International Journal of
Mathematical Modelling & Computations
Vol. 12, No. 03, Summer 2022, 173- 181
DOI:10.30495/ijm2c.2022.1932741.1214

Norm and Numerical Radius Inequalities for Hilbert Space Operators

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Abstract. In this paper, we present several numerical radius and norm inequalities for sum of Hilbert space operators. These inequalities improve some earlier related inequalities. For $A, B \in B(H)$, we prove that

$$\omega\left(B^* A\right) \leq \sqrt{\frac{1}{2}\|A\|^2\|B\|^2+\frac{1}{2}\omega\left(|B|^2|A|^2\right)} \leq 4\omega\left(A\right)\omega\left(B\right).$$

Received: 06 June 2021, Revised: 14 October 2021, Accepted: 31 January 2022.

Keywords: Bounded linear operators; Numerical radius; Operator norm; Inequality.

AMS Subject Classification: 47A12, 47A30, 47A63.

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1. Introduction

Let $(H, \langle ., . \rangle)$ be a complex Hilbert space. The numerical range of an operator A is the subset of the complex numbers $\mathbb{C}$ given by:

$$W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}.$$

For $A \in B(H)$, let

$$\|A\| = \sup\{\|Ax\| : \|x\| = 1\},$$
$$\omega(A) = \sup\{|\langle Ax, x\rangle| : \|x\| = 1\},$$

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denote the usual operator and the numerical radius of A, respectively. It is well known that $\omega(.)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $A{:}H \to H$. This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\frac{1}{2}\|A\| \le \omega(A) \le \|A\|. \tag{1}$$

Kittaneh has shown in [14], that if $A \in B(H)$,

$$\omega^2(A) \le \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|, \tag{2}$$

where $|A| = (A^*A)^{1/2}$. Obviously, the inequality (1.2) is sharper than the second inequality in (1.1). This can be seen by using the fact that

$$\frac{1}{2}\left\||A|^2 + |A^*|^2\right\| \le \frac{1}{2}\left\||A|^2\right\| + \frac{1}{2}\left\||A^*|^2\right\| = \|A\|^2.$$

In [10], El-Haddad and Kittaneh proved that for any $0 \le t \le 1, r \ge 1$

$$\omega^{2r}(A) \le \frac{1}{2}\left\|(1-t)|A|^{2r} + t|A^*|^{2r}\right\|,$$

and

$$\omega^{r}(A) \le \frac{1}{2}\left\||A^*|^{2r(1-t)} + |A|^{2rt}\right\|.$$

In fact, this is a generalization of the inequality (1.2). For some recent and interesting results concerning inequalities for the numerical radius, see [2, 3, 4, 5, 12].

In this paper, we introduce new norm and numerical radius inequalities for operators some of them are related to the inequalities (1.1). Among these inequalities, we obtain a considerable improvement of the triangle inequality for the usual operator norm.

2. Results

Remark 2.1 It has been shown in [10] that

$$\omega(AB) \le \frac{1}{2}\|AA^* + B^*B\|.$$

Let $A = U|A|$ be the polar decomposition of A. Hence,

$$\begin{aligned}\omega(A) &= \omega(U|A|)\\ &= \omega\left(U|A|^{1-t}|A|^t\right)\\ &\le \frac{1}{2}\left\|\left(U|A|^{1-t}\right)\left(U|A|^{1-t}\right)^* + |A|^t|A|^t\right\|\\ &= \frac{1}{2}\left\|U|A|^{2(1-t)}U^* + |A|^{2t}\right\|\\ &= \frac{1}{2}\left\||A^*|^{2(1-t)} + |A|^{2t}\right\|.\end{aligned}$$

It can be considered as an alternative proof of [7, Theorem 1].

In order to achieve the goal of this section, we need the following lemmas.

Lemma 2.2 *(Buzano's inequality [2]) If a, b, x are vectors in an inner product space, then*

$$|\langle a, x\rangle|\,|\langle x, b\rangle| \le \frac{\|a\|\,\|b\| + |\langle a, b\rangle|}{2}\|x\|^2.$$

Lemma 2.3 *([8]) If a, b are positive real numbers, then for any $0 \le t \le 1$ and $r \ge 1$*

$$a^{1-t}b^t \le (1-t)\,a + tb \le ((1-t)\,a^r + tb^r)^{\frac{1}{r}}.$$

Lemma 2.4 *([10]) Let $A \in B(H)$ and let $x, y \in H$ be any vectors. If $0 \le t \le 1$,*

$$|\langle Ax, y\rangle|^2 \le \left\langle |A|^{2(1-t)}x, x\right\rangle \left\langle |A^*|^{2t}y, y\right\rangle.$$

Our first result reads as follows.

Theorem 2.5 *Let $A, B \in B(H)$. Then for any $0 \le t \le 1$*

$$\|A + B\| \le \frac{1}{\sqrt{2}}\left(\omega\left(|A|^{2(1-t)} + i|B|^{2(1-t)}\right) + \omega\left(|A^*|^{2t} + i|B^*|^{2t}\right)\right).$$

In particular,

$$\|A + B\| \le \frac{1}{\sqrt{2}}\left(\omega\left(|A| + i\,|B|\right) + \omega\left(|A^*| + i\,|B^*|\right)\right).$$

Proof For any unit vectors $x, y \in H$, we have

$$
\begin{aligned}
&|\langle (A+B)x, y\rangle| \\
&\le |\langle Ax, y\rangle| + |\langle Bx, y\rangle| \\
&\quad \text{(by the triangle inequality)} \\
&\le \sqrt{\left\langle |A|^{2(1-t)}x, x\right\rangle \left\langle |A^*|^{2t}y, y\right\rangle} + \sqrt{\left\langle |B|^{2(1-t)}x, x\right\rangle \left\langle |B^*|^{2t}y, y\right\rangle} \\
&\quad \text{(by Lemma 2.4)} \\
&\le \frac{1}{2}\left(\left\langle |A|^{2(1-t)}x, x\right\rangle + \left\langle |A^*|^{2t}y, y\right\rangle + \left\langle |B|^{2(1-t)}x, x\right\rangle + \left\langle |B^*|^{2t}y, y\right\rangle\right) \\
&\quad \text{(by Lemma 2.3)} \\
&= \frac{\sqrt{2}}{2}\left(\left|\left\langle \left(|A|^{2(1-t)} + i|B|^{2(1-t)}\right)x, x\right\rangle\right| + \left|\left\langle \left(|A^*|^{2t} + i|B^*|^{2t}\right)y, y\right\rangle\right|\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
|\langle (A+B)x, y\rangle| \le \frac{1}{\sqrt{2}}\Big(&\left|\left\langle \left(|A|^{2(1-t)} + i|B|^{2(1-t)}\right)x, x\right\rangle\right| \\
&+ \left|\left\langle \left(|A^*|^{2t} + i|B^*|^{2t}\right)y, y\right\rangle\right|\Big).
\end{aligned}
$$

Now taking the supremum over $x, y \in H$ with $\|x\| = \|y\| = 1$ in the above inequality produces

$$
\|A+B\| \le \frac{1}{\sqrt{2}}\left(\omega\left(|A|^{2(1-t)} + i|B|^{2(1-t)}\right) + \omega\left(|A^*|^{2t} + i|B^*|^{2t}\right)\right),
$$

as required. ■

As a consequence of Theorem 2.5, we have the following result.

Corollary 2.6 *Let $A, B \in B(H)$ be two positive operator. Then*

$$
\|A+B\| \le \frac{2}{\sqrt{2}}\omega(A+iB).
$$

Remark 2.7 The inequality in Corollary 2.6 has recently been proved by a different way in [12, Corollary 2.2].

The next theorem, gives a result about the triangle inequality for the numerical radius.

Theorem 2.8 *Let $A, B \in B(H)$. Then*

$$
\begin{aligned}
\omega^2(A+B) \le \omega\left(|A^*|^{2t}|A|^{2(1-t)}\right) &+ \omega\left(|B^*|^{2t}|B|^{2(1-t)}\right) \\
&+ \frac{1}{2}\left\| |A|^{4(1-t)} + |A^*|^{4t} + |B|^{4(1-t)} + |B^*|^{4t} \right\|.
\end{aligned}
$$

Proof Let $a = |A|^{2(1-t)}x$, $b = |A^*|^{2t}x$, and $\|x\| = 1$ in Lemma 2.2. Then

$$\begin{aligned}
|\langle Ax, x\rangle|^2 &\leq \left\langle |A|^{2(1-t)}x, x\right\rangle \left\langle |A^*|^{2t}x, x\right\rangle \\
&\leq \frac{1}{2}\left(\left\||A|^{2(1-t)}x\right\| \left\||A^*|^{2t}x\right\| + \left|\left\langle |A|^{2(1-t)}x, |A^*|^{2t}x\right\rangle\right|\right) \\
&= \frac{1}{2}\left(\sqrt{\left\langle |A|^{4(1-t)}x, x\right\rangle \left\langle |A^*|^{4t}x, x\right\rangle} + \left|\left\langle |A^*|^{2t}|A|^{2(1-t)}x, x\right\rangle\right|\right) \\
&\leq \frac{1}{2}\left|\left\langle |A^*|^{2t}|A|^{2(1-t)}x, x\right\rangle\right| + \frac{1}{4}\left\langle \left(|A|^{4(1-t)} + |A^*|^{4t}\right)x, x\right\rangle \\
&\quad \text{(by Lemma 2.3)}.
\end{aligned}$$

Thus,

$$|\langle Ax, x\rangle| \leq \frac{1}{2}\sqrt{2\left|\left\langle |A^*|^{2t}|A|^{2(1-t)}x, x\right\rangle\right| + \left\langle \left(|A|^{4(1-t)} + |A^*|^{4t}\right)x, x\right\rangle}.$$

We have, by this inequality,

$$\begin{aligned}
&(|\langle Ax, x\rangle| + |\langle Bx, x\rangle|)^2 \\
&\leq \left(\frac{\sqrt{2\left|\left\langle |A^*|^{2t}|A|^{2(1-t)}x, x\right\rangle\right| + \left\langle \left(|A|^{4(1-t)} + |A^*|^{4t}\right)x, x\right\rangle} + \sqrt{2\left|\left\langle |B^*|^{2t}|B|^{2(1-t)}x, x\right\rangle\right| + \left\langle \left(|B|^{4(1-t)} + |B^*|^{4t}\right)x, x\right\rangle}}{2}\right)^2 \\
&\leq \frac{2\left|\left\langle |A^*|^{2t}|A|^{2(1-t)}x, x\right\rangle\right| + \left\langle \left(|A|^{4(1-t)} + |A^*|^{4t}\right)x, x\right\rangle + 2\left|\left\langle |B^*|^{2t}|B|^{2(1-t)}x, x\right\rangle\right| + \left\langle \left(|B|^{4(1-t)} + |B^*|^{4t}\right)x, x\right\rangle}{2} \\
&\quad \left(\text{by the convexity of the function } f(t) = t^2\right) \\
&= \left|\left\langle |A^*|^{2t}|A|^{2(1-t)}x, x\right\rangle\right| + \left|\left\langle |B^*|^{2t}|B|^{2(1-t)}x, x\right\rangle\right| + \frac{1}{2}\Big(\left\langle \left(|A|^{4(1-t)} + |A^*|^{4t}\right)x, x\right\rangle \\
&\qquad + \left\langle \left(|B|^{4(1-t)} + |B^*|^{4t}\right)x, x\right\rangle\Big).
\end{aligned}$$

Whence,

$$\begin{aligned}
|\langle (A+B)x, x\rangle|^2 &\leq \left|\left\langle |A^*|^{2t}|A|^{2(1-t)}x, x\right\rangle\right| + \left|\left\langle |B^*|^{2t}|B|^{2(1-t)}x, x\right\rangle\right| \\
&\quad + \frac{1}{2}\left\langle \left(|A|^{4(1-t)} + |A^*|^{4t} + |B|^{4(1-t)} + |B^*|^{4t}\right)x, x\right\rangle.
\end{aligned}$$

Now taking the supremum over $x \in H$ with $\|x\| = 1$ in the above inequality produces

$$\omega^2(A+B) \leq \omega\left(|A^*|^{2t}|A|^{2(1-t)}\right) + \omega\left(|B^*|^{2t}|B|^{2(1-t)}\right) + \frac{1}{2}\left\||A|^{4(1-t)} + |A^*|^{4t} + |B|^{4(1-t)} + |B^*|^{4t}\right\|,$$

as required. ■

The case $A = B = T$, implies that:

Corollary 2.9 *Let $T \in B(H)$. Then for any $0 \leq t \leq 1$,*

$$\omega^2(T) \leq \frac{1}{2}\omega\left(|T^*|^{2t}|T|^{2(1-t)}\right) + \frac{1}{4}\left\||T|^{4(1-t)} + |T^*|^{4t}\right\|.$$

Corollary 2.10 *Let $T \in B(H)$ and $0 \leq t \leq 1$. Then, for any $r \geq 1$*

$$\omega^{2r}(T) \leq \frac{1}{2}\omega^r\left(|T^*|^{2t}|T|^{2(1-t)}\right) + \frac{1}{4}\left\||T|^{4r(1-t)} + |T^*|^{4rt}\right\|.$$

Proof We have by [1, Theorem 2.3].

$$\begin{aligned}\omega^{2r}(T) &\leq \left(\frac{1}{2}\omega\left(|T^*|^{2t}|T|^{2(1-t)}\right) + \frac{1}{4}\left\||T|^{4(1-t)} + |T^*|^{4t}\right\|\right)^r \\ &\leq \frac{1}{2}\left(\omega^r\left(|T^*|^{2t}|T|^{2(1-t)}\right) + \left(\frac{1}{2}\left\||T|^{4(1-t)} + |T^*|^{4t}\right\|\right)^r\right) \\ &\leq \frac{1}{2}\omega^r\left(|T^*|^{2t}|T|^{2(1-t)}\right) + \frac{1}{4}\left\||T|^{4r(1-t)} + |T^*|^{4rt}\right\|.\end{aligned}$$

■

Remark 2.11. Note that the inequality in Corollary 2.10 improves on the existing inequality

$$\omega^{2r}(T) \leq \frac{1}{2}\|T\|^{2r} + \frac{1}{4}\left\||T|^{4r(1-t)} + |T^*|^{4rt}\right\|.$$

obtained by Sattary et al. [15]. To show this,

$$\begin{aligned}\omega^{2r}(T) &\leq \frac{1}{2}\omega^r\left(|T^*|^{2t}|T|^{2(1-t)}\right) + \frac{1}{4}\left\||T|^{4r(1-t)} + |T^*|^{4rt}\right\| \\ &\leq \frac{1}{2}\left\||T^*|^{2t}|T|^{2(1-t)}\right\|^r + \frac{1}{4}\left\||T|^{4r(1-t)} + |T^*|^{4rt}\right\| \\ &\leq \frac{1}{2}\left\||T^*|^{2t}\right\|^r\left\||T|^{2(1-t)}\right\|^r + \frac{1}{4}\left\||T|^{4r(1-t)} + |T^*|^{4rt}\right\| \\ &= \frac{1}{2}\||T^*|\|^{2rt}\||T|\|^{2r(1-t)} + \frac{1}{4}\left\||T|^{4r(1-t)} + |T^*|^{4rt}\right\| \\ &= \frac{1}{2}\|T\|^{2r} + \frac{1}{4}\left\||T|^{4r(1-t)} + |T^*|^{4rt}\right\|.\end{aligned}$$

Our next result reads as follows.

Theorem 2.11 *Let $A, B \in B(H)$. Then for any $r \geq 2$*

$$\omega^r(B^*A) \leq \frac{1}{2}\left(\|A\|^r\|B\|^r + \omega\left(|B|^r|A|^r\right)\right).$$

Proof From the Buzano's inequaity

$$|\langle a, x\rangle|\,|\langle b, x\rangle| \leq \frac{\|a\|\,\|b\| + |\langle a, b\rangle|}{2}\|x\|^2.$$

Put $x = y$, $a = |A|^r x$, and $b = |B|^r x$ with $\|x\| = 1$ in the above inequality, to get

$$\langle |A|^r x, x\rangle\,\langle |B|^r x, x\rangle \leq \frac{\||A|^r x\|\,\||B|^r x\| + |\langle |A|^r x, |B|^r x\rangle|}{2}.$$

For any unit vector x, we have

$$\begin{aligned}
|\langle B^*Ax, x\rangle|^r &= |\langle Ax, Bx\rangle|^r \\
&\leq \|Ax\|^r\|Bx\|^r \\
&= \langle Ax, Ax\rangle^{\frac{r}{2}}\langle Bx, Bx\rangle^{\frac{r}{2}} \\
&= \langle A^*Ax, x\rangle^{\frac{r}{2}}\langle B^*Bx, x\rangle^{\frac{r}{2}} \\
&= \left\langle |A|^2 x, x\right\rangle^{\frac{r}{2}}\left\langle |B|^2 x, x\right\rangle^{\frac{r}{2}} \\
&\leq \left\langle |A|^{\frac{2r}{2}} x, x\right\rangle\left\langle |B|^{\frac{2r}{2}} x, x\right\rangle \\
&= \langle |A|^r x, x\rangle\,\langle |B|^r x, x\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
|\langle B^*Ax, x\rangle|^r &\leq \frac{\||A|^r x\|\,\||B|^r x\| + |\langle |A|^r x, |B|^r x\rangle|}{2} \\
&= \frac{\||A|^r x\|\,\||B|^r x\| + |\langle |B|^r|A|^r x, x\rangle|}{2}.
\end{aligned}$$

Now, taking the supremum over $x \in H$ with $\|x\| = 1$, we obtain the desired inequality. ■

Remark 2.12 For any $A, B \in B(H)$,

$$|\langle B^*Ax, x\rangle|^r \leq \frac{\||A|^r x\|\,\||B|^r x\| + |\langle |B|^r|A|^r x, x\rangle|}{2}.$$

Therefore,

$$\begin{aligned}
\frac{\||A|^r x\|\,\||B|^r x\| + |\langle |B|^r|A|^r x, x\rangle|}{2} &= \frac{\sqrt{\left\langle |A|^{2r} x, x\right\rangle\left\langle |B|^{2r} x, x\right\rangle} + |\langle |B|^r|A|^r x, x\rangle|}{2} \\
&\leq \frac{1}{4}\left\langle \left(|A|^{2r} + |B|^{2r}\right) x, x\right\rangle + \frac{1}{2}\left|\langle |B|^r|A|^r x, x\rangle\right|.
\end{aligned}$$

This implies

$$|\langle B^* A x, x\rangle|^r \leq \frac{1}{4}\left\langle\left(|A|^{2r}+|B|^{2r}\right) x, x\right\rangle+\frac{1}{2}\left|\langle|B|^r|A|^r x, x\rangle\right|.$$

So, we find that

$$\omega^r\left(B^* A\right) \leq \frac{1}{4}\left\||A|^{2r}+|B|^{2r}\right\|+\frac{1}{2} \omega\left(|B|^r|A|^r\right).$$

Theorem 2.13 *For any $A, B \in B(H)$,*

$$\omega\left(B^* A\right) \leq \sqrt{\frac{1}{2}\|A\|^2\|B\|^2+\frac{1}{2} \omega\left(|B|^2|A|^2\right)} \leq 4 \omega(A) \omega(B).$$

Proof Since for any $S, T \in B(H)$,

$$\frac{1}{2}\|T\|^2 \leq 2 \omega^2(T) \text { and }\|S\|^2 \leq 4 \omega^2(S),$$

we have by Theorem 2.12,

$$\begin{aligned}
\omega\left(B^* A\right) & \leq \sqrt{\frac{1}{2}\|A\|^2\|B\|^2+\frac{1}{2} \omega\left(|B|^2|A|^2\right)} \\
& \leq \sqrt{8 \omega^2(A) \omega^2(B)+\frac{1}{2} \omega\left(|B|^2|A|^2\right)} \\
& \leq \sqrt{8 \omega^2(A) \omega^2(B)+\frac{1}{2}\left\||B|^2|A|^2\right\|} \\
& \leq \sqrt{8 \omega^2(A) \omega^2(B)+\frac{1}{2}\|A\|^2\|B\|^2} \\
& \leq 4 \omega(A) \omega(B).
\end{aligned}$$

■

References

[1] J. S. Aujla and F. C. Silva, Weak majorization inequalities and convex functions, Linear Algebra Appl., **369** (2003) 217–233.

[2] M. L. Buzano, Generalizzazione della diseguaglianza di Cauchy-Schwarz, Rend. Sem. Mat. Univ. Politech. Torino., **31 (1971/73)** (1974) 405–409, in Italian.

[3] S. S. Dragomir, Power inequalities for the numerical radius of a product of two operators in Hilbert spaces, Sarajevo J. Math., **5 (18)** (2009) 269–278.

[4] S. S. Dragomir, Some Gruss type inequalities in inner product spaces, J. Inequal. Pure & Appl. Math., **4 (2)** (2003), Article 42.

[5] S. S. Dragomir, Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces, Tamkang J. Math., **39 (1)** (2008) 1–7.

[6] S. S. Dragomir, A note on numerical radius and the Krein-Lin inequality, Mathematica, **60 (83)** (2018) 149–151.

[7] M. El-Haddad and F. Kittaneh, Numerical radius inequalities for Hilbert space operators. II, Studia Math., **182 (2)** (2007) 133–140.

[8] G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities, 2nd edition, Cambridge University Press, Cambridge, (1988).

[9] F. Kittaneh and H. R. Moradi, Cauchy-Schwarz type inequalities and applications to numerical radius inequalities, Math. Inequal. Appl., **23 (3)** (2020) 1117–1125.

[10] F. Kittaneh, Notes on some inequalities for Hilbert space operators, Publ. Res. Inst. Math. Sci., **24 (2)** (1988) 283–293.

[11] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, Studia Math., **168 (1)** (2005) 73–80.
[12] H. R. Moradi and M. Sababheh, New estimates for the numerical radius, Filomat, **35 (14)** (2020) 4957–4962.
[13] M. E. Omidvar and H. R. Moradi, Better bounds on the numerical radii of Hilbert space operators, Linear Algebra Appl., **604** (2020) 265–277.
[14] M. E. Omidvar and H. R. Moradi, New estimates for the numerical radius of Hilbert space operators, Linear Multilinear Algebra, **69 (5)** (2021) 946–956.
[15] M. Sattari, M. S. Moslehian and T. Yamazaki, Some generalized numerical radius inequalities for Hilbert space operators, Linear Algebra Appl., **470** (2015) 216–227.