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Analysis of a Three Age Group Preys with Control Measures of COVID-19 Spread in the Third Prey

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Abstract. In this paper we study the covid-19 disease with treatment and control to spread it with different measures. The model equations are analysed from the general MC Kendrick equations for age structured populations. The existence,positiveness,boundedness and stability of equilibria are studied as they depend on the prey's natural carrying capacity. The main result of this paper is the three age group population, how to control and avoid to infect the disease from predator with local,global stability and Hopf bifurcation method also utilised.Finally the result of this model prey predator where numerical examples using maple software of Rossler type.

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1. Introduction

Mathematical modelling is one way to explain many of the ideas and concepts in the sciences discussed [14]. Predator-prey interaction and competition are often viewed as the two main building blocks in mathematical population models [13]. Catering to the necessities and comforts of human beings invariably robs the ecological

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structure of the nature [8]. Mathematical models have been used to explain the dynamics of diseases use a similar underlying methodology based on theaged group population. Cushing investigated many equations derived from the Mc Kendrick model for age-structured populations [1].

Having pointed out the three prey populations with different age group as

i) First prey population age from infant to 14 years.
ii) Second prey population age from 15 to 39 years.
iii) Third prey population age from 40 and above.

In the first and second prey population the infection of disease will be less because of the high level of immunity and take some remedial measures as mentioned below in constant U. In the case of third prey population age group 40 and above, they need adequate nutrient is another important way to help reduce the risk and impact of virus infections, as well as to build a more resilient immune system. We consider the constant τ as strengthining the immunity power either by natural medicine or vaccination given by the government.

The model we consider in this paper is an improved model by the inclusion of time dependent control parameters(social distance, facemask,hand wash, elbow cough,self quarantine) and with the assumptions that the covid-19 individuals can also transmit the disease recklessly [3]. Our main goal are: We investigate the model under the assumption that the control measures are constants (social distance, facemask, hand wash, elbow cough, self quarantine) avoid to infect and spread the disease. We denote the constant τ as vaccination or natural medicine to boost the immunity level of prey population those who are under aged persons like 40 and above.

Based on the seminal work of Lotka and Volterra, different prey models are being developed, describing various physical interaction between prey and predator species [12]. Here we are applying the control strategies for third prey and predator population and carrying capacities for all the three preys [4]. We analyse the direction and stability of the Hopf-bifurcation arising the boundary equilibrium point [10]. In this paper the system are obtained and criteria for local stability and global stability of the system derived [2]. We present results from the exploration of three preys interact with one predator model including vaccination and treatment options like natural medicine under varying rates for incidence and disease related death [3]. Finally, we present the results of numerical simulations for each model under various parameter values.

2. Method for selection of parameter values

The model subdivides the total human population at time t, [9] denoted by $N(t)$ into the following sub-populations of three prey population $N_1(t), N_2(t), N_3(t)$ and one predator population $N_4(t)$. We consider the 4D model system given below, obtained by coupling the RM model with the Leslie-Gower model, which is schematically. Let us consider the following three prey one predator model given by

$$\begin{aligned}
\tfrac{dN_1}{dt} &= aN_1\left(1-\tfrac{N_1}{k_1}\right)-\alpha\, N_1N_4\\
\tfrac{dN_2}{dt} &= bN_2\left(1-\tfrac{N_2}{k_2}\right)-\beta\, N_2N_4\\
\tfrac{dN_3}{dt} &= cN_3\left(1-\tfrac{N_3}{k_3}\right)-\gamma\, N_3N_4-eN_3+U\tau\, N_3\\
\tfrac{dx_3}{dt} &= \eta\, N_1N_4+\delta\, N_2N_4+\mu\, N_3N_4-fN_4-U\tau\, N_4
\end{aligned} \tag{1}$$

with initial densities

$$N_1(0) > 0,\; N_2(0) > 0,\; N_3(0) > 0,\; N_4(0) > 0. \tag{2}$$

Here, $\boldsymbol{N}_1(t), \boldsymbol{N}_2(t), \boldsymbol{N}_3(t)$ and $\boldsymbol{N}_4(t)$ denote the first prey, second prey, third prey and predator respectively and parameters are all positive.

Model parameters are described below:
a,b,c are the intrinsic growth rate of first, second, third prey respectively.
$\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3$ are the carrying capacities of first, second, third prey respectively.
$\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are the interaction of three preys (N_1, N_2, N_3) respectively by the predator.
e is the natural death of third prey population.
U denotes the control measures given in the above Figure 1.
$\boldsymbol{\tau}$ denotes the rise of immunity power either by vaccination or take natural medicine.
$\boldsymbol{\eta}, \boldsymbol{\delta}, \boldsymbol{\mu}$ are the conversion coefficient of predator from three preys population.
f is the natural death of predator.

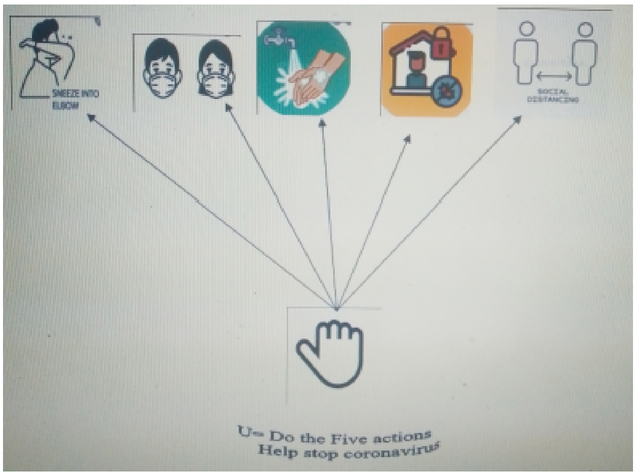

Figure 1. Control measure for Corona virus.

This model involves certain assumptions which consist of the followings:

(1) The first, second and third prey population the interaction take place by the predator.
(2) First and second prey population is recovered after the interaction taken place by the predator because of high immunity power from age group infant to 39 years.
(3) Lesslie-Gower model applied in the three prey population with intrinsic growth rate as a,b,c and carrying capacities as k_1, k_2, k_3 and there is no interaction among these three prey populations.
(4) Only the third prey population is affected by the virus because of the less immunity power, attained natural death. Also we raise the immunity power by introduce some constants $U\tau$ as control measure U (Figure 1) and vaccination or natural medicine as τ.

$$\tau = pepper, turmeric, cuminseeds, ginger, cardamon$$

(5) In third prey population after we applied the control measure as $U\tau$ then the affected prey become recovered due to raise the immunity power.

(6) Predator weakens in the first and second prey population but not in the third prey.
(7) A Lotka-Volterra functional response is taken to represent the interaction between prey and predator.

3. Positiveness and boundedness of theorem

In this section, we intend to establish the conditions to get positive as well as bounded solutions of the system.

3.1 *Positivity*

Theorem 3.1 *Every solution of system (1) with initial conditions (2) exists in the interval* $[0,\infty)$ *and* $N_1(t) > 0, N_2(t) > 0, N_3(t) > 0, N_4(t) > 0$, *for all* $t \geqslant 0$

Proof Since the right hand side of system (1) is completely continuous and locally Lipschitzian on C, the conditions (2) exists and is unique on $[0,\xi)$, where $0 < \xi \leqslant +\infty$ [11]. From system (1) with initial conditions (2), we have

$$
\begin{aligned}
N_1(t) &= N_1(0)\exp[\int_0^t \{a(1-\frac{N_1}{k_1}) - \alpha N_4\}dt] > 0,\\
N_2(t) &= N_2(0)\exp[\int_0^t \{b(1-\frac{N_2}{k_2}) - \beta N_4\}dt] > 0,\\
N_3(t) &= N_3(0)\exp[\int_0^t \{c(1-\frac{N_3}{k_3}) - \gamma N_4 - e + U\tau\}dt] > 0,\\
N_4(t) &= N_4(0)\exp[\int_0^t \{\eta N_1 + \delta N_2 + \mu N_3 - f - U\tau\}dt] > 0,
\end{aligned}
$$

which completes the proof. ■

3.2 *Boundedness*

Theorem 3.2 *Three preys are always bounded above for* $a > 0, k_1 > 0$.

Proof $If N_0 = 0$ then the result is trivial, if $N_1(0) > 0$ then $N_1(t) > 0$ for all t on adding equation (1) we obtain $\frac{dN_1}{dt} \leqslant aN_1(1-\frac{N_1}{k_1})$. Hence, $\lim_{t\to\infty}(\sup N_1(t)) \leqslant k_1$,. similarly, $\frac{dN_2}{dt} \leqslant bN_2(1-\frac{N_2}{k_2})$ and $\frac{dN_3}{dt} \leqslant cN_3(1-\frac{N_3}{k_3})$. Hence, $\lim_{t\to\infty}(\sup N_2(t)) \leqslant k_2$ and $\lim_{t\to\infty}(\sup N_3(t)) \leqslant k_3$. ■

Theorem 3.3 *Predator are always bounded above.*

Proof If $N_4(0) = 0$, the result is obvious. We obtain the equation (1). If $N_4(0) > 0$, then $\frac{dN_4}{dt} < 0$ if $d_1 N_4 > 1$. Thus, $\lim_{t\to\infty}(\sup N_3(t)) \leqslant \frac{1}{d_1}$. ■

Theorem 3.4 *The trajectories of system (1) are bounded.*

Proof Define the function $N = N_1 + N_2 + N_3 + N_4$ and take its time derivative along the solution of (1).

$$\frac{dN}{dt} = \frac{dN_1}{dt} + \frac{dN_2}{dt} + \frac{dN_3}{dt} + \frac{dN_4}{dt}.$$

Now

$$\begin{aligned}\frac{dN}{dt} + \rho N = aN_1(1 - \frac{N_1}{k_1}) + bN_2(1 - \frac{N_2}{k_2}) + cN_3(1 - \frac{N_3}{k_3})\\ - eN_3 - fN_4 + \rho N_1 + \rho N_2 + \rho N_3 + \rho N_4\\ = (\rho + a)N_1 + (\rho + b)N_2 + \frac{\rho + c}{e}N_3 + (\rho - f)N_4 - \frac{aN_1^2}{k_1} - \frac{bN_2^2}{k_2} - \frac{cN_3^2}{k_3}\end{aligned}$$

where ρ is a positive constant for $\rho > f$ given $\epsilon > 0$ there exists to such that t on $t \geqslant t_0$

$$\frac{dN}{dt} + \rho N \leqslant m + \epsilon, m = min\{(\rho + a), (\rho + b), (\frac{\rho + c}{e}), (\rho - f)\},$$

hence,

$$\frac{d}{dt}(Ne^{\rho t}) \leqslant (m + \epsilon)e^{\rho t} \Rightarrow N(t) \leqslant N(t_0)e^{-\rho(t-t_0)} + \frac{m + t}{\rho}(1 - e^{-\rho(t-t_0)}).$$

Letting $t \to 0$ then letting $\epsilon \to 0$,

$$\lim_{t\to\infty}(\sup N(t)) \leqslant \frac{m}{\rho}.$$

On the initial conditions, the system (1) are bounded. ■

4. Existence and stability analysis of equilibrium points

In this section we will study the existence and stability behaviour of the system (1) at equilibrium points [11].

- The trivial equilibrium point $E1(0, 0, 0, 0)$ is saddlepoint.
- The axial equilibrium points $E3(k_1, 0, 0, 0)$ and $E6(0, k_2, 0, 0)$ are saddle point.
- The planar equilibrium points $E2(0, 0, \Theta_1, \Theta_2)$, $E4(\Theta_3, 0, \Theta_4, \Theta_5)$, $E9(0, \Theta_{10}, \Theta_{11}, \Theta_{12})$ and $E10(\Theta_{13}, \Theta_{14}, 0, \Theta_{15})$ are unstable.
- The planar equilibrium of $E5(\Theta_6, 0, 0, \Theta_7), E7(k_1, k_2, 0, 0), andE8(0, \Theta_8, 0, \Theta_9)$ are locally asymptotically stable.
  We will analyze the equilibrium point $E5$, $E7$, $E8$ and interior equilibrium point $E11$.

The jacobian matrix of the system (1) at equilibrium point $E = (N_1(t), N_2(t), N_3(t), N_4(t))$ is given by

$$J_{11} = \begin{bmatrix} m_{11} & 0 & 0 & -N_1\alpha \\ 0 & m_{22} & 0 & -N_2\beta \\ 0 & m_{32} & m_{33} & -N_3\gamma \\ N_4\eta & N_4\delta & N_4\mu & m_{44} \end{bmatrix},$$

where $m_{11} = a - 2\frac{aN_1}{k_1} - \alpha N_4$, $m_{22} = b - 2\frac{bN_2}{k_2} - \beta N_4$, $m_{32} = -\frac{N_3 c}{k_3}$, $m_{33} = c - \frac{cN_2}{k_3} - \gamma N_4 - e + U\tau$, $m_{44} = -U\tau + \delta N_2 + \eta N_1 + \mu N_3 - f$.
Based on the nature of eigen values, the dynamical system (1) gets stable when all four eigen values are negative in case of real roots or negative real parts in case of complex roots of the characteristic equation for the above jacobian matrix. otherwise the dynamical system is unstable.

Theorem 4.1 *The dynamical system (1) is locally asymptotically stable at equilibrium point* $E5\,\{N_1 = \Theta_6, N_2 = 0, N_3 = 0, N_4 = \Theta_7\}$ *provided that* $\lambda_1 < 0$, $\lambda_2 < 0$ *with the conditions* $a\beta\eta k_1 > Ua\beta\tau + \alpha b\eta k_1 + a\beta f$,
$a\eta\gamma k_1 + \alpha\eta e k_1 > U\alpha\eta j k_1 + Ua\gamma\tau + \alpha\eta c k_1 + af\gamma$ *and* λ_3, λ_4 *have negative real parts.*

Proof The jacobian matrix is

$$E5 = \begin{bmatrix} a - 2\frac{a(U\tau+f)}{\eta k_1} + \frac{a(U\tau-\eta k_1+f)}{\eta k_1} & 0 & 0 & -\frac{(U\tau+f)\alpha}{\eta} \\ 0 & b + \frac{a\beta(U\tau-\eta k_1+f)}{\alpha\eta k_1} & 0 & 0 \\ 0 & 0 & c + \frac{\gamma a(U\tau-\eta k_1+f)}{\alpha\eta k_1} - e + U\tau & 0 \\ -\frac{a(U\tau-\eta k_1+f)}{\alpha k_1} & -\frac{a(U\tau-\eta k_1+f)\delta}{\alpha\eta k_1} & -\frac{a(U\tau-\eta k_1+f)\mu}{\alpha\eta k_1} & 0 \end{bmatrix}.$$

Now the eigenvalues are

$$\lambda_1 = \frac{Ua\beta\tau - a\beta\eta k_1 + \alpha b\eta k_1 + a\beta f}{\alpha\eta k_1},$$

$$\lambda_2 = \frac{U\alpha\eta\tau k_1 + Ua\gamma\tau - a\eta\gamma k_1 + \alpha c\eta k_1 - \alpha e\eta k_1 + af\gamma}{\alpha\eta k_1},$$

$$\lambda_3 = 1/2\,\frac{-Ua\tau - fa + \sqrt{\Theta_{11}}}{\eta k_1}, \quad \lambda_4 = -1/2\,\frac{Ua\tau + fa + \sqrt{\Theta_{11}}}{\eta k_1},$$

where

$$\Theta_{11} = 4U^2a\eta\tau^2k_1 - 4Ua\eta^2\tau{k_1}^2 + U^2a^2\tau^2 + 8Ua\eta f\tau k_1 - 4a\eta^2 f{k_1}^2 + 2Ua^2f\tau + 4a\eta f^2k_1 + a^2f^2,$$

hence, the above equilibrium point is locally asymptotically stable if $a\beta\eta k_1 > Ua\beta\tau + \alpha b\eta k_1 + a\beta f, a\eta\gamma k_1 + \alpha\eta e k_1 > U\alpha\eta j k_1 + Ua\gamma\tau + \alpha\eta c k_1 + af\gamma$ and λ_3, λ_4 have negative real parts. ■

Theorem 4.2 *The dynamical system (1) is locally asymptotically stable at equilibrium point* $E7\,\{N_1 = k_1, N_2 = k_2, N_3 = 0, N_4 = 0\}$ *provided that* λ_1, λ_2 *have negative real parts and* $\lambda_3 < 0, \lambda_4 < 0$ *with the conditions* $ck_2 + ek_3 > U\tau k_3 + ck_3$, $U\tau + f > \delta k_2 + \eta k_1$.

Proof The jacobian matrix is

$$E7 = \begin{bmatrix} -a & 0 & 0 & -k_1\alpha \\ 0 & -b & 0 & -k_2\beta \\ 0 & 0 & c - \frac{ck_2}{k_3} - e + U\tau & 0 \\ 0 & 0 & 0 & -U\tau + \delta\, k_2 + \eta\, k_1 - f \end{bmatrix}.$$

Now the eigenvalues are $\lambda_1 = -a < 0$, $\lambda_2 = -b < 0$, $\lambda_3 = \frac{U\tau\, k_3 - ck_2 + ck_3 - ek_3}{k_3}$, $\lambda_4 = -U\tau + \delta\, k_2 + \eta\, k_1 - f$. Hence, the above equilibrium point is locally asymptotically stable if λ_1, λ_2 have negative real parts and $\lambda_3 < 0, \lambda_4 < 0$ satisfies $ck_2 + ek_3 > U\tau k_3 + ck_3$, $U\tau + f > \delta k_2 + \eta k_1$. ■

Theorem 4.3 *The dynamical system (1) is locally asymptotically stable at equilibrium point* $E8\,\{N_1 = 0, N_2 = \Theta_8, N_3 = 0, N_4 = \Theta_9\}$ *provided that* $\lambda_1 < 0$, $\lambda_2 < 0$ *with the conditions* $\alpha\delta bk_2 > U\alpha b\tau + \beta\delta ak_2 + \alpha bf$, $U\beta\tau ck_2 + \delta\gamma bk_2k_3 + ek_2k_3\beta\delta + cfk_2\beta > Uk_2k_3\beta\delta\tau + Ubk_3\gamma\tau + ck_2\beta\delta k_3 + bfk_3\gamma$ *and* λ_3, λ_4 *have negative real parts.*

Proof The jacobian matrix is

$$E8 = \begin{bmatrix} a + \frac{\alpha\, b(U\tau - \delta\, k_2 + f)}{\beta\,\delta\, k_2} & 0 & 0 & 0 \\ 0 & b - 2\,\frac{b(U\tau + f)}{\delta\, k_2} + \frac{b(U\tau - \delta\, k_2 + f)}{\delta\, k_2} & 0 & -\frac{(U\tau + f)\beta}{\delta} \\ 0 & 0 & c - \frac{c(U\tau + f)}{\delta\, k_3} + \frac{\gamma\, b(U\tau - \delta\, k_2 + f)}{\beta\,\delta\, k_2} - e + U\tau & 0 \\ -\frac{b(U\tau - \delta\, k_2 + f)\eta}{\beta\,\delta\, k_2} & -\frac{b(U\tau - \delta\, k_2 + f)}{k_2\beta} & -\frac{b(U\tau - \delta\, k_2 + f)\mu}{\beta\,\delta\, k_2} & 0 \end{bmatrix}.$$

Now the eigenvalues are

$$\lambda_1 = \frac{U\alpha b\tau + a\beta\delta k_2 - \alpha b\delta k_2 + \alpha bf}{\beta\,\delta\, k_2},$$

$$\lambda_2 = \frac{U\beta\delta\tau k_2k_3 + Ub\gamma\tau k_3 - U\beta c\tau k_2 - b\delta\gamma k_2k_3 + \beta c\delta\, k_2k_3 - \beta\delta ek_2k_3 + bf\gamma k_3 - \beta cfk_2}{\beta\delta k_2k_3},$$

$$\lambda_3 = 1/2\frac{-U\tau b - bf + \sqrt{\Theta_{12}}}{\delta k_2}, \quad \lambda_4 = -1/2\frac{U\tau b + bf + \sqrt{\Theta_{12}}}{\delta k_2},$$

where

$$\Theta_{12} = 4U^2b\delta\tau^2k_2 - 4Ub\delta^2\tau{k_2}^2 + U^2b^2\tau^2 + 8Ub\delta f\tau k_2 - 4b\delta^2f{k_2}^2 + 2Ub^2f\tau + 4b\delta f^2k_2 + b^2f^2,$$

Hence, the above equilibrium point is locally asymptotically stable if $\alpha\delta bk_2 > U\alpha b\tau + \beta\delta ak_2 + \alpha bf$, $U\beta\tau ck_2 + \delta\gamma bk_2k_3 + \beta\delta ek_2k_3 + \beta cfk_2 > U\beta\delta\tau k_2k_3 + Ub\gamma\tau k_3 + \beta\delta ck_2k_3 + bf\gamma k_3$ and λ_3, λ_4 have negative real parts. ■

Theorem 4.4 *The interior equilibrium point* ($N_1 = N_1^*$, $N_2 = N_2^*$, $N_3 = N_3^*$, $N_4 = N_4^*$) *is locally asymptotically stable.*

Proof The variation of the Jacobian matrix is

$$J_{E11} = \begin{bmatrix} m_{11} & 0 & 0 & -N_1^*\alpha \\ 0 & m_{22} & 0 & -N_2^*\beta \\ 0 & m_{32} & m_{33} & -N_3^*\gamma \\ N_4^*\eta & N_4^*\delta & N_4^*\mu & m_{44} \end{bmatrix},$$

where $m_{11} = a - 2\frac{aN_1^*}{k_1} - \alpha N_4^*$, $m_{22} = b - 2\frac{bN_2^*}{k_2} - \beta N_4^*$, $m_{32} = -\frac{N_3^* c}{k_3}$, $m_{33} = c - \frac{cN_2^*}{k_3} - \gamma N_4^* - e + U\tau$, and $m_{44} = -U\tau + \delta N_2^* + \eta N_1^* + \mu N_3^* - f$. The characteristic equation is $\Lambda(\lambda) = \lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda^1 + A_0$ where

$A_3 = -(m_{11} + m_{22} + m_{33} + m_{44}) = 4I - (m_{11} + m_{22} + m_{33} + m_{44})$,

$A_2 = 5I^2 - (3m_{11} + 3m_{22} + 3m_{33} + 3m_{44})I + m_{22}m_{33} + m_{22}m_{44} + m_{33}m_{44} + m_{11}m_{22} + m_{11}m_{33} + m_{11}m_{44} - N_3^*N_4^*\mu\gamma + N_1^*N_4^*\eta\alpha$,

$A_1 = 4I^3 - (3m_{11} + 3m_{22} + 3m_{33} + 3m_{44})I^2 + (2m_{22}m_{33} + 2m_{22}m_{44} + 2m_{33}m_{44} + 2m_{22}m_{11} + 2m_{33}m_{11} + 2m_{44}m_{11} + 2N_4^*N_3^*\gamma\mu + 2N_1^*N_4^*\alpha\eta + N_2^*N_4^*\delta\beta)I - (m_{22}m_{33}m_{44} + m_{22}m_{33}m_{11} + m_{22}m_{44}m_{11} + m_{33}m_{44}m_{11} + N_3^*N_4^*m_{22}\gamma\mu + N_3^*N_4^*m_{11}\gamma\mu + N_2^*N_4^*m_{11}\delta\beta + N_1^*N_4^*m_{22}\alpha\eta + N_1^*N_4^*m_{33}\alpha\eta)$,

$A_0 = I^4 - (m_{11} + m_{22} + m_{33} + m_{44})I^3 + (m_{11}m_{22} + m_{11}m_{33} + m_{11}m_{44} + m_{22}m_{33} + m_{33}m_{44} + m_{22}m_{44} - N_3^*N_4^*\gamma\mu - N_2^*N_4^*\beta\delta - N_1^*N_4^*\alpha\eta)I^2 - (m_{11}m_{22}m_{33} + m_{11}m_{22}m_{44} + m_{11}m_{33}m_{44} + m_{22}m_{33}m_{44} + m_{11}N_3^*N_4^*\gamma\mu + m_{22}N_3^*N_4^*\gamma\mu + m_{11}N_2^*N_4^*\beta\delta + m_{33}N_2^*N_4^*\beta\delta + m_{32}N_2^*N_4^*\beta\mu + m_{22}N_1^*N_4^*\alpha\eta + m_{33}N_1^*N_4^*\alpha\eta)I + m_{11}m_{22}m_{33}m_{44} - m_{11}m_{22}N_3^*N_4^*\gamma\mu - m_{11}m_{33}N_2N_4\beta\delta - m_{11}m_{32}N_2^*N_4^*\beta\mu - m_{22}m_{33}N_1^*N_4^*\alpha\eta$.

By Routh Hurwitzs criterion, all the eigenvalues of j_{11} have negative real parts if

(i) $A_3 > 0$,
(ii) $A_3A_2 > A_1$,
(iii) $A_3A_2A_1 > A_1^2 + A_3^2A_0$.

Therefore the given system of the nonlinear differential equation (1) is locally asymptotically stable around the positive equilibrium point $N_1 = N_1^*$, $N_2 = N_2^*$, $N_3 = N_3^*$, and $N_4 = N_4^*$ if the conditions stated in the theorem holds. ■

4.1 *Global stability analysis*

We perform a global analysis of the system (1) around the positive equilibrium point $G(N_1^*, N_2^*, N_3^*, N_4^*)$ of the coexistence. The following theorem of Lyapunov function V is considered [6].

Theorem 4.5 *Let* $V=\frac{1}{2}(N_1 - N_1^*)^2 + \frac{1}{2}\psi_1(N_2 - N_2^*)^2 + \frac{1}{2}\psi_2(N_3 - N_3^*)^2 + \frac{1}{2}\psi_3(N_4 - N_4^*)^2$ *where* $\psi_1, \psi_2, \psi_3 > 0$ *are to be carefully chosen such that* $V'(G) = 0$ *then* $G(N_1^*, N_2^*, N_3^*, N_4^*)$ *and* $V = (N_1, N_2, N_3, N_4) > 0$ $\forall N_1, N_2, N_3, N_4|G$. *If the time derivative of* V *is* $\frac{dV}{dt} \leqslant 0$, $\forall N_1, N_2, N_3, N_4 \in \Gamma^+$ *then it follows that* $\frac{dV}{dt} = 0$, $\forall N_1, N_2, N_3, N_4 \in \Gamma^+$ *implies that* G^* *of the system is Lyapunov stable and* $\frac{dV}{dt} < 0$ $\forall N_1, N_2, N_3, N_4 \in \Gamma^+$ *near implies that* G^* *is globally stable.*

Proof $\frac{dV}{dt} = (N_1 - N_1^*)\frac{dN_1}{dt} + \psi_1(N_2 - N_2^*)\frac{dN_2}{dt} + \psi_2(N_3 - N_3^*)\frac{dN_3}{dt} + \psi_3(N_4 - N_4^*)\frac{dN_4}{dt}$
Now by substituting the model equations (1) we get,
$\frac{dV}{dt} = (N_1 - N_1^*)\{aN_1 - \frac{aN_1{}^2}{k_1} - \alpha N_1N_4\} + \psi_1(N_2 - N_2^*)\{bN_2 - \frac{bN_2{}^2}{k_2} - \beta N_4N_2\} + \psi_2(N_3 - N_3^*)\{cN_3 - \frac{cN_3^2}{k_3} - \gamma N_4N_3 - eN_3 + U\tau N_3\} + \psi_3(N_4 - N_4^*)\{\eta N_1N_4 + \delta N_2N_4 + \mu N_3N_4 - fN_4 - U\tau N_4\}$.
The above equation becomes
$\frac{dV}{dt} = (N_1 - N_1^*)\{a - \frac{aN_1}{k_1} - \alpha N_4\}\{(N_1 - N_1^*)\} + \psi_1(N_2 - N_2^*)$
$\{b - \frac{bN_2}{k_2} - \beta N_4\}\{(N_2 - N_2^*)\} + \psi_2(N_3 - N_3^*)\{c - \frac{cN_3}{k_3} - \gamma N_4 - e + U\tau\}\{(N_3 - N_3^*)\} + \psi_3(N_4 - N_4^*)\{\eta N_1 + \delta N_2 + \mu N_3 - f - U\tau\}\{(N_4 - N_4^*)\}$.

By rearranging we obtain,
$\frac{dV}{dt} = -(N_1 - N_1^*)^2\{a - \frac{aN_1}{k_1} - \alpha\, N_4\} - \psi_1(N_2 - N_2^*)^2\{b - \frac{bN_2}{k_2} - \beta\, N_4\} - \psi_2(N_3 - N_3^*)^2\{c - \frac{cN_3}{k_3} - \gamma\, N_4 - e + U\tau\} - \psi_3(N_4 - N_4^*)^2\{\eta\, N_1 + \delta\, N_2 + \mu\, N_3 - f - U\tau\}$.
Thus it is possible to set $\psi_1, \psi_2, \psi_3 > 0$ such that $V' \leqslant 0$ is an endemic positive equilibrium point is globally stable. Therefore it is noted that the parameters play an important roles in controlling the stability aspects of the system [7]. ■

5. Hopf bifurcation

In this section we investigate the Hopf bifurcation around the interior equilibrium point $E11$. The parameter c is basic and represent the growth rate of prey N_3 is identified as a bifurcation parameter. Hopf bifurcation occurs provided the jacobian matrix $J(E11)$ has a pair of purely imaginary eigenvalues and the other eigenvalues have negative real parts and $RE[\frac{d\lambda}{dc}]|_{c=c_0} \neq 0$ [5]. Assume that the characteristic equation at the interior equilibrium point E11 is as follows:

$$\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda^1 + A_0 = 0. \tag{3}$$

For purely imaginary eigenvalues, it is clear that coefficients of characteristic polynomial (4) must satisfy the following condition:

$$A_3A_2A_1 - A_3^2A_0 - A_1^2 = 0.$$

Suppose $\pm i\omega$ is a pair of purely imaginary eigenvalues corresponding to c_0. We derive from the characteristic equation (4) relative to c

$$[4\lambda^3 + 3A_3\lambda^2 + 2A_2\lambda + A_1]\frac{d\lambda}{dc} + (\lambda^3\frac{dA_3}{dc} + \lambda^2\frac{dA_2}{dc} + \lambda\frac{dA_1}{dc} + \frac{dA_0}{dc}) = 0,$$

hence,

$$\frac{d\lambda}{dc} = -(\frac{\lambda^3\frac{dA_3}{dc} + \lambda^2\frac{dA_2}{dc} + \lambda\frac{dA_1}{dc} + \frac{dA_0}{dc}}{4\lambda^3 + 3A_3\lambda^2 + 2A_2\lambda + A_1}). \tag{4}$$

We substitute $i\omega$ in to equation (5), we have

$$\frac{d\lambda}{dc}|_{i\omega} = -(\frac{-i\omega^3\frac{dA_3}{dc} + -\omega^2\frac{dA_2}{dc} + i\omega\frac{dA_1}{dc} + \frac{dA_0}{dc}}{-4i\omega^3 - 3A_3\omega^2 + 2A_2i\omega + A_1}),$$

hence,

$$Re(\frac{d\lambda}{dc}|_{i\omega}) = -(\frac{[A_1 - 3A_3\omega^2][\frac{dA_0}{dc} - \omega^2\frac{dA_2}{dc}] + [2A_2\omega - 4\omega^3][\omega^3\frac{dA_3}{dc} - \omega\frac{dA_1}{dc}]}{[A_1 - 3A_3\omega^2]^2 + [2A_2\omega - 4\omega^3]^2}).$$

Theorem 5.1 *Consider parameter c as bifurcation parameter. System (1) undergoes a Hopf bifurcation provided*

$$([A_1 - 3A_3\omega^2][\frac{dA_0}{dc} - \omega^2\frac{dA_2}{dc}] + [2A_2\omega - 4\omega^3][\omega^3\frac{dA_3}{dc} - \omega\frac{dA_1}{dc}]) \neq 0.$$

6. Numerical solution

The system of the nonlinear differential equation (1) for the numerical solution.

(1) First we take the parameter of the system as $(a, b, c, \alpha, \beta, k_1, k_2, k_3, U, \delta, \eta, \mu, \tau, \gamma, e, f) = (1.23, 1.05, 1.450, 2.236, 1.05, 1.25, 0.23, 0.005, 0.545, 1.236, 0.212, 2.024, 1.07, 2.1, 2.05, 2.11)$ at the population $(N_1, N_2, N_3, N_4) = (1.25, 10.46, 10.056, 10.13)$ The given system is asymptotically stable.

(2) First we take the parameter ρ of the system as $(a, b, c, \alpha, \beta, k_1, k_2, k_3, U, \delta, \eta, \mu, \tau, \gamma, e, f) = (1.3, 1.05, 1.450, 2.236, 1.05, 1.25, 0.23, 0.005, 0.545, 1.236, 0.212, 2.024, 1.07, 2.1, 2.05, 2.11)$ at the population $(N_1, N_2, N_3, N_4) = (2.35, 3.46, 2.06, 3.13)$. The given system is asymptotically stable.

(3) In Figure 4, we take the parameter of the system as mentioned above in point (1). Then the initial condition satisfies with $(N_1, N_2, N_3, N_4) = ((1.42, 0.1, 1, 1), (1.3, 0.6, 0.7, 0.2), (1.46, 1.7, 0.1), (1.32, 1.46, 1.7, 1.2))$ the contact rate of prey-predator interaction. If we interact the predator with first prey, only prey 1 and prey 3 will be moving away from predator because of control measure.

(4) In Figure 5, we take the parameter of the system as mentioned above in point (1). Then the initial condition satisfies with $(N_1, N_2, N_3, N_4) = ((1.42, 1.1, 1, 1), (1.3, 1.6, 0.7, 0.2), (1.32, 1.46, 1.7, 0.1), (0.32, 0.46, 1.7, 1))$ the contact rate of prey-predator interaction. If we interact the predator with second prey, only prey 1 will be moving away from predator, prey 2 will not affect by the predator because it was absent. But prey 3 will be moving away from predator after some effort will be taken because of control measure $(U\epsilon)$.

(5) In Figure 6, we take the parameter of the system as mentioned above in point (1). Then the initial condition satisfies with $(N_1, N_2, N_3, N_4) = ((1.42, 0.1, 1, 1), (1.3, 0.6, 0.7, 0.2), (1.32, 1.46, 1.7, 0.1), (1.32, 1.46, 1.7, 1))$ the contact rate of prey-predator interaction. If we interact the predator with third prey, prey 1 and prey 3 will be moving away from predator, after we applied the control measure predator interaction was absent.

(6) In Figure 7, we take the parameter of the system as mentioned above in point (1). Then the initial condition satisfies with $(N_1, N_2, N_3, N_4) = ((1.42, 0.1, 1, 1), (1.3, 0.6, 0.7, 0.2), (1.32, 1.46, 1.7, 0.1), (1.32, 1.46, 1.7, 1))$ the contact rate of prey-predator interaction. If we interact the predator with third prey then the prey 1 and prey 2 was absent only prey 3 will be appear, the result tends to a periodic.

7. Discussions and Conclusions

In this paper, we developed an eco-epidemiological of three prey one predator model with disease spread in the third prey population exhibits very interesting dynamics. Here we assumed that all the three preys grows logistically only the third prey is not capable of reproduction. So all our important analytical findings are numerically verified using mapple. The system with time delay undergoes a Hopf bifurcation around E^* at $c = c^*$ taking time delay c as bifurcation parameter.

Finally, we conclude that our system of three prey one predator model only the third prey disease will spread. After we take some control measure to reduce the spread of disease. We also noted that the control measure $U\tau$ plays a key role to

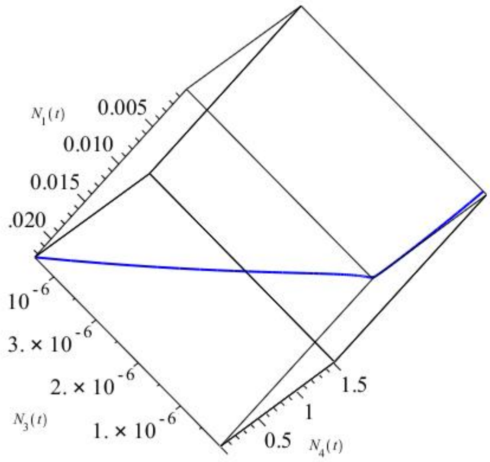

Figure 2. Rossler type of prey-predator is asymptotically stable.

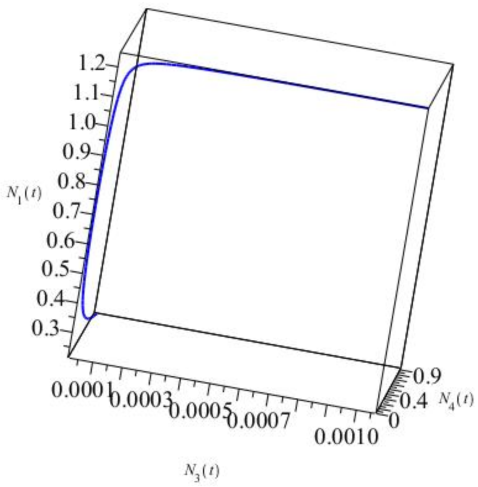

Figure 3. Rossler type of prey-predator is asymptotically stable.

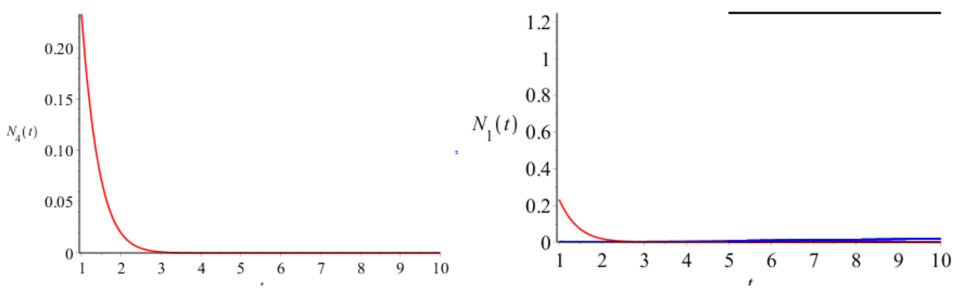

Figure 4. Rossler type of predator interact in the first prey.

control the stability of populations. There must be some time lag called the raise of immunity power where steps to be taken by the control measure during this process. At finally the result is periodic with the control measure of $U\tau$ applied in the third prey.

The equations may exhibit chaotic oscillation and chaotis attractor that may arise due to Rossler system. Hence we recommend that the predator will spread in the third prey population should be controlled by using the control measures to

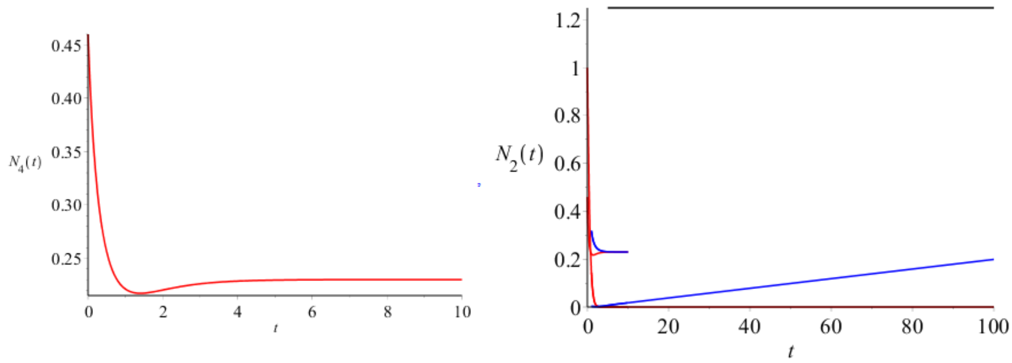

Figure 5. Rossler type of predator interact in the second prey.

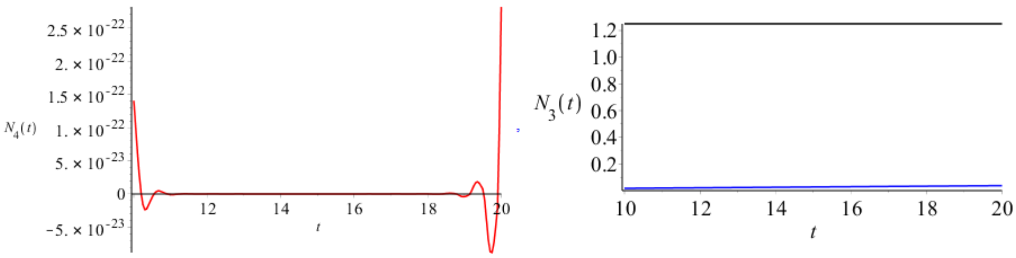

Figure 6. Rossler type of predator interact in the third prey.

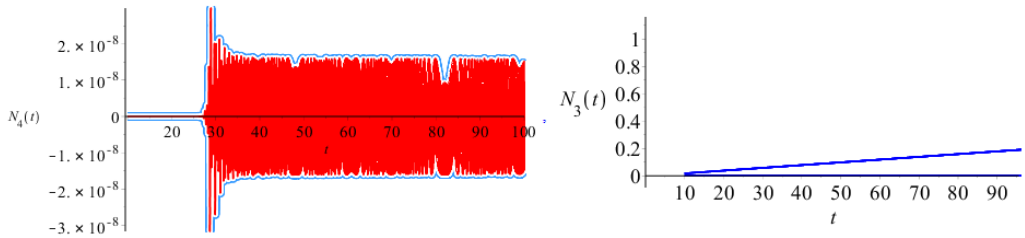

Figure 7. Rossler type of predator interact in the third prey with periodic solution.

avoid the infection and disease spread.

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