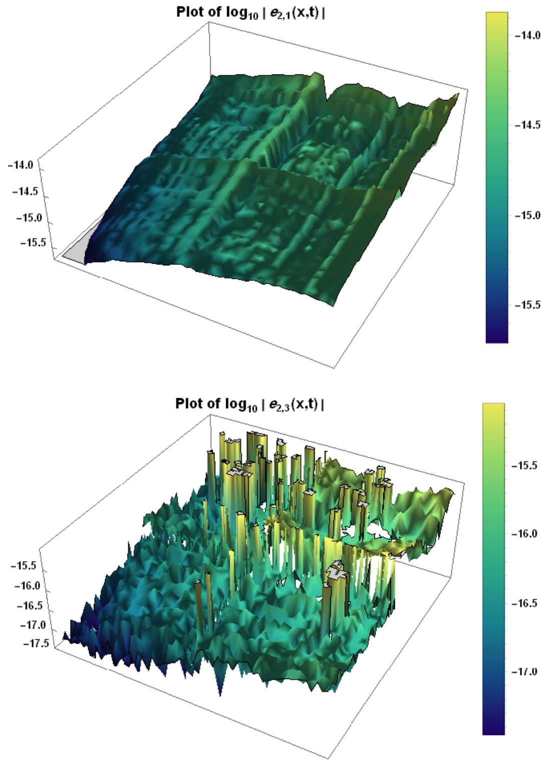

Figure 3. Logaritmic absolute error functions $log_{10}|e_{2,1}(x,t)|$ and $log_{10}|e_{2,3}(x,t)|$ in unit square domain for Example (5.3).

Table 6. Comparison of maximum absolute errors for Example (5.3).

$N = 2$ and $M = 1, 3$	Hybrid Block-Pulse Legendre Method [30]	Present Method
$N = 2, M = 1$	4.12×10^{-12}	1.52×10^{-13}
$N = 2, M = 3$	1.09×10^{-12}	1.17×10^{-14}