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New Inclusion Sets for the Eigenvalues of Stochastic Tensors

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Abstract. The purpose of this paper is to locate and estimate the eigenvalues of stochastic tensors. We present several estimation theorems about the eigenvalues of stochastic tensors. Meanwhile, we obtain the distribution theorem for the eigenvalues of tensor product of two stochastic tensors. We will conclude the paper with the distribution for the eigenvalues of generalized stochastic tensors.

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1. Introduction

Tensors have numerous applications in many branches of mathematics and physics. In late studies of numerical multilinear algebra eigenvalue problems for tensors have been brought to special attention. The concept of eigenvalues for tensors was first introduced and studied by Qi [6] and Lim [5] independently in 2005, and initiated the rapid developments of the spectral theory of tensors. Eigenvalue localization has been a hot topic in tensor theory and its applications. More references about this concept can be found in [2–4, 7, 10]. This article discusses location, distribution and estimate of the eigenvalues for stochastic tensor. In continue we introduce the concepts of generalized stochastic tensor and discuss the eigenvalue distribution for generalized stochastic tensors.

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We first add a comment on the notation that is used. Vectors are written as $(x, y, \ldots)$, matrices correspond to $(A, B, \ldots)$ and tensors are written as $(\mathbb{A}, \mathbb{B}, \ldots)$. The entry with row index i and column index j in a matrix A, i.e. $(A)_{ij}$ is symbolized by a_{ij} (also $(\mathbb{A})_{i_1 i_2 \ldots i_m} = a_{i_1 i_2 \ldots i_m}$). $\Re$ and $\mathcal{C}$ represents the real and complex field, respectively. For each nonnegative integer n, denote $[n] = \{1, 2, \ldots, n\}$. $\Re^n_+ \left(\Re^n_{++}\right)$ denotes the cone $\{x \in \Re^n \,:\, x_i \geq (>)\ 0,\ i = 1, \ldots, n\}$.

2. Preliminaries

In this section, we will cover some fundamental notions and properties on tensors. A tensor can be regarded as a higher order generalization of a matrix, which takes the form

$$\mathbb{A} = \left(a_{i_1,\ldots,i_m}\right), \quad a_{i_1,\ldots,i_m} \in \Re, \quad 1 \leqslant i_1, \ldots, i_m \leqslant n,$$

where $\Re$ is the real field. Such a multi-array $\mathbb{A}$ is said to be an mth order n-dimensional square real tensor with n^m entries $a_{i_1,\ldots,i_m}$. In this regard, a vector is a first order tensor and a matrix is a second order tensor. Tensors of order more than two are called higher order tensors. An mth order n-dimensional tensor $\mathbb{A}$ is called nonnegative if $a_{i_1 i_2 \ldots i_m} \geqslant 0$. We denote the set of all nonnegative mth order n-dimensional tensors by $\Re^{[m,n]}_+$. For a vector $x = (x_1, \ldots, x_n)^T$, let $\mathbb{A}x^{m-1}$ be a vector in $\Re^n$ whose ith component is defined as the following:

$$\left(\mathbb{A}x^{m-1}\right)_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}, \tag{1}$$

and let $x^{[m]} = (x_1^m, \ldots, x_n^m)^T$.

Definition 2.1 [6] A pair $(\lambda, x) \in \mathcal{C} \times (\mathcal{C}^n \backslash \{0\})$ is called an eigenvalue and an eigenvector of $\mathbb{A} \in \Re^{[m,n]}$, if they satisfy

$$\mathbb{A}x^{m-1} = \lambda x^{[m-1]}. \tag{2}$$

Definition 2.2 [8] Let $\mathbb{A}$ (and $\mathbb{B}$) be an order $m \geq 2$ (and order $k \geq 1$), dimension n tensor, respectively. The product $\mathbb{A}\mathbb{B}$ is defined to be the following tensor $\mathbb{C}$ of order $(m-1)(k-1)+1$ and dimension n:

$$c_{i\alpha_1 \ldots \alpha_{m-1}} = \sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \ldots i_m} b_{i_2 \alpha_1} \ldots b_{i_m \alpha_{m-1}},$$

where $(i \in [n],\, \alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1})$.

It is easy to check from the definition that $I_n \mathbb{A} = \mathbb{A} = \mathbb{A} I_n$, where I_n is the identity matrix of order n. When $k = 1$ and $\mathbb{B} = x \in \mathcal{C}^n$ is a vector of dimension n, then $(m-1)(k-1)+1 = 1$. Thus $\mathbb{A}\mathbb{B} = \mathbb{A}x$ is still a vector of dimension n, and we have

$$(\mathbb{A}x)_i = (\mathbb{A}\mathbb{B})_i = c_i = \sum_{i_2 \ldots i_m=1}^{n} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m} = \left(\mathbb{A}x^{m-1}\right)_i,$$

thus we have $\mathbb{A}x^{m-1} = \mathbb{A}x$. So the first application of the tensor product defined above is that now $\mathbb{A}x^{m-1}$ can be simply written as $\mathbb{A}x$.

Definition 2.3 [5] A tensor $\mathbb{A} \in \Re^{[m,n]}$ is called reducible, if there exists a nonempty proper index subset $I \subset \{1, \ldots, n\}$ such that

$$a_{i_1,\ldots,i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I,$$

if $\mathbb{A}$ is not reducible, then we call $\mathbb{A}$ irreducible.

3. Main results

Definition 3.1 [9] A nonnegative tensor $\mathbb{A}$ of order m dimension n is called stochastic provided that

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} = 1, \qquad i = 1, 2, \ldots, n.$$

Obviously, when $\mathbb{A}$ is stochastic, 1 is the spectral radius of $\mathbb{A}$ and e is an eigenvector corresponding to 1, where e is an all ones vector; if, further, $\mathbb{A}$ is irreducible, then e is the unique positive eigenvector corresponding to 1.

Theorem 3.2 *Suppose $\mathbb{A} \in \Re_{+}^{[m,n]}$ is a stochastic tensor and*

$$M = \min\left\{a_{ii\ldots i} \,:\, i = 1, 2, \ldots, n\right\},$$

then

$$\sigma(\mathbb{A}) \subset G(\mathbb{A}) = \{z : |z - M| \leq 1 - M\},$$

where $\sigma(\mathbb{A})$ is denoted the whole eigenvalues of tensor $\mathbb{A}$, $G(A)$ is the Gerschgorin disc of tensor $\mathbb{A}$.

Proof Let $\lambda \in \sigma(\mathbb{A})$ be arbitrary, thus there exists $x = (x_1, \ldots, x_n)^T \neq 0$ such that $\mathbb{A}x = \lambda x^{[m-1]}$. Set $y_i = \frac{x_i}{t_i}$ where $t_i\,(i = 1, \ldots, n)$ is positive number such that $|t_s| = \max\limits_{1\leq i\leq n} |t_i|$, and $|y_s| = \max\limits_{1\leq i\leq n} |y_i|$. $\mathbb{A}x = \lambda x^{[m-1]}$ implies that

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} y_{i_2} t_{i_2} \ldots y_{i_m} t_{i_m} = \lambda y_i^{m-1} t_i^{m-1},$$

$$\lambda y_s^{m-1} t_s^{m-1} = \sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} a_{si_2\ldots i_m} y_{i_2} t_{i_2} \ldots y_{i_m} t_{i_m} + a_{ss\ldots s} y_s^{m-1} t_s^{m-1}.$$

Multiply right each item of the above equation with $(y_s^*)^{m-1}$, then

$$\lambda t_s^{m-1} - a_{ss\ldots s} t_s^{m-1} = \frac{\sum\limits_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} a_{si_2\ldots i_m} y_{i_2} t_{i_2} \ldots y_{i_m} t_{i_m} (y_s^*)^{m-1}}{|y_s|^{2(m-1)}}.$$

By using trigonal inequality,

$$\left|\lambda t_s^{m-1} - a_{ss\ldots s} t_s^{m-1}\right| \leq \sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} |a_{si_2\ldots i_m} t_{i_2} \ldots t_{i_m}|,$$

thus

$$|\lambda - a_{ss\ldots s}| \leq \sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} |a_{si_2\ldots i_m}| = \sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} a_{si_2\ldots i_m} = 1 - a_{ss\ldots s}.$$

Therefore

$$\begin{aligned} |\lambda - M| &= |\lambda - a_{ss\ldots s} + a_{ss\ldots s} - M| \\ &\leq |\lambda - a_{ss\ldots s}| + |a_{ss\ldots s} - M| \\ &\leq 1 - a_{ss\ldots s} + a_{ss\ldots s} - M \\ &= 1 - M \end{aligned}$$

Thus the proof is completed. ∎

Definition 3.3 [8] Let $\mathbb{A}$ and $\mathbb{B}$ be two order k tensors with dimension n and m, respectively. Define the direct product $\mathbb{A} \otimes \mathbb{B}$ to be the following tensor of order k and dimension nm (the set of subscripts is taken as $[n] \times [m]$ in the lexicographic order):

$$(\mathbb{A} \otimes \mathbb{B})_{(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)} = a_{i_1 i_2\ldots i_k} b_{j_1 j_2 \ldots j_k}.$$

Theorem 3.4 *[8] Let $\mathbb{A}$ and $\mathbb{B}$ be two order k tensors with dimension n and m, respectively. Suppose that we have $\mathbb{A}u^{k-1} = \lambda u^{[k-1]}$, and $\mathbb{B}v^{k-1} = \mu v^{[k-1]}$, and we also write $w = u \otimes v$. Then we have:*

$$(\mathbb{A} \otimes \mathbb{B})\, w^{[k-1]} = (\lambda\mu)\, w^{[k-1]}.$$

Theorem 3.5 *Let $\mathbb{A}, \mathbb{B} \in \Re_+^{[m,n]}$ are stochastic tensors and*

$$M_1 = \min\{a_{ii\ldots i} : i = 1, 2, \ldots, n\}, M_2 = \min\{b_{ii\ldots i} : i = 1, 2, \ldots, n\},$$

then

$$\sigma(\mathbb{A} \otimes \mathbb{B}) \subset G(\mathbb{A} \otimes \mathbb{B}) = \{z : |z - M_1| \leq 1 - M_1\} . \{z : |z - M_2| \leq 1 - M_2\},$$

where $\sigma(\mathbb{A} \otimes \mathbb{B})$ is denoted the whole eigenvalues of tensor product for tensor $\mathbb{A}$ and tensor $\mathbb{B}$, $G(\mathbb{A} \otimes \mathbb{B})$ is the oval region of the product for elements of Gerschgorin *disc whose center is M_1 and radius is $1 - M_1$ and Gerschgorin disc whose center is M_2 and radius is $1 - M_2$.*

Proof From Theorem (3.2), we have

$$\{z : |z - M_1| \leq 1 - M_1\},$$

$$\{z : |z - M_2| \leq 1 - M_2\},$$

and by Theorem (3.4) we are done. Therefore, the eigenvalues of tensor product for tensor $\mathbb{A}$ and tensor $\mathbb{B}$ are located in the oval region $G(\mathbb{A} \otimes \mathbb{B})$. ■

Theorem 3.6 *Suppose $\mathbb{A} \in \Re_+^{[m,n]}$ is a stochastic tensor and*

$$M_i = \max\{a_{ii_2\ldots i_m} : 1 \leq i_2, \ldots, i_m \leq n\},$$

then

$$\sigma(\mathbb{A}) \subset G(\mathbb{A}) = \bigcup_{i=1}^{n} \left\{z : |z - a_{ii\ldots i}| \leq \sqrt{(m-1)(n-1) M_i (1 - a_{ii\ldots i})}\right\}.$$

Proof Let $\lambda \in \sigma(\mathbb{A})$ be arbitrary, thus there exists $x = (x_1, \ldots, x_n)^T \neq 0$ such that $\mathbb{A}x = \lambda x^{[m-1]}$. Set $|x_s| = \max\limits_{1 \leq i \leq n} |x_i|$ .Thus $\sum\limits_{i_2,\ldots,i_m=1}^{n} a_{si_2\ldots i_m} x_{i_2} \ldots x_{i_m} = \lambda x_s^{m-1}$, so $(\lambda - a_{ss\ldots s}) x_s^{m-1} = \sum\limits_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} a_{si_2\ldots i_m}$.

From Schwarz inequality and trigonal inequality, we have the following result:

$$\begin{aligned}
|\lambda - a_{ss\ldots s}| &= \left| \frac{\sum\limits_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} a_{si_2\ldots i_m} x_{i_2} \ldots x_{i_m} (x_s^*)^{m-1}}{|x_s|^{2(m-1)}} \right| \\
&\leq \sqrt{\sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} |a_{si_2\ldots i_m}|^2} \cdot \sqrt{\sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} \left|\frac{x_{i_2}}{x_s}\right|^2 \left|\frac{x_s^*}{x_s}\right|^2 \ldots \left|\frac{x_{i_m}}{x_s}\right|^2 \left|\frac{x_s^*}{x_s}\right|^2} \\
&\leq \sqrt{(m-1)(n-1)} F_s,
\end{aligned}$$

where $F_s = \sqrt{\sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} |a_{si_2\ldots i_m}|^2}$, $s = 1, 2, \ldots, n$. And since

$$\begin{aligned} F_s &= \sqrt{\sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} |a_{si_2\ldots i_m}|^2} \\ &\leq \sqrt{M_s \sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} |a_{si_2\ldots i_m}|} \\ &= \sqrt{M_s\left(1 - a_{ss\ldots s}\right)}, \end{aligned}$$

for $s = 1, 2, \ldots, n$. Thus $|\lambda - a_{ss\ldots s}| \leq \sqrt{(m-1)(n-1) M_s (1 - a_{ss\ldots s})}$. Because λ is an arbitrary eigenvalue of tensor $\mathbb{A}$, the theorem is proved. ■

Lemma 3.7 *[1] Assume that $a_1 \leq a_2 \leq \cdots \leq a_n < k$. Each of the ovals*

$$|z - a_i|\ |z - a_j| \leq (k - a_i)(k - a_j), \quad (i, j = 1, 2, \ldots, n;\ i < j) \tag{3}$$

is either identical with the oval

$$|z - a_1|\ |z - a_2| \leq (k - a_1)(k - a_2) \tag{4}$$

or lies in the interior of (4). The point $z = k$ is the only common point of the boundaries of two different ovals (3).

Theorem 3.8 *Let $a_{ii\ldots i}$ and $a_{jj\ldots j}$ be the smallest elements of the main diagonal of a stochastic tensor $\mathbb{A} \in \Re_+^{[m,n]}$. Then all the eigenvalues lie in the interior or on the boundary of the oval*

$$\{z : |z - a_{ii\ldots i}|\ |z - a_{jj\ldots j}| \leq (1 - a_{ii\ldots i})(1 - a_{jj\ldots j})\}. \tag{5}$$

Proof Let $\lambda \in \sigma(\mathbb{A})$ be arbitrary, thus there exists $x = (x_1, \ldots, x_n)^T \neq 0$ such that $\mathbb{A}x = \lambda x^{[m-1]}$. Thus

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{vi_2\ldots i_m} x_{i_2} \ldots x_{i_m} = \lambda x_v^{m-1}. \tag{6}$$

Let x_r and x_s be two of these n numbers $\{x_1, \ldots, x_n\}$ which have the greatest absolute value and assume that $|x_r| \geq |x_s|$. We consider the r-th and the s-th of the equations (6)

$$(\lambda - a_{ss\ldots s})\, x_s^{m-1} = \sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} a_{si_2\ldots i_m} x_{i_2} \ldots x_{i_m}, \tag{7}$$

$$(\lambda - a_{rr\ldots r})\, x_r^{m-1} = \sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(r,\ldots,r)}}^{n} a_{ri_2\ldots i_m} x_{i_2}\ldots x_{i_m}. \tag{8}$$

If $x_s = 0$ then also $x_t = 0$ for every $t \neq r$. It follows from the equation (7) that $\lambda = a_{rr\ldots r}$ since $x_r \neq 0$. This proves our theorem if $x_s = 0$. Now assume that $x_s \neq 0$. By multiplying the equations (7), (8) we obtain

$$\begin{aligned}(\lambda - a_{ss\ldots s})(\lambda - a_{rr\ldots r})\, x_s^{m-1} x_r^{m-1} &= \left(\sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} a_{si_2\ldots i_m} x_{i_2}\ldots x_{i_m} \right) \\ &\cdot \left(\sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(r,\ldots,r)}}^{n} a_{ri_2\ldots i_m} x_{i_2}\ldots x_{i_m} \right),\end{aligned}$$

hence

$$\begin{aligned}|\lambda - a_{ss\ldots s}|\, |\lambda - a_{rr\ldots r}|\, \left|x_s^{m-1}\right| \left|x_r^{m-1}\right| &\leq \left|x_r^{m-1}\right| \left(\sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(s,\ldots,s)}}^{n} a_{si_2\ldots i_m} \right) \\ &\cdot \left|x_s^{m-1}\right| \left(\sum_{\substack{i_2,\ldots,i_m=1 \\ (i_2,\ldots,i_m)\neq(r,\ldots,r)}}^{n} a_{ri_2\ldots i_m} \right),\end{aligned}$$

therefore

$$|\lambda - a_{ss\ldots s}|\, |\lambda - a_{rr\ldots r}| \leq (1 - a_{ss\ldots s})(1 - a_{rr\ldots r}). \tag{9}$$

It follows that each eigenvalue lies in the interior or on the boundary of at least one of the ovals (9), where $s, r = 1, 2, \ldots, n$ and $s \neq r$. But all these ovals lie in (5) by Lemma (3.7) for $k = 1$. ■

Definition 3.9 **(i)** A nonnegative tensor $\mathbb{A}$ of order m and dimension n is called first generalized stochastic if

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} = k, \quad i = 1, 2, \ldots, n.$$

(ii) A tensor $\mathbb{A}$ of order m and dimension n is called second generalized stochastic if

$$\sum_{i_2,\ldots,i_m=1}^{n} |a_{ii_2\ldots i_m}| = 1, \quad i = 1, 2, \ldots, n.$$

(iii) A tensor $\mathbb{A}$ of order m and dimension n is called third generalized stochastic if

$$\sum_{i_2,\ldots,i_m=1}^{n} |a_{ii_2\ldots i_m}| = k, \quad i = 1,2,\ldots,n.$$

Theorem 3.10 *Suppose* $\mathbb{A},\mathbb{B} \in \Re_+^{[m,n]}$ *are first generalized stochastic tensors, then*

$$\begin{aligned}\sigma(\mathbb{A}\otimes\mathbb{B}) \subset G(\mathbb{A}\otimes\mathbb{B}) = &\{z : |z-a_{ii\ldots i}|\ |z-a_{jj\ldots j}| \le (k-a_{ii\ldots i})(k-a_{jj\ldots j})\}\\ &.\{z : |z-b_{ii\ldots i}|\ |z-b_{jj\ldots j}| \le (k-b_{ii\ldots i})(k-b_{jj\ldots j})\}.\end{aligned}$$

Proof The proof is similar to Theorem (3.8). ■

Theorem 3.11 *Suppose* $\mathbb{A} \in \Re^{[m,n]}$ *is third generalized stochastic tensor, and* $M = \min\{|a_{ii\ldots i}| : i = 1,2,\ldots,n\}$*, then*

$$\sigma(\mathbb{A}) \subset G(\mathbb{A}) = \{z : |z-M| \le k+M\}.$$

Proof Let λ is arbitrary and $|x_s| = \max_{1\le i\le n} |x_i|$. Thus

$$|\lambda - a_{ss\ldots s}| \le \sum_{\substack{(i_2,\ldots,i_m)=1\\(i_2,\ldots,i_m)\ne(s,\ldots,s)}}^{n} |a_{si_2\ldots i_m}| = k - |a_{ss\ldots s}|,$$

therefore

$$\begin{aligned}|\lambda - M| &= |\lambda - a_{ss\ldots s} + a_{ss\ldots s} - M|\\ &\le |\lambda - a_{ss\ldots s}| + |a_{ss\ldots s} - M|\\ &\le k - |a_{ss\ldots s}| + |a_{ss\ldots s}| + M\\ &= k+M.\end{aligned}$$

■

Theorem 3.12 *Let* $\mathbb{A},\mathbb{B} \in \Re^{[m,n]}$ *be third generalized stochastic tensors,* $M_1 = \min\{|a_{ii\ldots i}| : i = 1,2,\ldots,n\}$ *and* $M_2 = \min\{|b_{ii\ldots i}| : i = 1,2,\ldots,n\}$*, then*

$$\sigma(\mathbb{A}\otimes\mathbb{B}) \subset G(\mathbb{A}\otimes\mathbb{B}) = \{z : |z-M_1| \le k+M_1\}.\{z : |z-M_2| \le k+M_2\}.$$

Proof The method is the same as Theorem (3.5), which is omitted. ■

Theorem 3.13 *Let* $a_{ii\ldots i}$ *and* $a_{jj\ldots j}$ *be the smallest elements of the main diagonal of a third generalized stochastic tensor* $\mathbb{A} \in \Re^{[m,n]}$*. Then*

$$\sigma(\mathbb{A}) \subset G(\mathbb{A}) = \{z : |z-a_{ii\ldots i}|\ |z-a_{jj\ldots j}| \le (k+|a_{ii\ldots i}|)(k+|a_{jj\ldots j}|)\}.$$

Theorem 3.14 *Let* $\mathbb{A},\mathbb{B} \in \Re_+^{[m,n]}$ *are first generalized stochastic tensors, then*

$$\begin{aligned}\sigma(\mathbb{A}\otimes\mathbb{B}) \subset G(\mathbb{A}\otimes\mathbb{B}) = &\{z : |z-a_{ii\ldots i}|\ |z-a_{jj\ldots j}| \le (k+a_{ii\ldots i})(k+a_{jj\ldots j})\}\\ &.\{z : |z-b_{ii\ldots i}|\ |z-b_{jj\ldots j}| \le (k+b_{ii\ldots i})(k+b_{jj\ldots j})\}.\end{aligned}$$

Example 3.15 Let $\mathbb{A} = (a_{ijk}) \in \Re^{[3,2]}$ for $1 \leqslant i, j, k \leqslant 2$ such that:

$$a_{ijk} = \begin{cases} 1 \text{ if} & i = j = k \\ 0 \text{ o.w} \end{cases}$$

Thus $\mathbb{A}$ is a first generalized stochastic tensor, and suppose $\mathbb{B} = (b_{ijk})$ for $1 \leqslant i, j, k \leqslant 2$ such that:

$$b_{ijk} = \begin{cases} 1 \text{ if} & i = 1, j = k = 2 \\ 0 \text{ o.w} \end{cases}$$

Then by theorem 2.14, we have

$$\sigma\left(\mathbb{A} \otimes \mathbb{B}\right) \subset \{z : |z-1|^2 \leqslant 4\}.\{z : |z|^2 \leqslant 1\}$$

where $(\mathbb{A} \otimes \mathbb{B}) \in \Re^{[3,4]}$ has 81 entries.

4. Conclusion

In this paper, we estimated of the eigenvalues for stochastic tensor. The purpose of this research was to location, distribution and estimate of the eigenvalues for stochastic tensor and introduce the concepts of generalized stochastic tensor and discuss the eigenvalue distribution for generalized stochastic tensors.

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