International Journal of
Mathematical Modelling & Computations
Vol. 13, No. 02, 2023, 01- 09
DOI: 10.30495/ijm2c.2023.1985644.1274

Casimir Energy in Non-relativistic Backgrounds: Numerical Approach

M. Belyad[a,*] and M. R. Tanhayi[a]

[a] *Department of Physics, Central Tehran Branch, Islamic Azad University, Tehran, P.O. Box 14676-86831, Iran.*

Abstract. In this paper we use numerical methods to investigate the Casimir effect for a scalar field in a specific boundary condition. In order to calculate the energy-momentum tensor, the holographic method is used, and, the background is Schrödinger-type metric which is close to the classical metric. We also compute the holographic entanglement entropy, and, for two steps the mutual information is also studied. By numerical analysis, we argue that the mutual information is always positive. Furthermore, for three entangling regions, we show that the corresponding tripartite information becomes negative.

Received: 01 March 2023; Revised: 01 June 2023; Accepted: 17 June 2023.

Keywords: Holographic method, Casimir Energy, Entanglement Entropy, n-Partite Information, Numerical Analysis

Index to information contained in this paper

1. Introduction

One of the manifestations of macroscopic zero-point energy in the Quantum Field Theory is the Casimir effect, which expresses the non-trivial properties of the vacuum state. In simple experimental terms, the attractive force between two parallel conductive plates that are electrically neutral and located in a vacuum is called the Casimir effect, which is caused by the presence of the vacuum. Theoretically, the Casimir effect can be considered as a result of the zero-point fluctuation spectrum in the presence and absence of these plates. It is noteworthy that the zero-point energy of any relativistic field is obtained under boundary conditions. In this view, the virtual particle-antiparticle pairs have the ability to create and annihilate in vacuum [1,2,3,4,5].

Basically the Casimir effect is a completely quantum effect and its computation needs to consider the quantum fields with some specific boundary conditions. Usually there are no analytic solution and in most cases numerical methods are needed. In this paper we study

*Corresponding author. Email: mozhgan.belyad@gmail.com

a scalar field and use numerical method to calculate Casimir effect. We use Anti-de Sitter space/Conformal Field Theory correspondence (AdS/CFT) to calculate the vacuum energy of a given system in the non-relativistic background.

In order to expand the initial studies of Casimir energy in reference [8], we can used the AdS/CFT correspondence, which expresses a relation between the quantum physics of strongly correlated many-body systems and classical gravitational dynamics of a higher dimension. According to this correspondence, the asymptotic metric determines the expectation value of all the individual components of the energy-momentum tensor and the corresponding energy is calculated from relation $E = \int\langle T_{00}\rangle$ [6,7,8].

Nowadays, more general classifications of metric than metrics with asymptotic AdS boundaries are used in investigating some feature of condense matter systems. For example, in recent AdS/CFT applications, a hypersurface violation of the dual quantum field Theory is shown by a metric that transforms covariantly under dilatation. We use this tool to extend in the context of Schrödinger holography and lifshitz spacetime [9,10,11].

2. Review

2-1. Lifshitz Metric

Holographic dual of physical systems in critical points with different space-time scales is given by Lifshitz metric. The time and space are scaled as $x_i \to \xi x_i$, $r \to \xi r$, $t \to \xi^z t$; where z is the dynamical critical component. The Lifshitz metric is:

$$ds^2 = L^2 r^{-2}\left(-r^{-2(z-1)}dt^2 + dr^2 + \sum_{i=1}^{d} dx_i^2\right) \tag{1}$$

Where L is the geometric radius. Due to an anisotropy between space and time, the above metric cannot be an ordinary solution for the Einstein equation; To break this anisotropy, some kinds of material fields are needed [8,27,46]. In general, the metric corresponding to the hyperscaling violating geometries is as follows:

$$ds^2 = L^2\, r_f^{-\frac{2\theta}{d}}\, r^{-2\left(\frac{d-\theta}{d}\right)}\left(-f(r)r^{-2(z-1)}dt^2 + \frac{1}{f(r)}dr^2 + \sum_{i=1}^{d} dx_i^2\right) \tag{2}$$

$$that: f(r) = 1 - \left(\frac{r}{r_h}\right)^{d+z-\theta}$$

Where θ is the hyperscaling violating factor, r_f is a dynamical scale. The appeared parameters are included the two conditions by considering null energy [26,16] and scale transmation:

$$(z-1)(d+z-\theta) \geq 0 \ \ , \ \ (d-\theta)(d(z-1)-\theta) \geq 0 \tag{3}$$

$$and \quad ds \to \xi^{\frac{d}{\theta}}\, ds$$

Because metric (2) is not invariant, the above scale transmation has been applied.

2-2. Schrödinger-type Metric

The non-relativistic Schrodinger-type metric with Galilean scaling and for 2+d dimensional theories is as:

$$ds^2 = r^{-2+2\theta/d}\left(-r^{-2(z-1)}dt^2 - 2dt\, d\chi + dr^2 + \sum_{i=1}^{d} dx_i^2\right). \tag{4}$$

Under the special conformal transformation, we will have:

$$\vec{x}' = \frac{\vec{x}}{1+t}, \qquad t' = \frac{t}{1+t}, \qquad r' = \frac{r}{1+t}, \qquad \chi' = \chi + \frac{c}{2}\frac{\vec{x}.\vec{x}+r^2}{1+t} \tag{5}$$
$$\Longrightarrow \qquad ds \to \left(\frac{r}{1+t}\right)^{\frac{\theta}{d}} ds$$

3. Energy-Momentum Tensor

The Ricci scalar curvature for the metric (4) is obtained as follows:

$$R = \frac{1}{r^{2\theta/d}}\frac{(d+1)(\theta-d)(d(d+2)-d\theta)}{d^2} \tag{6}$$

The Ricci scalar curvature is constant for $\theta = 0$. And the quasilocal energy-momentum tensor with a boundary metric of a given gravity theory $\gamma_{\mu\nu}$ is as follows:

$$T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma_{\mu\nu}} \tag{7}$$

Where S is action. The energy-momentum tensor is divergence at the boundary. To eliminate this divergency, a boundary term (that does not affect the bulk equations of motion) can be added to the action. In other words, using the AdS/CFT correspondence, (7) is as the expectation value of the stress tensor in the CFT:

$$T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{eff}}{\delta\gamma_{\mu\nu}}. \tag{8}$$

The zero-zero component of the energy tensor is calculated as follows:

$$T_{00} = \frac{2d^2(2+(d-3)z+2z^2) - 2d\big((d+1)+dz\big)\theta + 2d\theta^2 + (d+1)(\theta-d)(d(d+2)-d\theta)}{16\pi G d^2 r^{2z}} \tag{9}$$

Therefore, the energy between two infinite planes at r_{c_1}, r_{c_1} for d=3 with $\theta = 0, z = 1$ will be as follows:

$$E = \int T_{00}\sqrt{-g}d^5x = \frac{V_4}{32\pi G_N}\left(\frac{1}{r_{c_2}^6} - \frac{1}{r_{c_1}^6}\right). \tag{10}$$

4. Holographic Energy-Momentum Tensor

The energy-momentum tensor gives us information about the number of degrees of freedom and conformal anomalies. Considering the high importance of the energy-momentum tensor and its expectation value, here we briefly review the method of calculating the Brown-York stress tensor. The induced metric with a radial cutoff at the hypersurface $r = r_c$ from the metric (4) and the extrinsic curvature is given by:

$$\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \qquad , \qquad K_{\mu\nu} = -\gamma_\mu^\rho \Delta_\rho n_\nu. \tag{11}$$

Where n_μ is the unit normal vector of timelike surface. And the quasilocal stress tensor is expressed by:

$$\tau_{\mu\nu} = K_{\mu\nu} - \gamma_{\mu\nu}K \tag{12}$$

According to the AdS/CFT correspondence and the limit of $r_+ \to 0$, we will have:

$$\sqrt{-h}h^{\mu\rho}\langle T_{\rho\nu}\rangle = \lim_{r_c\to 0}\sqrt{-\gamma}\,\gamma^{\mu\rho}\tau_{\rho\nu} \tag{13}$$

Which in the hypersurface $r = r_c$, the induced metric will be as follows:

$$\gamma_{\mu\nu} = r_c^{-2+2\theta/d}\left(-r_c^{-2(z-1)}dt^2 - 2dt\,d\chi + \sum_{i=1}^{d-1} dx_i^2\right) \tag{14}$$

Now, by using the above equation, the expectation value of the zero-zero component of the stress tensor is obtained by:

$$\langle T_{00}\rangle = -\frac{d+1-z-\theta}{r_c^{d+1-\theta}}h_{00} = \frac{d+1-z-\theta}{r_c^{d-1-\theta+2z}}. \tag{15}$$

5. Casimir Energy

Now let us consider the zero-point energy in a box of size a at the boundary r_c which is located between χ_1 and χ_2. Putting these plates will affect the modes inside and outside the plate and a 1+1-dimensional box, one obtains:

$$E(\chi) = \frac{1}{2\pi}\frac{2-z-\theta}{r_c^{-\theta+2z}}\hbar a\int \omega dk \tag{16}$$

Where as usual we have define $\omega_k = \hbar k$. The above integral is clearly infinite. However, making use of the damping function as $e^{-\Lambda\omega_n}$ where Λ is defined as a natural size leading to a high frequency cutoff, $\omega \leq \pi\Lambda$ which leads to:

$$E(\chi,\Lambda) = \frac{1}{2\pi}\frac{2-z-\theta}{r_c^{-\theta+2z}}\hbar a\int ke^{-\Lambda k}dk \tag{17}$$

Now as usual in computing the Casimir energy one can obtain the renormalized Casimir energy in this setup by subtracting the above integral with same value which is calculated without the boundary, namely one has:

$$\begin{aligned} E^R(\chi) &= \lim_{\Lambda\to 0}\left[\frac{2-z-\theta}{2r_c^{-\theta+2z}}\left(\frac{1}{\pi}\int ke^{-\Lambda k}dk - \sum_n \frac{n\pi\hbar}{a}e^{-\Lambda n\pi/a}\right)\right] \\ &= -\frac{2-z-\theta}{r_c^{-\theta+2z}}\frac{\pi\hbar}{24a} \end{aligned} \tag{18}$$

CFT at the boundary $r = r_c$ causes the creation of this negative energy density, which can be considered the same as Casimir energy. Therefore, the Casimir force can be obtained as follows:

$$F = -\frac{d}{d\chi}E^R(\chi) = -\frac{2-z-\theta}{r_c^{-\theta+2z}}\frac{\pi\hbar}{24a^2} \tag{19}$$

For this one dimensional box, the null energy consideration implies that z=1 and $\theta < 1$. From these expressions, we observe that the hyperscaling violation has the effect of modifying the Casmir energy by shifting its value.

6- Holographic n-Partite Information

In this section, we want to obtain mutual information and tripartite information. Consider a strip as an entangling region located at $r = \epsilon$ in the given background:

$$-\frac{l}{2} \leq x_1 \leq \frac{l}{2} \quad , \quad 0 \leq x_a \leq L \quad (a = 2,\dots,d). \tag{20}$$

Holographic entanglement entropy is given as follows:

$$S(l) = \frac{L^{d-2}}{d-z-\theta}\frac{H}{R_\theta}\left(\frac{1}{\epsilon^{d-z-\theta}} - \frac{c_\theta}{l^{d-z-\theta}}\right) \tag{21}$$

For $\theta = 0$ and $z = d$:

$$S(l) = L^{d-2} H \log\left(\frac{2l}{\epsilon}\right) \tag{22}$$

Mutual information is a subset of n-partite information with $n = 2$ in quantum information literature. As we know, entanglement entropy measures the quantum correlation between two subsystems A and its complement. Therefore, mutual information is used to determine the amount of entanglement or shared information between two subsystems. Consider a system of two disjoint parts A_1 and A_2. Mutual information is given by:

$$I(A_1, A_2) = S(A_1) + S(A_2) - S(A_1 \cup A_2) \tag{23}$$

Where $S(A_1 \cup A_2)$, the union of two entangling regions, is the entanglement entropy [59]. For two operators as $\mathcal{O}_{A_1}$ and $\mathcal{O}_{A_2}$ in the regions A_1 and A_2 respectively, mutual information sets an upper bound in both quantum and classical correlations between them is as follows [57]:

$$I(A_1, A_2) \geq \frac{\left(\langle \mathcal{O}_{A_1}\mathcal{O}_{A_2}\rangle - \langle \mathcal{O}_{A_1}\rangle\langle \mathcal{O}_{A_2}\rangle\right)^2}{2\langle {\mathcal{O}^2}_{A_1}\rangle\langle {\mathcal{O}^2}_{A_2}\rangle} \tag{24}$$

Therefore, mutual information expresses not only the total correlation between two subsystems, rather from its definition, it can be seen that it is UV-cutoff independent and free of divergences.

There is a phase transition [58] that be raised from the union of two entangling regions and affects the calculation of mutual information. Here, the presence of a critical distance causes the two subsystems A_1 and A_2 to be completely separated; in other words, the mutual information vanishes as the distance between two subsystems increases, which is called disentangling transition. Within the holographic point of view, this phase transition has a simple explanation [59]. The minimal area surfaces for union entanglement entropy exists in two ways shown in Figure 2 for two strips with length l and distance h from each other, one of which is used according to the separation between two subsystems.

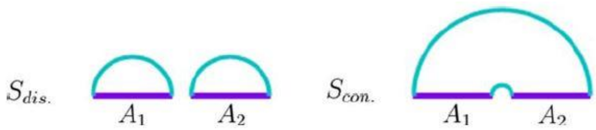

Figure 2. Showing two different structures of calculation $S(A_1 \cup A_2)$. According to the value of the parameters, the minimum state can be changed from one to another.

The corresponding minimal configurations can change from one shape to another, depending on the value of h/l, which leads to the definition of a critical ratio as follows:

$$r_{crit.} = \frac{h}{l} \tag{25}$$

Which:

$$S(A_1 \cup A_2) = \begin{cases} S_{con.} & 0 < \frac{h}{l} < r_{crit.} \\ S_{dis.} & r_{crit.} \leq \frac{h}{l} \end{cases} \tag{26}$$

Evolution of entanglement entropy for two minimal configurations is shown in figure 3. This plot shows that disconnected configuration has the minimal area with this range of parameters.

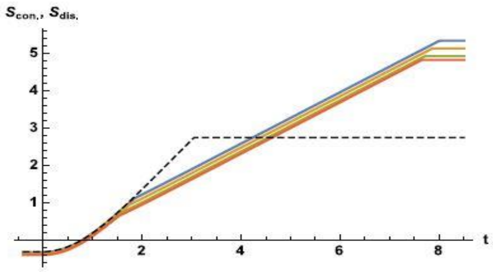

Figure 3. Evolution of entanglement entropy for two minimal configurations for $l = 4.5$ and $h = 2.1, 2.2, 2.4, 2.6$. The dashed curve represents disconnected configuration is independent of h.

Therefore, from (26), the holographic mutual information will have two answers as follows:

$$I(A_1, A_2) = \begin{cases} 2S(l) - S(h) - S(h+2l) & 0 < \frac{h}{l} < r_{crit.} \\ 0 & r_{crit.} \leq \frac{h}{l} \end{cases} \tag{27}$$

Therefore, the finite part of mutual information with the special case of logarithmic is obtained as follows:

$$I(A_1, A_2) = HL^{d-2} \log \frac{h(2l+h)}{l^2} \tag{28}$$

As seen in Figure 4, the holographic mutual information starts from the initial value at $\theta = 0$ and is always positive.

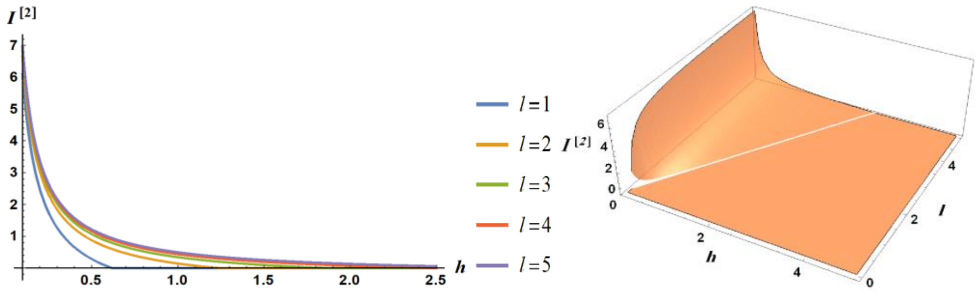

Figure 4. Numerical results for holographic mutual information: Left: Holographic mutual information as a function of h with $l = 1,2,3,4,5$. Right: 3D graph of holographic mutual information.

Now we consider a system with three strips, which is called tripartite information and is defined by:

$$\begin{aligned} I^{[3]}(A_1, A_2, A_3) = S(A_1) + S(A_2) + S(A_3) - S(A_1 \cup A_2) \\ - S(A_1 \cup A_3) - S(A_2 \cup A_3) + S(A_1 \cup A_2 \cup A_3). \end{aligned} \tag{29}$$

Similar to mutual information, the most important part is, finding the minimal surfaces for each of the union sentences, which was investigated in Einstein's gravity [7]. Here, we generalize the results to the non-relativistic Schrödinger background. For union sentences, we can write:

$$\begin{aligned} S(A_1 \cup A_2) &= \begin{cases} 2S(l) & \equiv S_1 \\ S(2l+h) + S(h) & \equiv S_2 \end{cases} \\ S(A_1 \cup A_3) &= \begin{cases} 2S(l) & \equiv S_3 \\ S(3l+2h) + S(2h+l) & \equiv S_4 \end{cases} \\ S(A_2 \cup A_3) &= \begin{cases} 2S(l) & \equiv S_5 \\ S(2l+h) + S(h) & \equiv S_6 \end{cases} \end{aligned} \tag{30}$$

And:

$$S(A_1 \cup A_2 \cup A_3) = \begin{cases} 3S(l) & \equiv S_7 \\ S(l) + S(2l+h) + s(h) & \equiv S_8 \\ S(2l+h) + S(h) + S(l) & \equiv S_9 \\ S(3l+2h) + S(2h+l) + S(l) & \equiv S_{10} \\ S(3l+2h) + 2S(h) & \equiv S_{11} \end{cases} \tag{31}$$

The graph related to the tripartite information in terms of hyperscaling violating parameter and as well as a function of the separation between the entangling regions is shown in Figure 5, which is negative as can be seen.

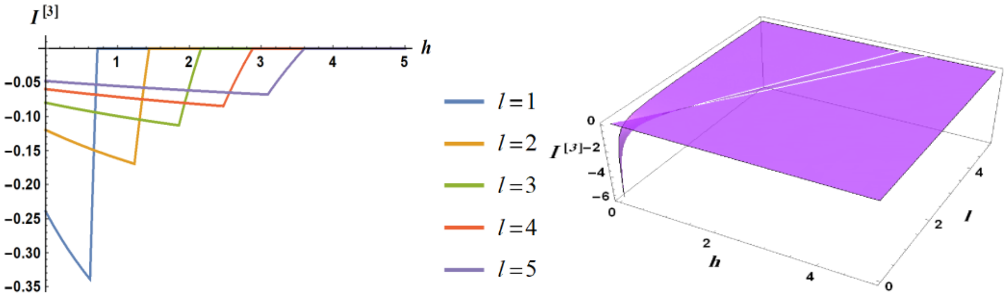

Figure 5. Numerical results for holographic mutual information: Left: Holographic tripartite information as a function of h with $l = 1,2,3,4,5$. Right: 3D graph of holographic tripartite information.

7- Conclusion

In this paper, we investigated the mutual and tripartite information for non-relativistic Schrodinger-type geometry. The non-negative mutual information indicates the subadditivity of entropy. On the other hand, this quantity is always negative for three regions.

The non-negativity of the mutual information leads to the following inequality:

$$S(A_1) + S(A_2) - S(A_1 \cup A_2) \geq 0 \tag{32}$$

Which includes a set of other inequalities for tripartite information. In standard holographic calculation of entanglement entropy for three entangling regions is as follows [12,61]:

$$S(A_1 \cup A_2) + S(A_2 \cup A_3) - S(A_1 \cup A_2 \cup A_3) - S(A_2) \geq 0 \tag{33}$$

It is obtained from equations (23) and (29):

$$I^{[3]}(A_1, A_2, A_3) = I^{[2]}(A_1, A_2) + I^{[2]}(A_1, A_3) - I^{[2]}(A_1, A_2 \cup A_3) \tag{34}$$

By generalizing relation (34) for n regions, we get a sum over all possible combinations as follows [44]:

$$I_n(A_1, A_2, A_3, \dots) = \sum_{\sigma} (-1)^{|\sigma|} S(\sigma) \tag{35}$$

Considering that tripartite information is always negative, it leads to the following inequality:

$$I^{[2]}(A_1, A_2) + I^{[2]}(A_1, A_3) \leq I^{[2]}(A_1, A_2 \cup A_3) \tag{36}$$

The above inequality in quantum information theory means that the sharing of entangling correlation between two systems A_1 and A_2 with A_3 leads to spoil the original entropy; this feature is called monogamy, which is a property of holographic theories for large-N limit. When bulk quantum effects are important, this property does not persist at finite N [44].

References

[1] Bordag, M., Klimchitskaya, G.L., Mohideen, U., Mostepanenko, V.M.: Advances in the Casimir Effect. University Press, Oxfords (2009).
[2] Milton, K.A.: The Casimir effect: recent controversies and progress. J. Phys. A **37**, R209 (2004). https://doi.org/10.1088/0305-4470/37/38/R01[hep-th/0406024]
[3] Milton, K.A.: The Casimir effect: physical manifestations of zero point energy. arXiv:9901011[hep-th]
[4] Milton, K.A., Parashar, P., Brevik, I., Kennedy, G.: Self-stress on a dielectric ball and Casimir-polder forces. Annals Phys. **412**, 168008 (2020). https://doi.org/10.1016/j.aop.2019.168008. arXiv:1909.05721[hep-th]
[5] Brevik, I., Parashar, P., Shajesh, K.V.: Casimir force for magnetodielectric media. Phys. Rev. A **98**(3),032509 (2018). https://doi.org/10.1103/PhysRevA.98.032509. arXiv:1808.02205[physics.class-ph]
[6] Lambrecht, A.: The Casimir effect: a force from nothing. Physics World, pp. 28–32, ISSN: 0953-8585. http://casimir-network.fr/IMG/pdf/Casimir 20effect.pdf (2002)
[7] Trang, T.N.: Casimir effect and vacuum fluctuations. http://www.hep.caltech.edu/phys199/lectures/lect5 6 cas.pdf (2003)
[8] Pejhan, H., Tanhayi, M.R., Takook, M.V.: Casimir effect for a scalar field via Krein quantization. Annals Phys. **341**, 195–204 (2014). arXiv:1204.6001[math-ph]
[9] Hasani, M., Tavakoli, F., Tanhayi, M.R.: Radial Casimir effect in a sphere through the Krein space quantization. Mod. Phys. Lett. A **27**, 1250096 (2012)
[10] Ghaffari, A., Karimaghaee, S., Tanhayi, M.R.: Vacuum energy in two dimensional box through the Krein quantization. Int. J .Theor. Phys. **56**, 887–897 (2017)
[11] Maldacena, J.M.: The large N limit of superconformal field theories and supergravity. Adv. Theor.Math. Phys. **2**, 231 (1998)
[12] Maldacena, J.M.: Int. J. Theor. Phys. **38**, 1113 (1999). arXiv:9711200[hep-th]
[13] Gubser, S.S., Klebanov, I.R., Polyakov, A.M.: Gauge theory correlators from non-critical string theory. Phys. Lett. B **428**, 105 (1998). arXiv:9802109[hep-th]
[14] Witten, E.: Anti-de Sitter space and holography. Adv. Theor. Math. Phys. **2**, 253 (1998). arXiv:9802150[hep-th]
[15] Aharony, O., Gubser, S.S., Maldacena, J.M., Ooguri, H., Oz, Y.: Large N field theories, string theory and gravity. Phys. Rept. **323**, 183 (2000). arXiv:9905111[hep-th]
[16] Horowitz, G.T., Myers, R.C.: Phys. Rev. **D59**, 026005 (1999). arXiv:9808079[hep-th]
[17] Son, D.T.: Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry. Phys. Rev. D **78**, 046003 (2008). arXiv:0804.3972[hep-th]
[18] Balasubramanian, K., McGreevy, J.: Gravity duals for non-relativistic CFTs. Phys. Rev. Lett. **101**, 061601 (2008). arXiv:0804.4053[hep-th]
[19] Goldberger, W.D.: AdS/CFT duality for non-relativistic field theory. JHEP **0903**, 069 (2009). arXiv:0806.2867[hep-th]
[20] Barbon, J.L.F., Fuertes, C.A.: On the spectrum of nonrelativistic AdS/CFT. JHEP **0809**, 030 (2008). arXiv:0806.3244[hep-th]
[21] Kachru, S., Liu, X., Mulligan, M.: Gravity duals of lifshitz-like fixed points. Phys. Rev. D **78**, 106005 (2008). arXiv:0808.1725[hep-th]
[22] Taylor, M.: Non-relativistic holography, arXiv:0812.0530[hep-th]
[23] Ryu, S., Takayanagi, T.: Aspects of holographic entanglement entropy. JHEP **0608**, 045 (2006). https://doi.org/10.1088/1126-6708/2006/08/045[hep-th/0605073]
[24] Hayden, P., Headrick, M., Maloney, A.: Holographic mutual information is monogamous. Phys. Rev. D **87**(4), 046003 (2013). https://doi.org/10.1103/PhysRevD.87.046003. arXiv:1107.2940[hep-th]
[25] Dong, X., Harrison, S., Kachru, S., Torroba, G., Wang, H.: Aspects of holography for theories with hyperscaling violation. arXiv:1201.1905[hep-th]
[26] Alishahiha, M., Colgain, E.O., Yavartanoo, H.: Charged black branes with hyperscaling violating factor. JHEP **1211**, 137 (2012). https://doi.org/10.1007/JHEP11(2012)137. arXiv:1209.3946[hep-th]
[27] Alishahiha, M., Astaneh, A.F., Mozaffar, M.R.M., Mollabashi, A.: Complexity growth with lifshitz scaling and hyperscaling violation. JHEP **1807**, 042 (2018). https://doi.org/10.1007/JHEP07(2018)042. arXiv:1802.06740[hep-th]
[28] Taylor, M.: Lifshitz holography. Class. Quant. Grav. **33**(3), 033001 (2016). https://doi.org/10.1088/0264- 9381/33/3/033001. arXiv:1512.03554[hep-th]
[29] Kim, B.S.: Schr¨odinger holography with and without hyperscaling violation. JHEP **1206**, 116 (2012). https://doi.org/10.1007/JHEP06(2012)116. arXiv:1202.6062[hep-th]
[30] Casini, H., Huerta, M.: Remarks on the entanglement entropy for disconnected regions. JHEP **0903**, 048 (2009).

arXiv:0812.1773[hep-th]
[31] Wolf, M.M., Verstraete, F., Hastings, M.B., Cirac, J.I.: Area laws in quantum systems: mutual information and correlations. Phys. Rev. Lett. **100**(7), 070502 (2008). https://doi.org/10.1103/PhysRevLett.100. 070502. arXiv:0704.3906[quant-ph]
[32] Headrick, M.: Entanglement Renyi entropies in holographic theories. Phys. Rev. D **82**, 126010 (2010). arXiv:1006.0047[hep-th]
[33] Fischler, W., Kundu, A., Kundu, S.: Holographic mutual information at finite temperature. Phys. Rev. D **87**, 126012 (2013). arXiv:1212.4764[hep-th]
[34] Mirabi, S., Tanhayi, M.R., Vazirian, R.: On the monogamy of holographic *n*-partite information. Phys. Rev. D **93**(10), 104049 (2016). https://doi.org/10.1103/PhysRevD.93.104049. arXiv:1603.00184[hep-th]
[35] Headrick, M., Takayanagi, T.: A holographic proof of the strong subadditivity of entanglement entropy. Phys. Rev. D **76**, 106013 (2007). https://doi.org/10.1103/PhysRevD.76.106013. arXiv:0704.3719[hepth]
[36] Allais, A., Tonni, E.: Holographic evolution of the mutual information. JHEP **1201**, 102 (2012). 10.1007/JHEP01(2012)102. arXiv:1110.1607[hep-th]
[37] Wall, A.C.: Maximin surfaces, and the strong subadditivity of the covariant holographic entanglement entropy. Class. Quant. Grav. **31**(22), 225007 (2014). https://doi.org/10.1088/0264-9381/31/22/225007. arXiv:1211.494[hep-th]
[38] Alishahiha, M., Mozaffar, M.R.M., Tanhayi, M.R.: On the time evolution of holographic n-partite information. JHEP **1509**, 165 (2015). https://doi.org/10.1007/JHEP09(2015)165. arXiv:1406.7677[hep-th]
[39] Flory, M., Erdmenger, J., Fernandez, D., Megias, E., Straub, E., Witkowski, P.: Time dependence of entanglement for steady state formation in AdS3/CFT2. J. Phys. Conf. Ser. **942**(1), 012010 (2017). https://doi.org/10.1088/1742-6596/942/1/012010. arXiv:1709.08614[hep-th]
[40] Tanhayi, M.R., Vazirian, R.: Higher-curvature corrections to holographic entanglement with momentum dissipation. Eur. Phys. J. C **78**(2), 162 (2018). https://doi.org/10.1140/epjc/s10052-018-5620-8. arXiv:1610.08080[hep-th]
[41] Aref'eva, I., Volovich, I.: Holographic photosynthesis. arXiv:1603.09107[hep-th]
[42] Tanhayi, M.R.: Universal terms of holographic entanglement entropy in theories with hyperscaling violation. Phys. Rev. D **97**(10), 106008 (2018). https://doi.org/10.1103/PhysRevD.97.106008. arXiv:1711.10526[hep-th]
[43] Erdmenger, J., Fernandez, D., Flory, M., Megias, E., Straub, A.K., Witkowski, P.: Time evolution of entanglement for holographic steady state formation. JHEP **1710**, 034 (2017). https://doi.org/10.1007/ JHEP10(2017)034. arXiv:1705.04696[hep-th]
[44] Hyden P., Headrick M., Maloney A.: Holographic mutual information is monogamous. Phys. Rev. **D 87**, 046003 (2013).https://doi.org/10.48550/arXiv.1107.2940