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Partial Pseudo-Triangular Entropy of Uncertain Random Variables with Application to Portfolio Risk Management

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Abstract. In this paper, the concept of partial pseudo-triangular entropy as a superior measure of indeterminacy for uncertain random variables is proposed. It is first proved that partial entropy and partial triangular entropy sometimes fail to measure the indeterminacy of an uncertain random variable. Then, the concept of partial pseudo-triangular entropy and its mathematical properties are investigated. To illustrate the outperformance of partial pseudo-triangular entropy as a measure of risk, a portfolio optimization problem is optimized via different types of entropy. Furthermore, a genetic algorithm (GA) is implemented in MATLAB to solve the corresponding problem. Numerical results show that partial pseudo-triangular entropy as a quantifier of portfolio risk outperforms partial entropy and partial triangular entropy in the uncertain random portfolio optimization problem.

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Index to information contained in this paper

1. Introduction

Entropy is a quantitative measurement of indeterminacy associated with a variable. Entropy of random variables was first proposed in logarithm form (Shannon [28]). A pioneer research carried out by scholars to associate entropy with a measure of risk in portfolio optimization showed that entropy is more common and better suited in portfolio optimization than variance (Philippatos and Wilson [26]). Moreover, researches showed that entropy as a measure of risk is better than variance in wealth allocation and by using entropy instead of variance in the portfolio optimization problem, all major difficulties with

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Markowitz's mean-variance portfolio optimization model can be eliminated (simonelli [31] and Mercurio et al. [25]).

In the mentioned literatures, the indeterminacy is considered under probability theory. The key assumption for using probability theory is that the probability distribution of historical data is similar to the past one and close enough to the frequency. Nevertheless, it is difficult to achieve this assumption generally. As an approach to deal with problems associated with non-random phenomena, fuzzy set theory was proposed (Zadeh [34]). As an improvement, Liu and Liu [21] presented a self-dual credibility measure for fuzzy events. Further researches by Liu [17] confirmed that using fuzzy set theory or subjective probability to model human uncertainty may lead to inaccurate results.

To better deal with non-random phenomena and in particular human uncertainty, Liu [14] founded uncertainty theory. Then, Liu [15] introduced entropy of uncertain variables in logarithm form. Since then, several scholars have been investigating entropy under uncertainty theory. Chen et al. [6] proposed the concept of cross-entropy to measure the divergence degree of uncertain variables and presented the minimum cross-entropy principle. Chen and Dai [5] proposed the maximum entropy principle for uncertain variables. Moreover, Dai and Chen [8] presented a formula to calculate the entropy of uncertain variables. As a supplement of logarithm entropy, several types of entropy for uncertain variables have been investigated by scholars (Tang and Gao [32], Yao et al. [33], Dai [9] and Abtahi et al. [1]).

In numerous situations, uncertainty and randomness may appear together in phenomena. In these situations, the concept of uncertain random variable and chance theory are used for modeling such phenomena. In order to describe such phenomena, Liu [19] proposed uncertain random variables. Liu [20] also discussed the concepts of chance distribution, expected value and variance of uncertain random variables. Then, Liu and Ralescu [22] proposed the risk index for uncertain random variables and established a formula for calculating this index. Guo and Wang [10] presented a formula to obtain variance of uncertain random variables. Liu and Ralescu [23] presented the concept of value at risk for uncertain random variables and Qin et al. [27] optimized portfolio selection problems of uncertain random returns based on value at risk models. Liu et al. [24] proposed the concept of tail value at risk for uncertain random variables and applied it to series systems, parallel systems, k-out-of-n systems, standby systems and structural system. Li et al. [13] proved some mathematical properties of tail value at risk for uncertain random variables and formulated several mean-TVaR models for hybrid portfolio optimization models.

Entropy of uncertain random variables was first proposed in logarithm form by Sheng et al. [30]. Then, Ahmadzade et al. [3] introduced a definition of partial entropy for uncertain random variables to measures how much the entropy of an uncertain random variable belongs to the uncertain variable and derived several properties. Further applications of entropy for uncertain random variables in portfolio optimization problems have been investigated by several scholars (Ahmadzadeh et al. [4], Chen et al. [7] and He et al. [11]).

As it mentioned, partial entropy and partial triangular entropy sometimes fail to measure the indeterminacy of an uncertain random variable. Therefore, in order to address this problem, in this paper, the concept of partial pseudo-triangular entropy for uncertain random variables is proposed. As an application, partial pseudo-triangular entropy as a measure of risk is applied in a portfolio optimization problem. The rest of this paper is organized as follows. In Section 2, some concepts of uncertainty theory and chance theory are reviewed. In Section 3, the concept of partial pseudo-triangular entropy for uncertain random variables together with its mathematical properties are proposed. Then in Section 4, a portfolio optimization problem based on different types of entropy are optimized via a mean–entropy model. Finally, conclusions are given in Section 5.

2. Preliminaries

This section comes with reviewing some concepts of uncertainty theory and chance theory including definition of uncertain variable, uncertainty distribution, uncertain random variable and chance distribution.

2.1. *Uncertainty theory*

Uncertainty theory was founded by Liu in 2007 to model human uncertainty. In lack of historical data, we should request experts to evaluate the degree of belief for the occurrence of an event. In this section, some necessary definitions and theorems in uncertainty theory are reviewed.
Assume that Γ is a nonempty set and $\mathcal{L}$ is a σ- algebra over Γ. Then, $(\Gamma,\mathcal{L})$ is named a measurable space. Each element Λ in $\mathcal{L}$ is called a measurable set. A measurable set can be considered as an event in uncertainty theory. That is, a number $\mathcal{M}\{\Lambda\}$ will be assigned to each event Λ to indicate the belief degree with which we believe Λ will happen. In order to deal with belief degree, the following axioms are suggested (Liu [14]):

Axiom 1 (Normality) For the universal set Γ, $\mathcal{M}\{\Gamma\}=1$.

Axiom 2 (Duality) For any event , $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}=1$.

Axiom 3 (Subadditivity) For every countable sequence of events $\Lambda_1,\Lambda_2,\dots$, we have

$$\mathcal{M}\{\bigcup_{i=1}^{\infty}\Lambda_i\}\le\sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

Then, $(\Gamma,\mathcal{L},\mathcal{M})$ is called an uncertainty space.

Axiom 4 (Product) Let $(\Gamma_i,\mathcal{L}_i,\mathcal{M}_i)$ be uncertainty spaces for $i=1,2,\dots$. Then, the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\{\prod_{i=1}^{\infty}\Lambda_i\}=\bigwedge_{i=1}^{\infty}\mathcal{M}_i\{\Lambda_i\},$$

where $\Lambda_i$are arbitrarily chosen events from $\mathcal{L}_i$ for $i=1,2,\dots$, respectively.

Definition 2.1 (Liu [14]) An uncertain variable is a function τ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\tau \in B\}$ is an event for any Borel set B of real numbers.

Definition 2.2 (Liu [16]) For any Borel sets $\mathrm{B}_1, \mathrm{B}_2, \dots, \mathrm{B}_n$ of real numbers, the uncertain variables $\tau_1, \tau_2, \dots, \tau_n$ are independent if

$$\mathcal{M}\{\bigcap_{i=1}^{n}(\tau_i \in B_i)\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\tau_i \in B_i\}.$$

Definition 2.3 (Liu [16]) An uncertain variable τ is called normal denoted by $N(m, \delta)$ if it has a normal uncertainty distribution

$$\gamma(x) = \left(1 + exp\left(\frac{\pi(m-x)}{\sqrt{3}\delta}\right)\right)^{-1}; \qquad x \in \mathbb{R}$$

where $m \in \mathbb{R}$ and δ $(\delta > 0)$.

Example 2.1 (Liu [16]) Let $\tau \sim N(m, \delta)$, then the inverse uncertainty distribution of normal uncertain variable τ is

$$\gamma^{-1}(r) = m + \frac{\sqrt{3}\delta}{\pi} \, ln\left(\frac{r}{1-r}\right); \qquad 0 < r < 1.$$

Theorem 2.1 (Liu [16]) Let τ be an uncertain variable with regular uncertainty distribution $\gamma(x)$. If the expected value of τ exists, then

$$\mathrm{E}[\tau] = \int_0^1 \gamma^{-1}(r)\, \mathrm{d}r,$$

where $\gamma^{-1}(r)$ is the inverse uncertainty function of τ with respect to r.

Definition 2.4 (Liu [15]) Let τ be an uncertain variable with uncertainty distribution $\gamma(x)$. Then, the logarithm entropy of uncertain variable τ is

$$\mathrm{H}[\tau] = \int_{-\infty}^{+\infty} \mathrm{L}(\gamma(x))\, \mathrm{d}x,$$

where $\mathrm{L}(s) = -(s)ln(s) - (1-s)\, ln(1-s)$.

Theorem 2.2 (Dai and Chen [8]) Let τ be an uncertain variable with inverse uncertainty distribution $\gamma^{-1}(r)$. Then, the logarithm entropy of τ is

$$\mathrm{H}[\tau] = \int_0^1 \gamma^{-1}(r) ln(\frac{r}{1-r})\, \mathrm{d}r.$$

Definition 2.5 (Tang and Gao [32]) Suppose that τ is an uncertain variable with uncertainty distribution γ. Then, the triangular entropy of τ is defined by

$$\mathrm{T}[\tau] = \int_{-\infty}^{+\infty} \mathrm{K}\big(\gamma(x)\big)\,\mathrm{d}x$$

where $\mathrm{K}(\mathrm{s}) = \begin{cases} \mathrm{s}, & if\ \ 0 \le \mathrm{s} \le \frac{1}{2} \\ 1-\mathrm{s}, & if\ \ \frac{1}{2} < \mathrm{s} \le 1. \end{cases}$

Theorem 2.3 (Tang and Gao [32]) Let τ be an uncertain variable with uncertainty distribution γ. Then, the triangular entropy of τ is

$$\mathrm{T}[\tau] = \int_{\frac{1}{2}}^{1} \gamma^{-1}(r)\,\mathrm{d}r - \int_{0}^{\frac{1}{2}} \gamma^{-1}(r)\,\mathrm{d}r.$$

Remark 2.1 Logarithm entropy and triangular entropy sometimes may fail to measure the indeterminacy of an uncertain variable.

Example 2.2 (Abtahi et al. [1]) Let τ be an uncertain variable with uncertainty distribution

$$\gamma(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}\ ;\ \ x \in \mathbb{R},$$

and inverse uncertainty distribution

$$\gamma^{-1}(r) = \tan\left(\pi\left(r - \frac{1}{2}\right)\right);\ 0 < r < 1.$$

Then, the logarithm entropy and triangular entropy of τ are infinite.

As a superior measure of indeterminacy compared to logarithm entropy and triangular entropy, Abtahi et al. [1] proposed the concept of pseudo-triangular entropy for uncertain variables. They also proved that the pseudo-triangular entropy of τ in Example 2 is finite.

Definition 2.6 (Abtahi et al. [1]) Suppose that τ is an uncertain variable with uncertainty distribution γ. Then, the pseudo-triangular entropy of τ is defined by

$$\mathrm{PS}[\tau] = \int_{-\infty}^{+\infty} \mathrm{C}\big(\gamma(x)\big)\,\mathrm{d}x,$$

where $\mathrm{C}(\mathrm{s}) = \begin{cases} (\mathrm{s})^2, & if\ \ 0 \le \mathrm{s} \le \frac{1}{2} \\ (1-\mathrm{s})^2, & if\ \ \frac{1}{2} < \mathrm{s} \le 1. \end{cases}$

Theorem 2.4 (Abtahi et al. [1]) Suppose that τ is an uncertaint variable with regular uncertainty distribution γ. Then, the pseudo-triangular entropy of τ is

$$\mathrm{PS}[\tau] = \int_{\frac{1}{2}}^{1} 2(1-r)\gamma^{-1}(r)\,\mathrm{d}r - \int_{0}^{\frac{1}{2}} 2(r)\gamma^{-1}(r)\,\mathrm{d}r\,.$$

Example 2.3 (Abtahi et al. [1]) Let $\tau \sim N(m, \delta)$, then the pseudo-triangular entropy of uncertain variable τ is

$$\mathrm{PS}[\tau] = \frac{(2\ln 2 - 1)\sqrt{3}}{\pi}\delta.$$

Theorem 2.5 (Liu [16]) Let $\tau_1, \tau_2, \dots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $\gamma_1, \gamma_2, \dots, \gamma_n$, respectively. If $f(\tau_1, \tau_2, \dots, \tau_n)$ is strictly increasing with respect to $\tau_1, \tau_2, \dots, \tau_m$ and strictly decreasing with respect to $\tau_{m+1}, \tau_{m+2}, \dots, \tau_n$, then

$$\tau = f(\tau_1, \tau_2, \dots, \tau_n)$$

has an inverse uncertainty distribution

$$\varphi^{-1}(r) = f\big(\gamma_1^{-1}(r), \dots, \gamma_m^{-1}(r), \gamma_{m+1}^{-1}(1-r), \dots, \gamma_n^{-1}(1-r)\big).$$

2.2. *Chance theory*

In order to handle phenomena including both uncertainty and randomness, chance theory was proposed by Liu [19]. In chance theory, the chance space is refer to the product of $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathrm{pr})$, in which $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space and $(\Omega, \mathcal{A}, \mathrm{pr})$ is a probability space. The chance measure of an uncertain random event $\Theta = \mathcal{L} \times \mathcal{A}$ is defined as

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma \,|(\gamma, \omega) \in \Theta\} \geq r\}\,\mathrm{d}r.$$

Liu [19] proved that a chance measure satisfies following properties:

(i) (Normality) $\mathrm{Ch}(\Gamma \times \Omega) = 1$.
(ii) (Duality) $\mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta^{\mathrm{c}}\} = 1$, for any event Θ.
(iii) (Monotonicity) $\mathrm{Ch}\{\Theta_1\} < \mathrm{Ch}\{\Theta_2\}$ for any real number set $\Theta_1 \subset \Theta_2$.
Furthermore, Hou [12] proved that for a sequence of events $\Theta_1, \Theta_2, \dots$, a chance measure satisfies subadditivity as follows,

$$\mathrm{Ch}\{\bigcup_{i=1}^{\infty}\Theta_{\mathrm{i}}\} \leq \sum_{i=1}^{\infty}\mathrm{Ch}(\Theta_{\mathrm{i}}).$$

Definition 2.7 (Liu [19]) An uncertain random variable is a function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathrm{pr})$ to the set of real numbers such that for any Borel set of B of real numbers, $\{\xi \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$.

Definition 2.8 (Liu [19]) Suppose ξ is an uncertain random variable. Then the chance distribution of ξ for any $x \in \mathbb{R}$ is defined by

$$\Phi(x) = \mathrm{Ch}\{\xi \leq x\}.$$

Definition 2.9 (Sheng et al. [29]) A chance distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and $\lim_{x\to-\infty}\Phi(x) = 0$, $\lim_{x\to+\infty}\Phi(x) = 1$.

Definition 2.10 (Liu [19]) Let ξ be an uncertain random variable. Then the expected value of ξ is defined by

$$\mathrm{E}[\xi] = \int_{0}^{+\infty}\mathrm{Ch}\{\xi \geq x\}\,\mathrm{d}x - \int_{-\infty}^{0}\mathrm{Ch}\{\xi \leq x\}\,\mathrm{d}x,$$

provided that at least one of the two integrals is finite.

Theorem 2.6 (Liu [20]) Let $\eta_1, \eta_2, \dots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \dots, \Psi_m$, and let $\tau_1, \tau_2, \dots, \tau_n$ be independent uncertain variables with uncertainty distributions $\gamma_1, \gamma_2, \dots, \gamma_n$, respectively. Then the uncertain random variable $\xi = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$ has a chance distribution

$$\Phi(x) = \int_{\mathbb{R}^m} F(x;\ y_1, \dots, y_m)\mathrm{d}\Psi_1(y_1) \dots\ \mathrm{d}\Psi_m(y_m),$$

where $F(x;\ y_1, \dots, y_m)$ is the uncertainty distribution of uncertain variable $f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$ for any real numbers $y_1, y_2, \dots, y_m$.

Theorem 2.7 (Ahmadzadeh et al. [2]) Let $\eta_1, \eta_2, \dots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \dots, \Psi_m$, and let $\tau_1, \tau_2, \dots, \tau_n$ be independent uncertain variables with uncertainty distributions $\gamma_1, \gamma_2, \dots, \gamma_n$, respectively. Suppose $\xi = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$, then

$$\mathrm{E}[\xi] = \int_{\mathbb{R}^m}\int_{0}^{1} F^{-1}(r, y_1, \dots, y_m)\mathrm{d}r\mathrm{d}\Psi_1(y_1) \dots\ \mathrm{d}\Psi_m(y_m),$$

where $F^{-1}(r, y_1, \dots, y_m)$ is the inverse uncertainty distribution of uncertain variable $f(y_1, y_2, \dots, y_m, \tau_1, \tau_2, \dots, \tau_m)$.

3. Partial pseudo-triangular entropy of uncertain random variables

In this section, the concepts of pseudo-triangular entropy of uncertain random variables are proposed. Besides, some mathematical properties of pseudo-triangular entropy and a formula to calculate it via inverse uncertainty distribution are derived. We first recall the definition of entropy, partial entropy and partial triangular entropy for uncertain random variables.

Definition 3.1 (Sheng et al. [30]) Let ξ be an uncertain random variable with chance distribution $\Phi(x)$. Then the entropy of ξ is defined by

$$\mathrm{H}[\xi] = \int_{-\infty}^{+\infty} \mathrm{L}\big(\Phi(x)\big)\,\mathrm{d}x,$$

where $\mathrm{L}(s) = -s\ln s - (1-s)\ln(1-s)$.

Theorem 3.1 (Ahmadzade et al. [3]) Let $\eta_1, \eta_2, \dots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \dots, \Psi_m$ and $\tau_1, \tau_2, \dots, \tau_n$ be independent uncertain variables with uncertainty distributions $\gamma_1, \gamma_2, \dots, \gamma_n$, respectively, and let f be a measurable function. Also let $\xi = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$ be an uncertain random variable. Then, $\xi = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$ has partial entropy

$$\mathrm{PH}[\xi] = \int_{\mathbb{R}^m} \int_{-\infty}^{+\infty} \mathrm{L}\big(F(x, y_1, \dots, y_m)\big)\mathrm{d}x\mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m),$$

where $\mathrm{L}(s) = -s\ln s - (1-s)\ln(1-s)$ and $F(x, y_1, \dots, y_m)$ is the uncertainty distribution of uncertain variable $f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$ for any real numbers $y_1, y_2, \dots, y_m$.

Theorem 3.2 (Ahmadzade et al. [3]) Let $\eta_1, \eta_2, \dots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \dots, \Psi_m$ and $\tau_1, \tau_2, \dots, \tau_n$ be independent uncertain variables with uncertainty distributions $\gamma_1, \gamma_2, \dots, \gamma_n$, respectively, and let f be a measurable function. Then, $\xi = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$ has partial entropy

$$\mathrm{PH}[\xi] = \int_{\mathbb{R}^m} \int_0^1 F^{-1}(r, y_1, \dots, y_m) ln\frac{r}{1-r}\mathrm{d}r\mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m),$$

where $F^{-1}(r, y_1, \dots, y_m)$ is the inverse uncertainty distribution of uncertain variable $f(y_1, y_2, \dots, y_m, \tau_1, \tau_2, \dots, \tau_m)$.

Theorem 3.3 (Ahmadzade et al. [3]) Let τ be an uncertain variable with uncertainty distribution function γ and let η be a random variable with probability distribution function Ψ. If $\xi = \eta + \tau$, then

$$\mathrm{PH}[\xi] = \mathrm{H}[\tau].$$

Theorem 3.4 (Ahmadzade et al. [3]) Let η_1 and η_2 be independent random variables and let τ_1 and τ_2 be independent uncertain variables. Also assume that $\xi_1 = f(\eta_1, \tau_1)$ and $\xi_2 = f(\eta_2, \tau_2)$. Then, for any real numbers a and b, we have

$$\mathrm{PH}[a\xi_1 + b\xi_2] = |a|\mathrm{PH}[\xi_1] + |b|\mathrm{PH}[\xi_2].$$

Theorem 3.5 (Ahmadzade et al. [4]) Suppose that $\eta_1, \eta_2, \dots, \eta_m$ are independent random variables, and $\tau_1, \tau_2, \dots, \tau_n$ are independent uncertain variables. Also let $\xi = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$ be an uncertain random variable. Then, partial triangular entropy of uncertain random variable ξ is defined as follows,

$$\mathrm{PT}[\xi] = \int_{\mathbb{R}^m} \int_{-\infty}^{+\infty} \mathrm{K}\big(F(x, y_1, \dots, y_m)\big) \mathrm{d}x \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m),$$

where $\mathrm{K}(\mathrm{s}) = \begin{cases} \mathrm{s}, & if \;\; 0 \le \mathrm{s} \le \frac{1}{2} \\ 1 - \mathrm{s}, & if \;\; \frac{1}{2} < \mathrm{s} \le 1, \end{cases}$

and $F(x, y_1, \dots, y_m)$ is the uncertainty distribution of uncertain variable $f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$ for any real numbers $y_1, y_2, \dots, y_m$.

Theorem 3.6 (Ahmadzade et al. [4]) Let $\eta_1, \eta_2, \dots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \dots, \Psi_m$ and $\tau_1, \tau_2, \dots, \tau_n$ be independent uncertain variables with uncertainty distributions $\gamma_1, \gamma_2, \dots, \gamma_n$, respectively, and let f be a measurable function. Then, $\xi = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$ has partial triangular entropy

$$\begin{aligned} \mathrm{PT}[\xi] = & \int_{\mathbb{R}^m} \int_{\frac{1}{2}}^{1} F^{-1}(r, y_1, \dots, y_m) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m) \\ & - \int_{\mathbb{R}^m} \int_{0}^{\frac{1}{2}} F^{-1}(r, y_1, \dots, y_m) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m), \end{aligned}$$

where $F^{-1}(r, y_1, \dots, y_m)$ is the inverse uncertainty distribution of uncertain variable $f(y_1, y_2, \dots, y_m, \tau_1, \tau_2, \dots, \tau_m)$.

Theorem 3.7 (Ahmadzade et al. [4]) Let τ be an uncertain variable with uncertainty distribution function γ and let η be a random variable with probability distribution function Ψ. If $\xi = \eta + \tau$, then

$$\mathrm{PT}[\xi] = \mathrm{T}[\tau].$$

Theorem 3.8 (Ahmadzade et al. [4]) Let η_1 and η_2 be independent random variables and let τ_1 and τ_2 be independent uncertain variables. Also assume that $\xi_1 = f(\eta_1, \tau_1)$ and $\xi_2 = f(\eta_2, \tau_2)$. Then, for any real numbers a and b, we have

$$\mathrm{PT}[a\xi_1 + b\xi_2] = |a|\mathrm{PT}[\xi_1] + |b|\mathrm{PT}[\xi_2].$$

Remark 3.2 Partial entropy and partial triangular entropy sometimes may fail to measure the indeterminacy of an uncertain random variable.

Example 3.4 Let τ be an uncertain variable with uncertainty distribution

$$\gamma(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2};\ \ x \in \mathbb{R},$$

and inverse uncertainty distribution

$$\gamma^{-1}(r) = \tan\left(\pi\left(r - \frac{1}{2}\right)\right);\ 0 < r < 1,$$

also let η be a random variable. Consider $\xi = \eta + \tau$. Since, $\mathrm{H}[\tau]$ and $\mathrm{T}[\tau]$ are infinite (Abtahi et al. 2022), Theorem 10 and Theorem 14 imply that $\mathrm{PH}[\xi] = \mathrm{H}[\tau] = \infty$ and $\mathrm{PT}[\xi] = \mathrm{T}[\tau] = \infty$, respectively. Since, partial entropy and partial triangular entropy failed to measure the indeterminacy of uncertain random variable ξ, a new measure of indeterminacy for uncertain random variables will be proposed.

Definition 3.2 Suppose that $\eta_1, \eta_2, \dots, \eta_m$ are independent random variables, and $\tau_1, \tau_2, \dots, \tau_n$ are independent uncertain variables. Also let $\xi = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$ be an uncertain random variable. Then, partial pseudo-triangular entropy of uncertain random variable ξ is defined as follows,

$$\mathrm{PPS}[\xi] = \int_{\mathbb{R}^m}\int_{-\infty}^{+\infty} \mathrm{C}\big(F(x, y_1, \dots, y_m)\big)\mathrm{d}x\mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m),$$

where $\mathrm{C}(\mathrm{s}) = \begin{cases} (\mathrm{s})^2, & if\ \ 0 \le \mathrm{s} \le \frac{1}{2} \\ (1-\mathrm{s})^2, & if\ \ \frac{1}{2} < \mathrm{s} \le 1, \end{cases}$

and $F(x, y_1, \dots, y_m)$ is the uncertainty distribution of uncertain variable $f(\mathrm{y}_1, \mathrm{y}_2, \dots, \mathrm{y}_m, \tau_1, \tau_2, \dots, \tau_n)$.

Theorem 3.9 Suppose that $\eta_1, \eta_2, \dots, \eta_m$ are independent random variables, and $\tau_1, \tau_2, \dots, \tau_n$ are independent uncertain variables. Also let $\xi = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$ be an uncertain random variable. Then, the partial pseudo-triangular entropy of uncertain random variable ξ is defined as follows,

$$\mathrm{PPS}[\xi] = -\int_{\mathbb{R}^m}\int_{0}^{1} F^{-1}(r, y_1, \dots, y_m)\mathrm{C}'(r)\mathrm{d}r\mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m),$$

where $\mathrm{C}(r) = \begin{cases} (r)^2, & if\ \ 0 \le r \le \frac{1}{2} \\ (1-r)^2, & if\ \ \frac{1}{2} < r \le 1, \end{cases}$

and $F^{-1}(r, y_1, \dots, y_m)$ is the inverse uncertainty distribution of uncertain variable $f(y_1, y_2, \dots, y_m, \tau_1, \tau_2, \dots, \tau_m)$.

Proof It is clear that $\mathrm{C}(r)$ is a derivable function with

$$\mathrm{C}'(r) = \begin{cases} 2(r), & if \;\; 0 \le r \le \dfrac{1}{2} \\ -2(1-r), & if \;\; \dfrac{1}{2} < r \le 1. \end{cases}$$

Since,

$$\mathrm{C}\big(F(x, y_1, \dots, y_m)\big) = \int_0^{F(x,y_1,\dots,y_m)} \mathrm{C}'(r)\,\mathrm{d}r = -\int_{F(x,y_1,\dots,y_m)}^{1} \mathrm{C}'(r)\,\mathrm{d}r,$$

we have,

$$\mathrm{PPS}[\xi] = \int_{\mathbb{R}^m} \int_{-\infty}^{+\infty} \mathrm{C}\big(F(x, y_1, \dots, y_m)\big) \mathrm{d}x \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m)$$

$$= \int_{\mathbb{R}^m} \int_{-\infty}^{0} \mathrm{C}\big(F(x, y_1, \dots, y_m)\big) \mathrm{d}x \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m) + \int_{\mathbb{R}^m} \int_{0}^{+\infty} \mathrm{C}\big(F(x, y_1, \dots, y_m)\big) \mathrm{d}x \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m)$$

$$= \int_{\mathbb{R}^m} \int_{-\infty}^{0} \int_0^{F(x,y_1,\dots,y_m)} \mathrm{C}'(r) \mathrm{d}r \mathrm{d}x \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m) - \int_{\mathbb{R}^m} \int_{-\infty}^{0} \int_{F(x,y_1,\dots,y_m)}^{1} \mathrm{C}'(r) \mathrm{d}r \mathrm{d}x \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m).$$

It follows from Fubini's theorem that

$$= \int_{\mathbb{R}^m} \int_0^{F(0,y_1,\dots,y_m)} \int_{F^{-1}(r,y_1,\dots,y_m)}^{0} \mathrm{C}'(r) \mathrm{d}x \mathrm{d}r \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m)$$

$$- \int_{\mathbb{R}^m} \int_{F(0,y_1,\dots,y_m)}^{1} \int_0^{F^{-1}(r,y_1,\dots,y_m)} \mathrm{C}'(r) \mathrm{d}x \mathrm{d}r \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m)$$

$$= -\int_{\mathbb{R}^m} \int_0^{F(0,y_1,\dots,y_m)} F^{-1}(r, y_1, \dots, y_m) \mathrm{C}'(r) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m)$$

$$- \int_{\mathbb{R}^m} \int_{F(0,y_1,\dots,y_m)}^{1} F^{-1}(r, y_1, \dots, y_m) \mathrm{C}'(r) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_m(y_m)$$

$$= -\int_{\mathbb{R}^m}\int_0^1 F^{-1}(r, y_1, \ldots, y_m)\mathrm{C}'(r)\mathrm{d}r\mathrm{d}\Psi_1(y_1) \ldots \mathrm{d}\Psi_m(y_m).$$

The proof is completed.

Theorem 3.10 Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$ and $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables with uncertainty distributions $\gamma_1, \gamma_2, \ldots, \gamma_n$, respectively, and let f be a measurable function. Then, $\xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)$ has partial pseudo-triangular entropy

$$\mathrm{PPS}[\xi] = \int_{\mathbb{R}^m}\int_{\frac{1}{2}}^1 2(1-r)F^{-1}(r, y_1, \ldots, y_m)\mathrm{d}r\mathrm{d}\Psi_1(y_1) \ldots \mathrm{d}\Psi_m(y_m)$$
$$- \int_{\mathbb{R}^m}\int_0^{\frac{1}{2}} 2(r)F^{-1}(r, y_1, \ldots, y_m)\mathrm{d}r\mathrm{d}\Psi_1(y_1) \ldots \mathrm{d}\Psi_m(y_m).$$

Proof According to Theorem 16 we have

$$\mathrm{PPS}[\xi] = -\int_{\mathbb{R}^m}\int_0^1 F^{-1}(r, y_1, \ldots, y_m)\mathrm{C}'(r)\mathrm{d}r\mathrm{d}\Psi_1(y_1) \ldots \mathrm{d}\Psi_m(y_m)$$

$$= -\left(\int_{\mathbb{R}^m}\int_0^{\frac{1}{2}} F^{-1}(r, y_1, \ldots, y_m)\mathrm{C}'(r)\mathrm{d}r\mathrm{d}\Psi_1(y_1) \ldots \mathrm{d}\Psi_m(y_m)\right.$$
$$\left. + \int_{\mathbb{R}^m}\int_{\frac{1}{2}}^1 F^{-1}(r, y_1, \ldots, y_m)\mathrm{C}'(r)\mathrm{d}r\mathrm{d}\Psi_1(y_1) \ldots \mathrm{d}\Psi_m(y_m)\right)$$

$$= -\left(\int_{\mathbb{R}^m}\int_0^{\frac{1}{2}} 2(r)F^{-1}(r, y_1, \ldots, y_m)\mathrm{d}r\mathrm{d}\Psi_1(y_1) \ldots \mathrm{d}\Psi_m(y_m)\right.$$
$$\left. - \int_{\mathbb{R}^m}\int_{\frac{1}{2}}^1 2(1-r)F^{-1}(r, y_1, \ldots, y_m)\mathrm{d}r\mathrm{d}\Psi_1(y_1) \ldots \mathrm{d}\Psi_m(y_m)\right)$$

$$= \int_{\mathbb{R}^m}\int_{\frac{1}{2}}^1 2(1-r)F^{-1}(r, y_1, \ldots, y_m)\mathrm{d}r\mathrm{d}\Psi_1(y_1) \ldots \mathrm{d}\Psi_m(y_m)$$
$$- \int_{\mathbb{R}^m}\int_0^{\frac{1}{2}} 2(r)F^{-1}(r, y_1, \ldots, y_m)\mathrm{d}r\mathrm{d}\Psi_1(y_1) \ldots \mathrm{d}\Psi_m(y_m).$$

Theorem 3.11 Let τ be an uncertain variable with uncertainty distribution function γ and let η be a random variable with probability distribution function Ψ. If $\xi = \eta + \tau$, then

$$\mathrm{PPS}[\xi] = \mathrm{PS}[\tau].$$

Proof It is obvious that $F^{-1}(r,y) = \gamma^{-1}(r) + y$. Therefore, by applying theorem 17 we have

$$\mathrm{PPS}[\xi] = \int_{-\infty}^{+\infty}\int_{\frac{1}{2}}^{1} 2(1-r)F^{-1}(r,y)\mathrm{d}r\mathrm{d}\Psi(y) - \int_{-\infty}^{+\infty}\int_{0}^{\frac{1}{2}} 2(r)F^{-1}(r,y)\mathrm{d}r\mathrm{d}\Psi(y)$$

$$= \int_{-\infty}^{+\infty}\int_{\frac{1}{2}}^{1} 2(1-r)(\gamma^{-1}(r)+y)\mathrm{d}r\mathrm{d}\Psi(y) - \int_{-\infty}^{+\infty}\int_{0}^{\frac{1}{2}} 2(r)(\gamma^{-1}(r)+y)\mathrm{d}r\mathrm{d}\Psi(y)$$

$$= \int_{-\infty}^{+\infty}\int_{\frac{1}{2}}^{1} 2(1-r)\gamma^{-1}(r)\mathrm{d}r\mathrm{d}\Psi(y) + \int_{-\infty}^{+\infty}\int_{\frac{1}{2}}^{1} 2(1-r)y\mathrm{d}r\mathrm{d}\Psi(y)$$

$$- \int_{-\infty}^{+\infty}\int_{0}^{\frac{1}{2}} 2(r)\gamma^{-1}(r)\mathrm{d}r\mathrm{d}\Psi(y) - \int_{-\infty}^{+\infty}\int_{0}^{\frac{1}{2}} 2(r)y\mathrm{d}r\mathrm{d}\Psi(y)$$

$$= \int_{\frac{1}{2}}^{1} 2(1-r)\gamma^{-1}(r)\mathrm{d}r + \frac{1}{4}\mathrm{E}[\eta] - \int_{0}^{\frac{1}{2}} 2(r)\gamma^{-1}(r)\mathrm{d}r - \frac{1}{4}\mathrm{E}[\eta]$$

$$= \mathrm{PS}[\tau].$$

Theorem 3.12 Let τ be an uncertain variable with uncertainty distribution function γ and let η be a random variable with probability distribution function Ψ. If $\xi = \eta\tau$, then

$$\mathrm{PPS}[\xi] = \mathrm{PS}[\tau]\mathrm{E}[\eta].$$

Proof It is obvious that $F^{-1}(r,y) = \gamma^{-1}(r)y$. Therefore, by applying theorem 16 we have

$$\mathrm{PPS}[\xi] = \int_{-\infty}^{+\infty}\int_{\frac{1}{2}}^{1} 2(1-r)F^{-1}(r,y)\mathrm{d}r\mathrm{d}\Psi(y) - \int_{-\infty}^{+\infty}\int_{0}^{\frac{1}{2}} 2(r)F^{-1}(r,y)\mathrm{d}r\mathrm{d}\Psi(y)$$

$$= \int_{-\infty}^{+\infty}\int_{\frac{1}{2}}^{1} 2(1-r)(\gamma^{-1}(r)y)\mathrm{d}r\mathrm{d}\Psi(y) - \int_{-\infty}^{+\infty}\int_{0}^{\frac{1}{2}} 2(r)(\gamma^{-1}(r)y)\mathrm{d}r\mathrm{d}\Psi(y)$$

$$= \mathrm{PS}[\tau]\mathrm{E}[\eta].$$

Theorem 3.13 Suppose that $\eta_1, \eta_2, \dots, \eta_n$ are independent random variables, and $\tau_1, \tau_2, \dots, \tau_n$ are independent uncertain variables. Assume

$$\xi_1 = f_1(\eta_1, \tau_1), \xi_2 = f_2(\eta_2, \tau_2), \dots, \xi_n = f_n(\eta_n, \tau_n).$$

If $f(z_1, z_2, \dots, z_n)$ is strictly increasing with respect to $z_1, z_2, \dots, z_m$ and strictly decreasing with respect to $z_{m+1}, z_{m+2}, \dots, z_n$, then $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$ has partial pseudo-triangular entropy

$$\mathrm{PPS}[\xi] = \int_{\mathbb{R}^n} \int_{\frac{1}{2}}^{1} 2(1-r) f\left(F_1^{-1}(r, y_1), \dots, F_m^{-1}(r, y_m), F_{m+1}^{-1}(1-r, y_{m+1}), \dots, F_n^{-1}(1-r, y_n)\right) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_n(y_n)$$

$$- \int_{\mathbb{R}^n} \int_{0}^{\frac{1}{2}} 2(r) f\left(F_1^{-1}(r, y_1), \dots, F_m^{-1}(r, y_m), F_{m+1}^{-1}(1-r, y_{m+1}), \dots, F_n^{-1}(1-r, y_n)\right) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_n(y_n),$$

where $F_i^{-1}(r, y_i)$ is the inverse uncertainty distribution of uncertain variable $f_i(\eta_i, \tau_i)$ for any real number $y_i; i = 1, 2, \dots, n$.

Proof By applying Theorem 17, the proof of theorem is straightforward.

Theorem 3.14 Let η_1 and η_2 be independent random variables with probability distributions Ψ_1 and Ψ_2, respectively, and let τ_1 and τ_2 be independent uncertain variables with uncertainty distributions γ_1 and γ_2, respectively. Then,

(i) If $\xi_1 = \eta_1 \tau_1$ and $\xi_2 = \eta_2 \tau_2$,

$$\mathrm{PPS}[\xi_1 \xi_2] = \mathrm{E}[\eta_1] \mathrm{E}[\eta_2] \mathrm{PS}[\tau_1 \tau_2].$$

(ii) If $\xi_1 = \eta_1 + \tau_1$ and $\xi_2 = \eta_2 + \tau_2$,

$$\mathrm{PPS}[\xi_1 \xi_2] = \mathrm{PS}[\tau_1 \tau_2] + \mathrm{E}[\eta_1] \mathrm{PS}[\tau_2] + \mathrm{E}[\eta_2] \mathrm{PS}[\tau_1].$$

Proof of part (i) it is clear that $F_1^{-1}(r, y_1) = y_1 \gamma_1^{-1}(r)$ and $F_2^{-1}(r, y_2) = y_2 \gamma_2^{-1}(r)$. By applying Theorem 20, we have

$$\mathrm{PPS}[\xi_1 \xi_2] = \int_{\mathbb{R}^2} \int_{\frac{1}{2}}^{1} 2(1-r) F^{-1}(r, y_1, y_2) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2)$$

$$- \int_{\mathbb{R}^2} \int_{0}^{\frac{1}{2}} 2(r) F^{-1}(r, y_1, y_2) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2)$$

$$= \int_{\mathbb{R}^2} \int_{\frac{1}{2}}^{1} 2(1-r) F_1^{-1}(r, y_1) F_2^{-1}(r, y_2) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2)$$

$$-\int_{\mathbb{R}^2}\int_0^{\frac{1}{2}} 2(r)F_1^{-1}(r,y_1)F_2^{-1}(r,y_2)\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$=\int_{\mathbb{R}^2}\int_{\frac{1}{2}}^1 2(1-r)\big(y_1\gamma_1^{-1}(r)\big)\big(y_2\gamma_2^{-1}(r)\big)\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$-\int_{\mathbb{R}^2}\int_0^{\frac{1}{2}} 2(r)\big(y_1\gamma_1^{-1}(r)\big)\big(y_2\gamma_2^{-1}(r)\big)\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$=\mathrm{E}[\eta_1\eta_2]\mathrm{PS}[\tau_1\tau_2]=\mathrm{E}[\eta_1]\mathrm{E}[\eta_2]\mathrm{PS}[\tau_1\tau_2].$$

Proof of part (ii) it is clear that $F_1^{-1}(r,y_1)=y_1+\gamma_1^{-1}(r)$ and $F_2^{-1}(r,y_2)=y_2+\gamma_2^{-1}(r)$. By applying Theorem 20, we have

$$\mathrm{PPS}[\xi_1\xi_2]=\int_{\mathbb{R}^2}\int_{\frac{1}{2}}^1 2(1-r)F^{-1}(r,y_1,y_2)\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$-\int_{\mathbb{R}^2}\int_0^{\frac{1}{2}} 2(r)F^{-1}(r,y_1,y_2)\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$=\int_{\mathbb{R}^2}\int_{\frac{1}{2}}^1 2(1-r)F_1^{-1}(r,y_1)F_2^{-1}(r,y_2)\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$-\int_{\mathbb{R}^2}\int_0^{\frac{1}{2}} 2(r)F_1^{-1}(r,y_1)F_2^{-1}(r,y_2)\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$=\int_{\mathbb{R}^2}\int_{\frac{1}{2}}^1 2(1-r)\big(y_1+\gamma_1^{-1}(r)\big)\big(y_2+\gamma_2^{-1}(r)\big)\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$-\int_{\mathbb{R}^2}\int_0^{\frac{1}{2}} 2(r)\big(y_1+\gamma_1^{-1}(r)\big)\big(y_2+\gamma_2^{-1}(r)\big)\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$=\mathrm{PS}[\tau_1\tau_2]+\frac{1}{4}\mathrm{E}[\eta_1\eta_2]+\mathrm{E}[\eta_1]\mathrm{PS}[\tau_2]+\mathrm{E}[\eta_2]\mathrm{PS}[\tau_1]-\frac{1}{4}\mathrm{E}[\eta_1\eta_2]$$

$$=\mathrm{PS}[\tau_1\tau_2]+\mathrm{E}[\eta_1]\mathrm{PS}[\tau_2]+\mathrm{E}[\eta_2]\mathrm{PS}[\tau_1].$$

Theorem 3.15 Let η_1 and η_2 be independent random variables with probability distributions Ψ_1 and Ψ_2, respectively, and let τ_1 and τ_2 be independent uncertain variables with uncertainty distributions γ_1 and γ_2, respectively. If $\xi_1=\eta_1\tau_1$ and $\xi_2=\eta_2\tau_2$, then

$$\mathrm{PPS}[\frac{\xi_1}{\xi_2}] = \mathrm{E}[\eta_1]\mathrm{E}[\frac{1}{\eta_2}]\mathrm{PS}[\frac{\tau_1}{\tau_2}].$$

Proof. It is clear that $F_1{}^{-1}(r, y_1) = y_1\gamma_1{}^{-1}(r)$ and $F_2{}^{-1}(r, y_2) = y_2\gamma_2{}^{-1}(r)$. By applying Theorem 20, we have

$$\mathrm{PPS}[\frac{\xi_1}{\xi_2}] = \int_{\mathbb{R}^2}\int_{\frac{1}{2}}^{1} 2(1-r)F^{-1}(r, y_1, y_2)\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$-\int_{\mathbb{R}^2}\int_{0}^{\frac{1}{2}} 2(r)F^{-1}(r, y_1, y_2)\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$= \int_{\mathbb{R}^2}\int_{\frac{1}{2}}^{1} 2(1-r)\frac{F_1{}^{-1}(r, y_1)}{F_2{}^{-1}(r, y_2)}\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$-\int_{\mathbb{R}^2}\int_{0}^{\frac{1}{2}} 2(r)\frac{F_1{}^{-1}(r, y_1)}{F_2{}^{-1}(r, y_2)}\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$= \int_{\mathbb{R}^2}\int_{\frac{1}{2}}^{1} 2(1-r)\frac{\left(y_1\gamma_1{}^{-1}(r)\right)}{\left(y_2\gamma_2{}^{-1}(r)\right)}\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$-\int_{\mathbb{R}^2}\int_{0}^{\frac{1}{2}} 2(r)\frac{\left(y_1\gamma_1{}^{-1}(r)\right)}{\left(y_2\gamma_2{}^{-1}(r)\right)}\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$= \mathrm{E}[\frac{\eta_1}{\eta_2}]\mathrm{PS}[\frac{\tau_1}{\tau_2}] = \mathrm{E}[\eta_1]\mathrm{E}[\frac{1}{\eta_2}]\mathrm{PS}[\frac{\tau_1}{\tau_2}].$$

Theorem 3.16 Let η_1 and η_2 be independent random variables and also let τ_1 and τ_2 be independent uncertain variables. Assume that $\xi_1 = f_1(\eta_1, \tau_1)$ and $\xi_2 = f_2(\eta_2, \tau_2)$. Then for any real numbers a and b, we have

$$\mathrm{PPS}[a\xi_1 + b\xi_2] = |a|\mathrm{PPS}[\xi_1] + |b|\mathrm{PPS}[\xi_2].$$

Proof. The theorem will be proved via three steps.

Step 1 We prove $\mathrm{PPS}[a\xi_1] = |a|\mathrm{PPS}[\xi_1]$.

If $a > 0$, then the inverse uncertainty distribution of $af_1(\tau_1, y_1)$ is

$$F^{-1}(r, y_1) = aF_1{}^{-1}(r, y_1),$$

where $F_1{}^{-1}(r, y_1)$ is the inverse uncertainty distribution of $f_1(\tau_1, y_1)$. It follows from Theorem 20 that

$$\mathrm{PPS}[a\xi_1] = \int_{-\infty}^{+\infty}\int_{\frac{1}{2}}^{1} 2(1-r)aF_1{}^{-1}(r,y_1)\mathrm{d}r\mathrm{d}\Psi_1(y_1) - \int_{-\infty}^{+\infty}\int_{0}^{\frac{1}{2}} 2(r)aF_1{}^{-1}(r,y_1)\mathrm{d}r\mathrm{d}\Psi_1(y_1)$$

$$= a\mathrm{PPS}[\xi_1].$$

If $a < 0$, then the inverse uncertainty distribution of $af_1(\tau_1, y_1)$ is

$$F^{-1}(r,y_1) = aF_1{}^{-1}(1-r,y_1),$$

therefore we have,

$$\mathrm{PPS}[a\xi_1] = \int_{-\infty}^{+\infty}\int_{\frac{1}{2}}^{1} 2(1-r)aF_1{}^{-1}(1-r,y_1)\mathrm{d}r\mathrm{d}\Psi_1(y_1)$$

$$- \int_{-\infty}^{+\infty}\int_{0}^{\frac{1}{2}} 2(r)aF_1{}^{-1}(1-r,y_1)\mathrm{d}r\mathrm{d}\Psi_1(y_1)$$

By changing variable $u = 1 - r$, we have

$$\mathrm{PPS}[a\xi_1] = -\int_{-\infty}^{+\infty}\int_{\frac{1}{2}}^{0} 2(u)aF_1{}^{-1}(u,y_1)\mathrm{d}u\mathrm{d}\Psi_1(y_1) + \int_{-\infty}^{+\infty}\int_{1}^{\frac{1}{2}} 2(1-u)aF_1{}^{-1}(u,y_1)\mathrm{d}u\mathrm{d}\Psi_1(y_1)$$

$$= \int_{-\infty}^{+\infty}\int_{0}^{\frac{1}{2}} 2(r)aF_1{}^{-1}(r,y_1)\mathrm{d}r\mathrm{d}\Psi_1(y_1)$$

$$- \int_{-\infty}^{+\infty}\int_{\frac{1}{2}}^{1} 2(1-r)aF_1{}^{-1}(r,y_1)\mathrm{d}r\mathrm{d}\Psi_1(y_1)$$

$$= -a\mathrm{PPS}[\xi_1].$$

Thus, we have $\mathrm{PPS}[a\xi_1] = |a|\mathrm{PPS}[\xi_1]$.

Step 2 we prove $\mathrm{PPS}[\xi_1 + \xi_2] = \mathrm{PPS}[\xi_1] + \mathrm{PPS}[\xi_2]$.

Since, the inverse distribution of $f_1(\tau_1, y_1) + f_2(\tau_2, y_2)$ is $F^{-1}(r, y_1, y_2) = F_1{}^{-1}(r, y_1) + F_2{}^{-1}(r, y_2)$, by applying Theorem 20 we have

$$\mathrm{PPS}[\xi_1 + \xi_2] = \int_{\mathbb{R}^2}\int_{\frac{1}{2}}^{1} 2(1-r)F^{-1}(r,y_1,y_2)\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$- \int_{\mathbb{R}^2}\int_{0}^{\frac{1}{2}} 2(r)F^{-1}(r,y_1,y_2)\mathrm{d}r\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)$$

$$= \int_{\mathbb{R}^2} \int_{\frac{1}{2}}^{1} 2(1-r)\left({F_1}^{-1}(r, y_1) + {F_2}^{-1}(r, y_2)\right) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2)$$

$$- \int_{\mathbb{R}^2} \int_{0}^{\frac{1}{2}} 2(r)\left({F_1}^{-1}(r, y_1) + {F_2}^{-1}(r, y_2)\right) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2)$$

$$= \int_{\mathbb{R}^2} \int_{\frac{1}{2}}^{1} 2(1-r)\left({F_1}^{-1}(r, y_1)\right) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) - \int_{\mathbb{R}^2} \int_{0}^{\frac{1}{2}} 2(r)\left({F_1}^{-1}(r, y_1)\right) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2)$$

$$+ \int_{\mathbb{R}^2} \int_{\frac{1}{2}}^{1} 2(1-r)\left({F_2}^{-1}(r, y_2)\right) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) - \int_{\mathbb{R}^2} \int_{0}^{\frac{1}{2}} 2(r)\left({F_2}^{-1}(r, y_2)\right) \mathrm{d}r \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2)$$

$$= \int_{-\infty}^{+\infty} \int_{\frac{1}{2}}^{1} 2(1-r)\left({F_1}^{-1}(r, y_1)\right) \mathrm{d}r \mathrm{d}\Psi_1(y_1) - \int_{-\infty}^{+\infty} \int_{0}^{\frac{1}{2}} 2(r)\left({F_1}^{-1}(r, y_1)\right) \mathrm{d}r \mathrm{d}\Psi_1(y_1)$$

$$+ \int_{-\infty}^{+\infty} \int_{\frac{1}{2}}^{1} 2(1-r)\left({F_2}^{-1}(r, y_2)\right) \mathrm{d}r \mathrm{d}\Psi_2(y_2) - \int_{-\infty}^{+\infty} \int_{0}^{\frac{1}{2}} 2(r)\left({F_2}^{-1}(r, y_2)\right) \mathrm{d}r \mathrm{d}\Psi_2(y_2)$$

$$= \mathrm{PPS}[\xi_1] + \mathrm{PPS}[\xi_2].$$

Step 3 For any real numbers a and b, we have

$$\mathrm{PPS}[a\xi_1 + b\xi_2] = \mathrm{PPS}[a\xi_1] + \mathrm{PPS}[b\xi_2] = |a|\mathrm{PPS}[\xi_1] + |b|\mathrm{PPS}[\xi_2].$$

The proof is completed.

4. Uncertain random portfolio optimization

In this section, in order to solve the portfolio optimization problem of uncertain random variables, a mean-entropy model via partial pseudo-triangular entropy is proposed. Suppose that there are n securities with uncertain random returns $\xi_i = \tau_i + \eta_i$; $i = 1, 2, \ldots, n$. Moreover, let x_i's be investment proportions in security $i = 1, 2, \ldots, n$. To make sure that the uncertain random portfolio risk is under control, we minimize entropy as the objective function. Moreover, we set expected value greater than some preset value c.

In order to optimize the portfolio optimization problem, a mean-entropy model based on partial pseudo-triangular entropy is presented as follows,

$$\begin{aligned}&Min\ \ \mathrm{PPS}[x_1\xi_1 + \cdots + x_n\xi_n]\\&S.t.\\&\quad \mathrm{E}[x_1\xi_1 + \cdots + x_n\xi_n] \geq \mathrm{C}\\&x_1 + x_2 + \cdots + x_n = 1, \quad 0 \leq x_i \leq 1;\ \forall i = 1,2,\dots,n,\end{aligned} \tag{4.1}$$

where the predetermined parameter C is designated by investor.

By applying the expected value formula of uncertain random variables $\xi_i = \tau_i + \eta_i;\ i = 1, 2, \dots, n$ in Theorem 7, we have

$$\begin{aligned}\mathrm{E}[x_1\xi_1 + \cdots + x_n\xi_n] &= \int_{\mathbb{R}^n}\int_0^1 \left(x_1{F_1}^{-1}(r, y_1) + \cdots + x_n{F_n}^{-1}(r, y_n)\right) \mathrm{d}r\mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_n(y_n)\\&= \int_{\mathbb{R}^n}\int_0^1 \big(x_1({\gamma_1}^{-1}(r) + y_1) + \cdots\\&\qquad\qquad + x_n({\gamma_n}^{-1}(r) + y_n)\big)\mathrm{d}r\mathrm{d}\Psi_1(y_1) \dots \mathrm{d}\Psi_n(y_n)\\&= x_1\big(\mathrm{E}(\tau_1) + \mathrm{E}(\eta_1)\big) + \cdots + x_n\big(\mathrm{E}(\tau_n) + \mathrm{E}(\eta_n)\big).\end{aligned}$$

Now, according to Theorem 18 and Theorem 23, the partial pseudo-triangular entropy of uncertain random variables $\xi_i = \tau_i + \eta_i;\ i = 1, 2, \dots, n$ is obtained as follows,

$$\mathrm{PPS}[x_1\xi_1 + \cdots + x_n\xi_n] = \mathrm{PPS}[x_1\xi_1] + \cdots + \mathrm{PPS}[x_n\xi_n] = x_1\mathrm{PS}[\tau_1] + \cdots + x_n\mathrm{PS}[\tau_n],$$

where $\mathrm{PS}[\tau_i]$ is the pseudo-triangular entropy of uncertain variable $\tau_i;\ i = 1, 2, \dots, n$.

Thus, Model (4.1) is equivalent to the following model,

$$\begin{aligned}&Min\ \ x_1\mathrm{PS}[\tau_1] + \cdots + x_n\mathrm{PS}[\tau_n]\\&S.t.\\&x_1\big(\mathrm{E}(\tau_1) + \mathrm{E}(\eta_1)\big) + \cdots + x_n\big(\mathrm{E}(\tau_n) + \mathrm{E}(\eta_n)\big) \geq \mathrm{C}\\&x_1 + x_2 + \cdots + x_n = 1, \quad 0 \leq x_i \leq 1;\ \forall i = 1,2,\dots,n,\end{aligned} \tag{4.2}$$

Now, in order to further investigate the outperformance of partial pseudo-triangular entropy as a quantifier of portfolio risk in comparison with partial entropy and partial triangular entropy in portfolio risk management let us consider the following example.

Example 5 Suppose there is an investment portfolio containing five securities. According to expert's evaluation and the data from Tehran Stock Exchange, five securities are assumed to be uncertain random variables with $\xi_i = \tau_i + \eta_i, i = 1, 2, \ldots, 5$ depicted in Table 1. Moreover, the parameter C in Model (4.2) is designated to 2 by investor.

The optimal solutions are obtained by implementing a genetic algorithm (GA) in MATLAB. The investment proportions in securities and the objective values are illustrated in Table 2. According to Table 2, the objective value for partial pseudo-triangular entropy has the lowest value amongst different types of entropy. Therefore, a portfolio based on partial pseudo-triangular entropy is less risky than partial entropy and partial triangular entropy. Furthermore, investment proportion in securities with smaller parameter δ is more other securities.

Table 1. Uncertain random returns.

No	Uncertain term $N(m, \delta)$	Random term $\mathcal{N}(\mu, \sigma^2)$
1	$\tau_1 \sim N(1.5, 0.7)$	$\eta_1 \sim \mathcal{N}(0.6, 0.09)$
2	$\tau_2 \sim N(1, 0.5)$	$\eta_2 \sim \mathcal{N}(1, 0.25)$
3	$\tau_3 \sim N(-0.5, 0.8)$	$\eta_3 \sim \mathcal{N}(1.7, 0.04)$
4	$\tau_4 \sim N(1, 0.3)$	$\eta_4 \sim \mathcal{N}(-0.7, 0.49)$
5	$\tau_5 \sim N(1.5, 0.4)$	$\eta_5 \sim \mathcal{N}(0.4, 0.36)$

Table 2. Investment proportion in securities.

Entropy	Objective value	Investment proportion
Partial	0.6548	(0.005, 0.032, 0.001, 0.543, 0.418)
Partial triangular	0.2544	(0.001, 0.055, 0.001, 0.73, 0.212)
Partial pseudo-triangular	0.0792	(0.006, 0.063, 0.002, 0.367, 0.561)

5. Conclusions

In this paper, a superior supplement measure of indeterminacy for uncertain random variables was proposed. It was first proved that partial entropy and partial triangular entropy sometimes fail to measure the indeterminacy of an uncertain random variable. Then, the concepts of partial pseudo-triangular entropy for uncertain random variables and its mathematical properties were investigated. To show the outperformance of partial pseudo-triangular entropy compared to partial entropy and partial triangular entropy in portfolio risk management, a numerical example was presented. To solve the corresponding problem, a genetic algorithm (GA) was implemented in MATLAB and optimal solutions were obtained. The example illustrated that a portfolio based on partial pseudo-triangular entropy is less risky than a portfolio based on partial entropy and partial triangular entropy. Furthermore, investment proportion in securities with smaller parameter δ is more than other securities.

References

[1] S. H. Abtahi, G. Yari, F. Hosseinzadeh Lotfi and R. Farnoosh, Pseudo-triangular entropy of uncertain variables: An entropy-based approach to uncertain portfolio optimization, International Journal of Mathematical Modelling & Computations, **12(1 (WINTER))** (2022), 61-77.

[2] H. Ahmadzade, Y. Sheng and F. Hassantabar Darzi, Some results of moments of uncertain random variables, Iranian Journal of Fuzzy Systems, **14(2)** (2017a) 1-21.

[3] H. Ahmadzade, R. Gao, M. H. Dehghan and Y. Sheng, Partial entropy of uncertain random variables, Journal of Intelligent & Fuzzy Systems, **33(1)** (2017b) 105-112.

[4] H. Ahmadzade, R. Gao, M. H. Dehghan and R. Ahmadi, Partial triangular entropy of uncertain random variables and its application, Journal of Ambient Intelligence and Humanized Computing, **9(5)** (2018) 1455-1464.

[5] X. Chen and W. Dai, Maximum entropy principle for uncertain variables, International Journal of Fuzzy Systems, **13(3)** (2011) 232-236.

[6] X. Chen, S. Kar and D. A. Ralescu, Cross-entropy measure of uncertain variables, Information Sciences, **201** (2012) 53-60.

[7] L. Chen, R. Gao, Y. Bian and H. Di, Elliptic entropy of uncertain random variables with application to portfolio selection, Soft Computing, **25(3)** (2021) 1925-1939.

[8] W. Dai and X. Chen, Entropy of function of uncertain variables, Mathematical and Computer Modelling, **55(3-4)** (2012) 754-760.

[9] W. Dai, Quadratic entropy of uncertain variables, Soft Computing, **22(17)** (2018) 5699-5706.

[10] H. Guo and X. Wang, Variance of uncertain random variables, Journal of Uncertainty Analysis and Applications, **2(1)** (2014) 1-7.

[11] Z. He, H. Ahmadzade, K. Rezaei, H. Rezaei and H. Naderi, Tsallis entropy of uncertain random variables and its application, Soft Computing, **25(17)** (2021) 11735-11743.

[12] YC. Hou, Subadditivity of chance measure, J Uncertain Anal Appl, **2** (2014) 1–8.

[13] Q. Li, Z. Qin and Y. Yan, Uncertain random portfolio optimization model with tail value-at-risk, Soft Computing, (2022) 1-10.

[14] B. Liu, Uncertainty theory, 2nd edition, Springer, Berlin, (2007).

[15] B. Liu, Some research problems in uncertainty theory, Journal of Uncertain systems, **3(1)** (2009) 3-10.

[16] B. Liu, Uncertainty theory: a branch of mathematics for modeling human uncertainty, Springer, Berlin, (2010).

[17] B. Liu, Why is there a need for uncertainty theory, Journal of Uncertain Systems, **6(1)** (2012) 3-10.

[18] B. Liu, Toward uncertain finance theory, Journal of Uncertainty Analysis and Applications, **1(1)** (2013) 1-15.

[19] Y, Liu, Uncertain random variables: A mixture of uncertainty and randomness, Soft Computing, **17(4)** (2013a) 625-634.

[20] Y. Liu, Uncertain random programming with applications, Fuzzy Optimization and Decision Making, **12(2)** (2013b) 153-169.

[21] B. Liu and Y. K. Liu, Expected value of fuzzy variable and fuzzy expected value models, IEEE transactions on Fuzzy Systems, **10(4)** (2002) 445-450.

[22] Y. Liu and D. A. Ralescu, Risk index in uncertain random risk analysis, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **22(04)** (2014) 491-504.

[23] Y. Liu and D. A. Ralescu, Value-at-risk in uncertain random risk analysis, Information Sciences, **391**(2017) 1-8.

[24] Y. Liu, D. A. Ralescu, C. Xiao and W. Lio, Tail value-at-risk in uncertain random environment, Soft Computing, **24(4)** (2020) 2495-2502.

[25] P. J. Mercurio, Y. Wu and H. Xie, An entropy-based approach to portfolio optimization, Entropy, **22(3)** (2020) 332.

[26] G. C. Philippatos and C. J. Wilson, Entropy, market risk, and the selection of efficient portfolios, Applied Economics, **4(3)** (1972) 209-220.
[27] Z Qin, Y. Dai Y and H. Zheng, Uncertain random portfolio optimization models based on value-at-risk, Journal of Intelligent & Fuzzy Systems, **32(6)** (2017) 4523-4531.
[28] C. E. Shannon, A mathematical theory of communication, ACM SIGMOBILE mobile computing and communications review, **5(1)** (2001) 3-55.
[29] Y. Sheng and K. Yao, Some formulas of variance of uncertain random variable, Journal of Uncertainty Analysis and Applications, **2(1)** (2014) 1-10.
[30] Y. Sheng, G. Shi and D. A. Ralescu, Entropy of Uncertain Random Variables with Application to Minimum Spanning Tree Problem, International journal of uncertainty, fuzziness and knowledge-based systems, **25(04)** (2017) 497-514.
[31] M. R. Simonelli, Indeterminacy in portfolio selection, European Journal of Operational Research, **163(1)** (2005) 170-176.
[32] W. Tang and W. Gao, Triangular entropy of uncertain variables, International Information Institute (Tokyo), Information, **16(2)** (2013) 1279.
[33] K. Yao, J. Gao and W. Dai, Sine entropy for uncertain variables. International journal of uncertainty, fuzziness and knowledge- based systems, **21(05)** (2013) 743-753.
[34] L. A. Zadeh, Fuzzy sets. Inform Control **8** (1965) 338– 353.