International Journal of
Mathematical Modelling & Computations
Vol. 11, No. 04, Fall 2021, 1- 37

Stability Analysis of a Staged Progression Susceptibility Model for Infectious Diseases

J. P. II Kouenkam[a,b], G. Chendjou[a], J. Mbang[a,b,c,*] and Y. Emvudu[a,b,c]

[a]*Laboratory of Applied Mathematics, Department of Mathematics, Faculty of Science, University of Yaounde I, P.O. Box 812 Yaounde, Cameroon,*
[b]*IRD, Sorbonne University, UMMISCO, F-93143, Bondy, France,*
[c]*LIRIMA, Project Team GRIMCAPE, Yaounde, Cameroon.*

Abstract. The aim of this paper is to provide a stability analysis for models with a general structure and mass action incidence; which include stage progression susceptibility, differential infectivity as well, and the loss of immunity induced by the vaccine also. We establish that the global dynamics are completely determined by the basic reproduction number R_0. More specifically, we prove that when R_0 is smaller or equal to one, the disease free equilibrium is globally asymptotically stable; while when it is greater than one, there exist a unique endemic equilibrium. We also provide sufficient conditions for the global asymptotic stability of the endemic equilibrium.

Received: 03 June 2021, Revised: 04 September 2021, Accepted: 13 September 2021.

Keywords: Differential susceptibility; Differential infectivity; Waning vaccine induced immunity; Lyapunov methods.

Index to information contained in this paper

1. Introduction

Infectivity or infectiousness can vary greatly in time for diseases that progress through a long infectious period. Among infected and infectious persons, amelioration of the disease state can occurs. It can result from a host's immune action or more commonly from treatment. On the other hand, heterogeneity in the sensitive host population can be the result of many factors such as genetic variations, social

*Corresponding author. Email: mbangjoh@yahoo.fr

behaviors, immunization states. For instance, the hepatitis B vaccine is gradually effective, according to the number of doses of vaccine already taken. As a result, vaccinated people can still contract the disease and susceptibility varies during the vaccination process. There have been several studies of models with a general structure including variable infectiousness and infectivity. But little study on the importance of variable susceptibility.

The global dynamics of a mathematical model for infectious diseases, which progress through separate steps, with various degrees of infectivity in infected hosts and possibility of improving was analyzed in [5]. The structure of this model can deal with various complex interactions between different classes of infectiousness. It can handle models with different classes of latency and infectiousness. Thus, this model encompassed the deterministic stage progression model in [4] and therefore the ones in [7]. Note that difference in susceptibility was not considered in [5].

In [6], compartmental differential susceptibility models in a general setting were considered. Homogeneous infectiousness of infected individuals was assumed, so that they were aggregated into one group of infected individuals. On the other side, the susceptible population was divided into multiple subgroups according to the susceptibility of individuals in each group. It was assumed that there was no flow between susceptibles subgroups. That is to say, each susceptible individual remained in his group until he is infected. In addition, the influx was supposed to be distributed into all the subgroups of susceptibles. Modified versions of models in [6] were studied in [2]. Therein, the infectiousness was considered non homogenous. Indeed, infected individuals were supposed to progress through distinct states of the disease. Moreover in [2], based on the ages of susceptible persons, a model including a stage progression (with flow among the susceptible compartments) among five classes of susceptible individuals was formulated to describe the dynamics of Hepatitis B virus infection.

Stage progression susceptibility models were also formulated to describe the dynamic of Hepatitis B virus infection in [9, 14]. Actually, in order to complete the vaccination process against Hepatitis B, three or four doses must generally be taken several times, and there are fixed time intervals between two successive doses. In these both papers, the differentiation in susceptibility therefore resulted from the fact that people who have not completed the vaccination process may still contract the disease. Considering the time to obtain immunity and the possibility to be infected before. However, they are less susceptible to develop the infection than fully susceptible persons. Such consideration was also present in [10, 12, 13], in which the efficiency of vaccine was assumed to be non-complete. It should be noted that, contrary to assumption that the vaccine induces immunity for life as regarded in [9, 14], some authors considered that immunity further to vaccination was not permanent when modelling the transmission of Hepatitis B virus [8, 18].

In this work, we consider stage progression susceptibility models in a general setting. We suppose that the influx is distributed in the subgroups of susceptible people, and we take into account flows between the infected and infectious compartments. In fact the model is constructed so that it encompasses the models in [2, 5, 9, 18]. Since it is presented in a general setting we add hypotheses for biological soundness. Our goal is to establish the global dynamics of the general model. The organization of the paper is as follows. In the next section, we derive a stage progression differential susceptibility and infectivity epidemic model that extends and generalizes the model in [9] in the absence of vertical transmission. We formulate a general differential susceptibility and infectivity epidemic model of which the form can be taken by the proposed model. Also, we establish its basic properties in section 3. Thereafter, in section 4, a simple formula for the basic reproduction ratio $\mathcal{R}_0$ is given, the global stability of the disease free equilibrium when $\mathcal{R}_0 \leqslant 1$ is proven and the existence and uniqueness of an endemic equilibrium when $\mathcal{R}_0 > 1$ is established. In section 5, we provide the expression of the disease free equilibrium of the proposed model, also some conditions under which its endemic equilibrium is globally asymptotically stable. Finally the work is summarized in section 6.

2. A stage progression susceptibility model with differential susceptibility

Suppose an infectious disease spreads in a population. Assume there is a vaccine against the infection and that the vaccine-induced immunity does not last lifelong. Suppose infected individuals are completely removed or isolated, or that they acquire lifelong total immunity after recovery. Assume, as it is generally considered, the infectiousness is independent of the susceptible classes. Furthermore, the susceptible individuals are taken into account on the basis of their inherent susceptibilities. In fact, the total host population is partitioned into : n groups of susceptible individuals S_i, $i = 1, 2, \ldots, n$; m groups of infectious/infected individuals I_j, $j = 1, 2, \ldots, m$; a group for successfully vaccinated persons V; a group for individuals who are removed from the infection process R.

Investigating transmission dynamics is our main interest. Hence we neglect demographic effects in the population. Assume that a constant influx $\tilde{\Lambda}$ is distributed into the n groups of susceptible people, such that Λ_i enter the group $S_i, i = 1, \cdots, n$, and $\sum_{i=1}^{n} \Lambda_i = \tilde{\Lambda}$. For any i, $1 \leqslant i \leqslant n-1$, $\theta_i \geqslant 0$, denotes the progression rate from the group S_i to the group S_{i+1}. Only individuals in the group S_n can get the status of vaccinated at a rate π. Vaccinated persons lose their immunity and return to the group S_n at a rate σ. As a matter of fact, for some diseases a boost dose of the vaccine is often necessary after a period has elapsed.

We model new infections using the mass-action law. $\alpha_i \in]0; 1]$, $i = 1, \cdots, n$, denotes the susceptibility of persons in the class S_i and $\beta_j \geqslant 0$, $j = 1, \cdots, m$, the infectiousness of persons in the class I_j. Hence, $\beta_{ij} := \alpha_i \beta_j$ is the transmission rate of I_j for S_i. As soon as susceptibles in a group S_i, are infected, they enter the infection group I_1 with infection rate $\lambda_i := \sum_{j=1}^{m} \beta_{ij} I_j$. Thus the total incidence is given by $\lambda := \sum_{i=1}^{n} \lambda_i S_i$. From the group I_1 they progress through a series of stages I_j, $j = 2, \cdots, m$, with the possibility of recovering at the rate $\gamma_j \geqslant 0$ at each stage. For $1 \leqslant j \leqslant m-1$, $k_j > 0$ denotes the progression rate from the jth stage to the $(j+1)$th stage, and $\delta_{i+1} \geqslant 0$ denotes the amelioration rate from the $(j+1)$th stage to the jth stage. Furthermore, infected individuals from the group I_{m-2} can directly progress to the group I_m at a rate $k'_{m-2} \geqslant 0$, and those in the group I_m can directly return in the group I_{m-2} at a rate $\delta'_m \geqslant 0$.

Moreover we take into account the differences between the death rates. Thus, for $i = 1, \cdots, n$, μ_i denotes the death rate of the class S_i; and for $j = 1, \cdots, m$, ν_j denotes the one of the class I_j. Besides, μ_v and μ_r are respectively the death rates of peoples in compartments V and R. The population transfer among compartments is schematically depicted in the transfer diagram in Figure 1. The parameters are gathered in Table 1. For convenience, we set $m-2 = c$, $m-1 = p$ and $m = f$, in order to get closer to the notations used in [9]. In that paper, a compartmental model is proposed to study the propagation of the hepatitis B virus. One of the novel features of that model is that the population of carriers is divided into four groups, denoted (E), (C), (P) and (F).

Finally, we assume that the biologically relevant hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$ that follow are satisfied [2]. Notice that the accessibility is defined here according to the graph theory.

- **H$_1$** : Any "susceptible" compartment is accessible from a "susceptible" compartment with recruitment.
- **H$_2$** : Any infected-infectious compartment is accessible from at least one compartment which is an "entry-point" for infection.

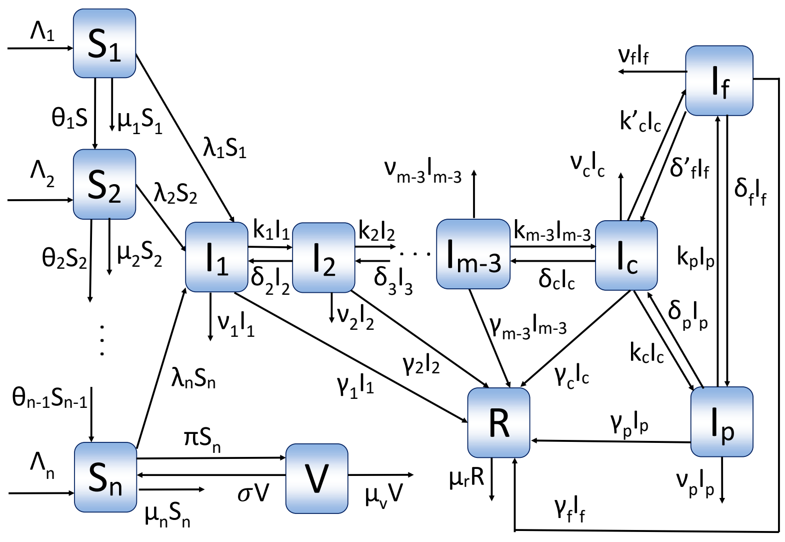

Figure 1. Diagram transmission of a general model with one vaccinated class, one removed class, n susceptible classes and m infected/infectious classes. The transitions between the compartments (C), (P) and (F) in [9] justify the triangular shape in this transfer diagram.

The transmission dynamics of infection are governed by the following system of differential equations:

$$\begin{cases} \dot{S}_1 = \Lambda_1 - a_1 S_1 - \lambda_1 S_1, & \\ \dot{S}_i = \Lambda_i - a_i S_i + \theta_{i-1} S_{i-1} - \lambda_i S_i, & i = 2, \cdots, n-1 \\ \dot{S}_n = \Lambda_n + \sigma V - a_n S_n + \theta_{n-1} S_{n-1} - \lambda_n S_n, & \\ \dot{V} = \pi S_n - (\mu_v + \sigma) V, & \\ \dot{I}_1 = \sum_{i=1}^{n} \lambda_i S_i + \delta_2 I_2 - b_1 I_1, & \\ \dot{I}_j = k_{j-1} I_{j-1} + \delta_{j+1} I_{j+1} - b_j I_j, & j = 2, \cdots, m-3 \\ \dot{I}_c = k_{m-3} I_{m-3} + \delta_p I_p + \delta'_f I_f - b_c I_c, & \\ \dot{I}_p = k_c I_c + \delta_f I_f - b_p I_p, & \\ \dot{I}_f = k'_c I_c + k_p I_p - b_f I_f, & \\ \dot{R} = \sum_{j=1}^{m} \gamma_j I_j - \mu_r R, & \end{cases} \tag{1}$$

where

$a_i = \mu_i + \theta_i,\ i = 1, \cdots, n-1$; $a_n = \mu_n + \pi$; $b_j = k_j + \delta_j + \nu_j + \gamma_j,\ j = 2, \cdots, m-3$;
$b_1 = k_1 + \nu_1 + \gamma_1$; $b_c = k_c + k'_c + \delta_c + \nu_c + \gamma_c$; $b_p = k_p + \delta_p + \nu_p + \gamma_p$ and $b_f = \delta_f + \delta'_f + \nu_f + \gamma_f$.

Table 1. Description of parameters.

Parameters	**Description**
$\Lambda_i,\ i = 1, \cdots, n$	Recruitment rates of persons who enter the compartment S_i
$\theta_i,\ i = 1, \cdots, n-1$	Progression rate from the group S_i to the group S_{i+1}
$\alpha_i,\ i = 1, \cdots, n,$	Susceptibility of persons in the groupe S_i
$\beta_j,\ j = 1, \cdots, m$	Infectiousness of persons in the class I_j
$k_j,\ j = 1, \cdots, m-1$	Progression rate from the stage I_j to the stage I_{j+1}
$\gamma_j,\ j = 2, \cdots, m$	Recovering rate at the stage I_j
$\mu_i\ i = 1, \cdots, n$	Mortality rate of susceptible individuals in groupe S_i
$\nu_j,\ j = 1, \cdots, m$	Mortality rate of infected individuals in class I_j.
δ_{i+1}	Amelioration rate from the stage I_{j+1} to the stage I_j
k'_{m-2}	Rate of direct progression from the stage I_{m-2} to the stage I_m
δ'_m	Amelioration rate from the stage I_m to the stage I_{m-2}
π	Vaccination rate of individuals in the groupe S_n
σ	Rate of waning vaccine-induced immunity
μ_v	Mortality rate of vaccinated persons
μ_r	Mortality rate of removed

In this paper, we use the bilinear mass action incidence to model new infections. However, The proofs of stability results we will establish are still true in both following cases. On the one hand, when new infections are modeled using the standard incidence mass action and the total population size N is constant, and on the other hand when the model deals with proportions instead of numbers of individuals. From the point of view of the structure of models, we have the following.

Remark 2.1.

- *If we consider only one class for susceptible individuals ($n = 1$), if we remove the class of vaccinated persons V ($\pi = \sigma = 0$) and the transitions between the compartments I_{m-2} et I_m ($k'_c = \delta'_f = 0$), and if we consider that only the infected persons in the last stage of infection can recover ($\gamma_j = 0,\ i = 1, \cdots, m-1$), then from Model (1) we get the stage progression infectivity model with amelioration in [5], therefore the one investigated in [4].*
- *When we remove the stage progression susceptibility ($\theta_i = 0,\ i = 1, \cdots, n-1$) and the class of vaccinated persons V, Model (1) corresponds to the situation where there is no flow between susceptible classes. Moreover, if we do not consider the amelioration of the state of infected persons ($\delta_j = 0,\ j = 2, \cdots, m$) and the transitions between the compartments I_{m-2} et I_m, then from Model (1) we obtain the differential susceptibility model studied in [2], and consequently that proposed in [6].*
- *When $\Lambda_i = 0,\ i = 2 \cdots, n$ and $\theta_i \neq 0,\ i = 1, \cdots, n-1$, Model (1) corresponds to the case where all the recruitment is in the first class of susceptibles S_1, and susceptible people progress successively through different stages. In addition, if we remove the class of vaccinated persons V, then Model (1) generalizes the one studied in [9] in the absence of vertical transmission.*

From now, the following notations will be used throughout this paper. We use the ordering in $\mathbb{R}^n$ generated by the cone $\mathbb{R}^n_+$. We will write: $x \leqslant y$, if $y - x \in \mathbb{R}^n_+$; $x < y$ if $x \leqslant y$ and $x \neq y$; $x \ll y$ if $x_i < y_i$ for all index i. For any vector x in $\mathbb{R}^n$, $diag(x)$ will denote the $n \times n$ diagonal matrix, whose diagonal elements are the components of x. We will denote by $\langle\ \mid\ \rangle$ the usual inner product on $\mathbb{R}^n$. Let $\{e_1, \cdots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. We will denote by $\mathbb{1}$ the vector given by $\mathbb{1} = (1, \cdots, 1)^t = e_1 + \cdots + e_n$, where the superscript t denotes transpose. Finally, we will use the same notations regarding $\mathbb{R}^m$.

Setting $S = (S_1, S_2, \ldots, S_n)$ and $I = (I_1, I_2, \ldots, I_m)$, the model (1) can be written under the form

$$\begin{cases} \dot{S} = \Lambda + \sigma V e_n + \tilde{A}_S S - \langle \beta \mid I \rangle diag(\alpha) S, \\ \dot{V} = \langle \eta, S \rangle - (\mu_v + \sigma) V, \\ \dot{I} = \langle \beta \mid I \rangle P diag(\alpha) S + \tilde{A}_I I, \\ \dot{R} = \langle \gamma, I \rangle - \mu_r R, \end{cases} \tag{2}$$

with:

$$\Lambda = (\Lambda_1, \Lambda_2, \cdots, \Lambda_n)^t,\ \eta = (0, 0, \cdots, 0, \pi)^t,\ \gamma = (\gamma_1, \gamma_2, \cdots, \gamma_m)^t,\ \mu = (\mu_1, \mu_2, \cdots, \mu_n)^t,\ \nu = (\nu_1, \nu_2, \cdots, \nu_m)^t,$$

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)^t,\ \beta = (\beta_1, \beta_2, \cdots, \beta_m)^t,\ \tilde{A}_S = -diag(\mu + \eta) + A_S,\ \tilde{A}_I = -diag(\nu + \gamma) + A_I.$$

P, A_S and A_I being respectively the $m \times n$, $n \times n$ and $m \times m$ matrices defined below

$$P = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_S = \begin{pmatrix} -\theta_1 & 0 & 0 & \cdots & 0 \\ \theta_1 & -\theta_2 & 0 & \cdots & 0 \\ 0 & \theta_2 & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & -\theta_{n-1} & 0 \\ 0 & \cdots & 0 & \theta_{n-1} & 0 \end{pmatrix},$$

$$A_I = \begin{pmatrix} -k_1 & \delta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_1 & -(k_2 + \delta_2) & \ddots & \ddots & \vdots & 0 & 0 & 0 \\ 0 & k_2 & \ddots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \delta_{m-3} & 0 & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & -(k_{m-3} + \delta_{m-3}) & \delta_c & 0 & 0 \\ 0 & \ddots & \ddots & 0 & k_{m-3} & -(k'_c + k_c + \delta_c) & \delta_p & \delta'_f \\ 0 & \ddots & 0 & 0 & 0 & k_c & -(k_p + \delta_p) & \delta_f \\ 0 & \cdots & 0 & 0 & 0 & k'_c & k_p & -(\delta_f + \delta'_f) \end{pmatrix}.$$

Since in System (2) the variable R only appear explicitly in its equation, we may reduce this system to the following equivalent system.

$$\begin{cases} \dot{S} = \Lambda + \sigma V e_n + \tilde{A}_S S - \langle \beta \mid I \rangle diag(\alpha) S, \\ \dot{V} = \langle \eta, S \rangle - (\mu_v + \sigma) V, \\ \dot{I} = \langle \beta \mid I \rangle P diag(\alpha) S + \tilde{A}_I I. \end{cases} \tag{3}$$

A_S is a compartmental Metzler matrix, whose column sums are zero, i.e., the sum of the elements of each column is zero, the same for A_I. It is obvious to see that $\tilde{A}_S$ and $\tilde{A}_I$ are also compartmental matrices. Furthermore, because $\mu \gg 0$ and $\nu \gg 0$, each susceptible compartment and each infected-infectious compartment is outflow-connected. Thus using Gershgorin theorem [16], it follows that the matrices $\tilde{A}_S$ and $\tilde{A}_I$ are asymptotically stable. Thus, the matrices $\tilde{A}_S$ and $\tilde{A}_I$ are Metzler and asymptotically stable. So when there is no transmission, the infected and the infectious disappear.

Set

$$_S = \tilde{A}_S + \frac{\sigma}{\mu_v + \sigma} e_n \eta^t.$$

It is obvious to show that $e_n\eta^t \geqslant 0$. Hence $_S$ is also a compartmental matrix. Furthermore, $_S$ is asymptotically stable. Indeed keeping in mind that q is a sthocastic vector, for all j, $j = 1, \dots, n$ the sum of all entries of the jth column of $_S$ is:

$$-(\mu_j + \eta_j) + \frac{\sigma}{\mu_v + \sigma} \sum_{i=1}^{n} q_i \eta_j = -(\mu_j + \eta_j) + \frac{\sigma}{\mu_v + \sigma} \eta_j = -\mu_j - \eta_j \left(\frac{\mu_v}{\mu_v + \sigma} \right) < 0.$$

We will use the following properties repeatedly in the sequel : a Metzler matrix M is stable if and only if $-M^{-1} > 0$. Also if a Metzler matrix M is stable, then $-M^{-1}x \gg 0$ for all $x \gg 0$.

3. Basic properties of the model

3.1 *Some useful lemmas*

Lemma 3.1. *Let a vector $x \in \mathbb{R}^n_+$ and the matrix*

$$M(x) :=_S - diag(x).$$

For any $i, j \in \mathbb{N}$, $i \neq j$, any $k \in \mathbb{N}$, $k \geqslant 2$, if $\left\langle A_S^p e_j \mid e_i \right\rangle = 0$ for all $p \in \mathbb{N}$, $1 \leqslant p \leqslant k-1$, then

$$\left\langle M(x)^k e_j \mid e_i \right\rangle = \left\langle A_S^k e_j \mid e_i \right\rangle.$$

Proof The proof is made by induction on k. Firstly, notice that for all vector a one has:

$$diag(a)e_i = a_i e_i \tag{4}$$

$$\langle diag(a) A_S e_j \mid e_i \rangle = \sum_{l=1}^{n} a_l \langle e_l \mid A_S e_j \rangle \langle e_l \mid e_i \rangle = a_i \langle A_S e_j \mid e_i \rangle \tag{5}$$

Besides, using the expression of $_S$, then the one of $\tilde{A}_S$, we can rewrite the matrix $M(x)$ as

$$M(x) = A_S - diag(\mu + \eta + x) + B,$$

where, keeping the definition of the vector η in mind,

$$B := \frac{\sigma}{\mu_v + \sigma} e_n \eta^t = diag\left(0, 0, \dots, \frac{\sigma\pi}{\mu_v + \sigma}\right) = \frac{\sigma\pi}{\mu_v + \sigma} diag(e_n).$$

Denote $(e_n)_j =$ the jth component of the vector e_n. We easily get

$$\langle Be_j \mid e_i \rangle = \frac{\sigma\pi(e_n)_j}{\mu_v + \sigma} \langle e_j \mid e_i \rangle. \tag{6}$$

Let us prove the claim for $k=2$. Assume that

$$\langle A_S e_j \mid e_i \rangle = 0.$$

We can easily show the following

$$diag(e_n)diag(e_n) = diag(e_n), \tag{7}$$

$$diag(\mu+\eta+x)diag(e_n) = diag(e_n)diag(\mu+\eta+x) = (\mu_n+\eta_n+x_n)diag(e_n). \tag{8}$$

Keeping relations (4), (5), (7), (8) and the last expression of B in mind, we have

$$\begin{aligned}
&\langle M(x)^2 e_j \mid e_i \rangle \\
&= \left\langle \left(A_S - diag(\mu+\eta+x) + \frac{\sigma\pi}{\mu_v+\sigma} diag(e_n) \right) \right. \\
&\qquad \left. \times \left(A_S - diag(\mu+\eta+x) + \frac{\sigma\pi}{\mu_v+\sigma} diag(e_n) \right) e_j \mid e_i \right\rangle \\
&= \left\langle A_S^2 e_j - (\mu_j+\eta_j+x_j) A_S e_j + \frac{\sigma\pi}{\mu_v+\sigma} A_S diag(e_n) e_j \right. \\
&\qquad \left. - (\mu_i+\eta_i+x_i) A_S e_j (\mu_j+\eta_j+x_j)^2 e_j \mid e_i \right\rangle \\
&+ \left\langle -2(\mu_n+\eta_n+x_n) \frac{\sigma\pi}{\mu_v+\sigma} diag(e_n) e_j + \frac{\sigma\pi}{\mu_v+\sigma} diag(e_n) A_S e_j \right. \\
&\qquad \left. + \left(\frac{\sigma\pi}{\mu_v+\sigma} \right)^2 diag(e_n) e_j \mid e_i \right\rangle.
\end{aligned}$$

Since $i \neq j$ and by hypothesis we respectively have

$$\langle e_j \mid e_i \rangle = 0 \quad \text{and} \quad \langle A_S e_j \mid e_i \rangle = 0. \tag{9}$$

Moreover, we have

$$\langle A_S diag(e_n) e_j \mid e_i \rangle = 0. \tag{10}$$

Because $A_S diag(e_n)$ is a matrix whose columns are zero, except its nth column which is the nth column of the matrix A_s. And this column is zero by the definition of A_s.

Also, denoting $(A_s)_{n,j}$ the coefficient of the nth line and the jth column of the matrix (A_s), we have

$$\langle diag(e_n) A_S e_j \mid e_i \rangle = \langle (A_S)_{n,j}\, e_j \mid e_i \rangle = (A_S)_{n,j} \langle e_j \mid e_i \rangle = 0. \tag{11}$$

Since $diag(e_n) A_S$ is a matrix whose lines are zero, except its nth line which is the nth line of the matrix A_s.

Likewise, $(e_n)_j$ being the jth component of the vector e_n, we have

$$\langle diag(e_n)e_j \mid e_i\rangle = \langle (e_n)_j e_j \mid e_i\rangle = (e_n)_j \langle e_j \mid e_i\rangle = 0. \tag{12}$$

Therefore, plugging (9), (10), (11) and (12) in the expanding expression of $\left\langle M(x)^2 e_j \mid e_i\right\rangle$, we get

$$\left\langle M(x)^2 e_j \mid e_i\right\rangle = \left\langle A_S^2 e_j \mid e_i\right\rangle.$$

Now, assume the truth of the statement for some $k \in \mathbb{N},\ k \geqslant 2$. Suppose,

$$\left\langle A_S^p e_j, e_i\right\rangle = 0, \quad \text{for all } p,\ 0 \leqslant p \leqslant k. \tag{13}$$

Since $M(x)^t e_i = \sum_{l=1}^{n} \langle M(x)e_l, e_i\rangle e_l$, we have

$$\begin{aligned}\left\langle M(x)^{k+1} e_j \mid e_i\right\rangle &= \left\langle M(x)^k e_j \mid M(x)^t e_i\right\rangle \\ &= \left\langle M(x)^k e_j \mid \sum_{l=1}^{n} \langle M(x)e_l, e_i\rangle e_l\right\rangle \\ &= \sum_{l=1}^{n} \langle M(x)e_l \mid e_i\rangle \left\langle M(x)^k e_j \mid e_l\right\rangle.\end{aligned}$$

By (13) we have

$$\left\langle A_S^p e_j, e_i\right\rangle = 0, \quad \text{for all } p,\ 0 \leqslant p \leqslant k-1.$$

Then by induction assumption, we get $\left\langle M(x)^k e_j \mid e_l\right\rangle = \left\langle A_S^k e_j \mid e_l\right\rangle$. Hence,

$$\left\langle M(x)^{k+1} e_j \mid e_i\right\rangle = \sum_{l=1}^{n} \langle M(x)e_l \mid e_i\rangle \left\langle A_S^k e_j \mid e_l\right\rangle.$$

Substituting $M(x) = A_S - diag(\mu + \eta + x) + B$ into the right hand side of above relation and using relations (4) and (6), we obtain

$$\begin{aligned}&\left\langle M(x)^{k+1} e_j \mid e_i\right\rangle \\ &= \sum_{l=1}^{n} \left(\langle A_S e_l \mid e_i\rangle - (\mu_l + \eta_l + x_l)\langle e_l \mid e_i\rangle + \frac{\sigma\pi}{\mu_v + \sigma}(e_n)_l \langle e_l \mid e_i\rangle \right) \left\langle A_S^k e_j \mid e_l\right\rangle \\ &= \sum_{l=1}^{n} \langle A_S e_l \mid e_i\rangle \left\langle A_S^k e_j \mid e_l\right\rangle \\ &\qquad + \sum_{l=1}^{n} \left(-(\mu_l + \eta_l + x_l) + \frac{\sigma\pi}{\mu_v + \sigma}(e_n)_l \right) \langle e_l \mid e_i\rangle \left\langle A_S^k e_j \mid e_l\right\rangle.\end{aligned}$$

Since $\langle e_i \mid e_l\rangle \neq 0$ only if $i = l$, reducing the second sum on the right, we obtain

$$\begin{aligned}&\left\langle M(x)^{k+1}e_j \mid e_i\right\rangle\\ &= \sum_{l=1}^{n} \langle A_S e_l \mid e_i\rangle \left\langle A_S^k e_j \mid e_l\right\rangle + \left(\frac{\sigma\pi}{\mu_v+\sigma}(e_n)_i - (\mu_i+\eta_i+x_i)\right)\left\langle A_S^k e_j \mid e_i\right\rangle.\end{aligned}$$

By (13), we have $\left\langle A_S^k e_j \mid e_i\right\rangle = 0$. It follows that

$$\left\langle M(x)^{k+1}e_j \mid e_i\right\rangle = \sum_{l=1}^{n} \langle A_S e_l \mid e_i\rangle \left\langle A_S^k e_j \mid e_l\right\rangle.$$

On the other hand, by some properties of the inner product, we have

$$\begin{aligned}\left\langle A_S^{k+1}e_j \mid e_i\right\rangle &= \left\langle A_S^k e_j \mid A_S^t e_i\right\rangle\\ &= \left\langle A_S^k e_j \mid \sum_{l=1}^{n} \langle A_S e_l \mid e_i\rangle e_l\right\rangle\\ &= \sum_{l=1}^{n} \langle A_S e_l \mid e_i\rangle \left\langle A_S^k e_j \mid e_l\right\rangle.\end{aligned}$$

Hence

$$\left\langle M(x)^{k+1}e_j \mid e_i\right\rangle = \left\langle A_S^{k+1}e_j \mid e_i\right\rangle.$$

Therefore by induction the claim is true for all $k \in \mathbb{N},\ k \geqslant 2$. ■

Lemma 3.2. *For any vector $x \in \mathbb{R}_+^n$, we have*

$$-\left[_S - diag(x)\right]^{-1}\Lambda \gg 0.$$

Lemma 3.3. *For any vector $c \geqslant 0$, we have*

$$-\left(\tilde{A}_I\right)^{-1} Pc \gg 0.$$

Using Lemma 3.1, the proof of Lemmas 3.2 and 3.3 are easy adaptations of the analogous ones in [2].

3.2 *A compact positively invariant absorbing set*

Proposition 3.1. ***(Positive invariance of the nonnegative orthant)*** *The nonnegative orthant $\mathbb{R}_+^{m+n+2}$ is positively invariant for the System (3).*

Proof System (3) can be rewritten in the following form:

$$\begin{cases}\dot{S} = \Lambda + \sigma V e_n + \tilde{A}_S S - \langle\beta \mid I\rangle diag(\alpha) S,\\ \dot{V} = \langle\eta, S\rangle - (\mu_v+\sigma)V,\\ \dot{I} = AI,\end{cases} \tag{14}$$

where: $A = Pdiag(\alpha)S\beta^t + \tilde{A}_I$.

On the boundary $S = 0$ the first equation of System (14) becomes $\dot{S} = \Lambda + \sigma V e_n \geqslant 0$ and on the boundary $V = 0$ the second equation of System (14) becomes $\dot{V} = \langle \eta, S\rangle \geqslant 0$. Therefore, for all $t > 0$ and initial data $S(0) \geqslant 0$ and $V(0) \geqslant 0$, the components S and V of the solutions of System (14) are positive, whenever they exist.

In addition, $\tilde{A}_I$ is a Metzler matrix, furthermore $Pdiag(\alpha)S\beta^t + \tilde{A}_I > 0$. Accordingly A is a Metzler matrix. It is well known that a linear Metzler matrix let invariant the nonnegative orthant. Thus the nonnegative orthant $\mathbb{R}^m_+$ is positively invariant for the system $\dot{I} = AI$. ■

Let $P(t) = \langle S \mid \mathbb{1}\rangle + V + \langle I \mid \mathbb{1}\rangle$. After replacing $\tilde{A}_S$ and $\tilde{A}_I$ by their respective expressions, and adding all equations in System (3), we find that

$$\dot{P} = \langle \Lambda \mid \mathbb{1}\rangle - \langle \mu \mid S\rangle - \mu_v V - \langle \nu \mid I\rangle. \tag{15}$$

Set: $\mu_0 = \min\{\mu_1, \cdots, \mu_n\}$ and $\nu_0 = \min\{\nu_1, \cdots, \nu_m\}$ and let φ_0 be defined as

$$\varphi_0 = \min\{\mu_0, \mu_v, \nu_0\}.$$

It follows from (15) that

$$\dot{P} \leqslant \langle \Lambda \mid \mathbb{1}\rangle - \varphi_0 P. \tag{16}$$

Lemma 3.4. ***(Boundedness and dissipativity of the trajectories)*** *For any small $\varepsilon > 0$, let*

$$\Omega_\varepsilon = \left\{(S, V, I) \in \mathbb{R}^{n+m+1},\ 0 \leqslant P \leqslant \frac{\langle \Lambda \mid \mathbb{1}\rangle}{\varphi_0} + \varepsilon\right\}.$$

The trajectories of the System (3) are bounded and the compact set Ω_ε is a positively invariant compact absorbing set for (3).

Proof Relation (16) implies that:

$$\limsup_{t\to+\infty} P(t) \leqslant \frac{\langle \Lambda \mid \mathbb{1}\rangle}{\varphi_0}.$$

Hence, the trajectories of the System (3) are bounded and Ω_ε is an absorbing set.

Let H be the function defined as follows:

$$\begin{aligned} H : \mathbb{R}^{n+m+1}_+ &\longrightarrow \mathbb{R} \\ x = (S, V, I) &\longmapsto \langle S \mid \mathbb{1}\rangle + V + \langle I \mid \mathbb{1}\rangle. \end{aligned}$$

H is a differentiable application and its gradient is given by:

$$\nabla H(x) = \mathbb{1}, \quad \forall x \in \mathbb{R}^{n+m+1}_+.$$

Let X be the field defined in the following manner:

$$\begin{aligned} X : \mathbb{R}^{n+m+1}_+ &\longrightarrow \mathbb{R}^{n+m+1}_+ \\ x = (S, V, I) &\longmapsto (X_1, X_2, X_3) \end{aligned}$$

with:

$$\begin{cases} X_1 = \Lambda + \sigma V e_n + \tilde{A}_S S - \langle \beta \mid I \rangle diag(\alpha) S, , \\ X_2 = \langle \eta, S \rangle - (\mu_v + \sigma) V, \\ X_3 = \langle \beta \mid I \rangle P diag(\alpha) S + \tilde{A}_I I. \end{cases}$$

For all $x \in H^{-1}\left(\dfrac{\langle \Lambda \mid \mathbb{1} \rangle}{\varphi_0} + \varepsilon\right)$, we have

$$\langle X(x) \mid \nabla H(x) \rangle = \dot{S} + \dot{V} + \dot{I} \quad \text{and} \quad P = \frac{\langle \Lambda \mid \mathbb{1} \rangle}{\varphi_0} + \varepsilon.$$

Thus,

$$\langle X(x) \mid \nabla H(x) \rangle \ = \dot{P} \leqslant \langle \Lambda \mid \mathbb{1} \rangle - \varphi_0 P \leqslant -\varepsilon \leqslant 0.$$

By using the theorem of the barrier (which derives from lemma on page 324 in [17]), we conclude that Ω_ε is a positively invariant set. ■

Moreover, without infection the dynamic of System (3) comes down to the following differential equation

$$\dot{x} = A_0 x + B_0. \tag{17}$$

where

$$x = \begin{pmatrix} S \\ V \end{pmatrix}, \qquad A_0 = \begin{pmatrix} \tilde{A}_S & \sigma e_n \\ \eta^t & -(\mu_v + \sigma) \end{pmatrix}, \qquad B_0 = \begin{pmatrix} \Lambda \\ 0 \end{pmatrix}$$

Any equilibrium of (17) satisfies the following equations:

$$\begin{cases} \Lambda + \sigma V_0^* q + \tilde{A}_S S_0^* = 0, \\ \langle \eta \mid S_0^* \rangle - (\mu_v + \sigma) V_0^* = 0. \end{cases} \tag{18}$$

The second equation of (18) yields

$$V_0^* = \frac{\langle \eta, S_0^* \rangle}{\mu_v + \sigma}. \tag{19}$$

Plugging this in the first equation of (18) and using the relation $\langle \eta, S_0^* \rangle = \eta^t S_0^*$, we obtain

$$-\left[\tilde{A}_S + \frac{\sigma}{\mu_v + \sigma} e_n \eta^t\right] S_0^* = \Lambda$$

i.e.,

$$-_S S_0^* = \Lambda$$

Since $_S$ is a stable Metzler matrix we have $-_S^{-1} > 0$. Therefore

$$S_0^* = -_S^{-1} \Lambda = -\left[\tilde{A}_S + \frac{\sigma}{\mu_v + \sigma} e_n \eta^t\right]^{-1} \Lambda.$$

It follows from Lemma 3.2 that $S_0^* \gg 0$. Which in turn implies that $V_0^* > 0$, since $\eta > 0$.

Proposition 3.2. *The set* Ω *such as*

$$\Omega = \{(S, V, I) \in \Omega_\varepsilon, \quad S(t) \leqslant S_0^*, \quad V(t) \leqslant V_0^*\}$$

is a positively invariant compact set for System (3).

Proof From System (3) and System (18), we deduce that on the boundary $S = S_0^*$, we have

$$\dot{S} = -\sigma V_0^* q + \sigma V e_n - \langle \beta \mid I \rangle diag(\alpha) S_0^* = -\sigma \left(V_0^* - V\right) q - \langle \beta \mid I \rangle diag(\alpha) S_0^* \leqslant 0.$$

Also, keeping Equation (19) in mind, on the boundary $V = V_0^*$, we have

$$\dot{V} = \langle \eta, S \rangle - (\mu_v + \sigma) V_0^* = \langle \eta, S \rangle - \langle \eta, S_0^* \rangle = -\langle \eta, S_0^* - S \rangle \leqslant 0.$$

Therefore, Ω is a positively invariant set for System 3. ∎

4. Basic reproduction number and equilibria

4.1 *Basic reproductive number, local stability of the disease free equilibrium*

A key concept in epidemiology is the basic reproduction number, commonly denoted by $\mathcal{R}_0$. Usually, $\mathcal{R}_0$ is defined as the expected number of secondary individuals produced, in a completely susceptible population, by a typical infected individual during its entire period of infectiousness [3].

The disease free equilibrium of System (3) is $X_0^* = (S_0^*, V_0^*, 0, 0)$, where

$$S_0^* = -{}_S^{-1}\Lambda = -\left[A_S - diag(\mu + \eta) + \frac{\sigma}{\mu_v + \sigma} e_n \eta^t\right]^{-1} \Lambda \ \text{ and } \ V_0^* = \frac{\langle \eta, S_0^* \rangle}{\mu_v + \sigma}.$$

To derive the basic reproductive number of our model, we use the next generation approach. It is defined by Van den Driessche and Watmough as the spectral radius of the next generation matrix [15]. Using the notation in [15] on the System (3) the derivatives $\mathcal{DF}$ and $\mathcal{DV}$ are given by:

$$\mathcal{DF} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \langle \beta \mid I \rangle P diag(\alpha) & 0 & P diag(\alpha) S \beta^t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathcal{DV} = \begin{pmatrix} -\tilde{A}_S + \langle \beta \mid I \rangle diag(\alpha) & -\sigma e_n & \beta^t diag(\alpha) S & 0 \\ -\eta^t & (\mu_v + \sigma) & 0 & 0 \\ 0 & 0 & -\tilde{A}_I & 0 \\ 0 & 0 & -\mathbb{1}^t diag(\gamma) & \mu_r \end{pmatrix}.$$

Denote $\tilde{F}$the matrix for the new infection terms and $\tilde{V}$ the matrix for the remaining transfer terms. We have $\tilde{F} = P diag(\alpha) S_0^* \beta^t$ and $\tilde{V} = -\tilde{A}_I$. So the basic reproduction number $\mathcal{R}_0$ which is the spectral radius of the matrix $\tilde{F}\tilde{V}^{-1}$ is given by

$$\mathcal{R}_0 = \rho\left(P diag(\alpha) S_0^* \beta^t (-\tilde{A}_I)^{-1}\right).$$

By writing

$$P diag(\alpha) S_0^* \beta^t (-\tilde{A}_I)^{-1} = [P diag(\alpha) S_0^*] \left[\beta^t (-\tilde{A}_I)^{-1}\right],$$

it is obvious to see that $Pdiag(\alpha)S_0^*\beta^t(-\tilde{A}_I)^{-1}$ is a rank one matrix. Therefore,

$$\rho\left(Pdiag(\alpha)S_0^*\beta^t(-\tilde{A}_I)^{-1}\right) = \beta^t(-\tilde{A}_I)^{-1}Pdiag(\alpha)S_0^*.$$

Hence the basic reproduction number of the System (3) is defined by:

$$\mathcal{R}_0 = \left\langle \beta \mid (-\tilde{A}_I)^{-1}Pdiag(\alpha)S_0^* \right\rangle$$

By the Theorem 2 in [15], we claim the following.

Proposition 4.1. *The disease free equilibrium X_0^* is asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

4.2 *Global stability of the disease free equilibrium*

Theorem 4.1. *If $\mathcal{R}_0 \leqslant 1$, then the disease free equilibrium X_0^* is globally asymptotically stable on the nonnegative orthant $\mathbb{R}_+^{n+m+1}$.*

Proof Let consider the following Lyapunov function: $U = \left\langle \beta \mid -A_I^{-1}I \right\rangle$. We have

$$\begin{aligned}
\dot{U} &= \left\langle \beta \mid -\tilde{A}_I^{-1}\dot{I} \right\rangle \\
&= \left\langle \beta \mid -\tilde{A}_I^{-1}\left(\langle \beta \mid I\rangle Pdiag(\alpha)S + \tilde{A}_I I\right) \right\rangle \\
&= \left\langle \beta \mid \langle \beta \mid I\rangle \left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha)S \right\rangle - \langle \beta \mid I\rangle,
\end{aligned}$$

i.e.,

$$\dot{U} = \langle \beta \mid I\rangle \left(\left\langle \beta \mid \left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha)S \right\rangle - 1\right). \tag{20}$$

Firstly, suppose $\mathcal{R}_0 < 1$. On the set Ω, we have $S \leqslant S_0^*$. It follows from (20) that in this case

$$\dot{U} \leqslant \langle \beta \mid I\rangle \left(\langle \beta \mid \left(-A_I^{-1}\right) Pdiag(\alpha)S_0^* \rangle - 1\right),$$

i.e.,

$$\dot{U} \leqslant \langle \beta \mid I\rangle (\mathcal{R}_0 - 1).$$

Hence, $\dot{U} \leqslant 0$ and $\dot{U} = 0$ if and only if $\langle \beta \mid I\rangle = 0$. The dynamics of I then obeys $\dot{I} = \tilde{A}_I I$. Since $\tilde{A}_I$ is a stable Metzler matrix, $\dot{I} = 0$ only if $I = 0$, which in turn implies $R = 0$, $S = S_0^*$ and $V = V_0^*$. Therefore the largest invariant set in Ω contained in $\{(S, V, I, R) \in \Omega, \quad \dot{U} = 0\}$ is reduced to the disease free equilibrium X_0^*. Given that Ω is an invariant compact set. By LaSalle invariance's principle [1, 11], the disease free equilibrium is globally asymptotically stable in Ω.

Now, suppose $\mathcal{R}_0 = 1$. Substituting 1 by the expression of $\mathcal{R}_0$ in the right hand side of (20), we get

$$\begin{aligned}
\dot{U} =& \langle \beta \mid I\rangle \left(\left\langle \beta \mid \left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha)S \right\rangle - \left\langle \beta \mid (-\tilde{A}_I)^{-1}Pdiag(\alpha)S_0^* \right\rangle\right) \\
=& - \langle \beta \mid I\rangle \left\langle \beta \mid \left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha)\left(S_0^* - S\right) \right\rangle.
\end{aligned}$$

We have $S \leqslant S_0^*$ and $\alpha \geqslant 0$, then $diag(\alpha)\,(S_0^* - S) \geqslant 0$. It follows from Lemma 3.3 that the vector $\left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha)\,(S_0^* - S) \gg 0$. Given that $\beta > 0$, it follows that $\left\langle \beta \mid \left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha)\,(S_0^* - S)\right\rangle > 0$.

Thus $\dot{U} \leqslant 0$ and $\dot{U} = 0$ if and only if $\langle \beta \mid I \rangle = 0$. We can conclude as previously, that the disease free equilibrium is globally asymptotically stable on the nonnegative orthant $\mathbb{R}_+^{n+m+1}$ when $\mathcal{R}_0 = 1$. ■

4.3 *Existence and uniqueness of an endemic equilibrium*

Proposition 4.2. *Model (3) has a unique endemic equilibrium X^* in the nonnegative orthant if and only if $\mathcal{R}_0 > 1$. Furthermore X^* is in the positive invariant compact Ω.*

Proof The endemic equilibrium X^* of System (3) can be deduced by the system

$$\begin{cases} \Lambda + \sigma V^* q + \tilde{A}_S S^* - \langle \beta \mid I^* \rangle diag(\alpha) S^* = 0, \\ \langle \eta, S^* \rangle - (\mu_v + \sigma) V^* = 0, \\ \langle \beta \mid I^* \rangle P diag(\alpha) S^* + \tilde{A}_I I^* = 0, \\ \mathbb{1}^t diag(\gamma) I^* - \mu_r R^* = 0. \end{cases} \tag{21}$$

From the second and the last equations of (21), we respectively have:

$$V^* = \frac{\langle \eta, S^* \rangle}{\mu_v + \sigma} = \frac{\eta^t S^*}{\mu_v + \sigma}. \tag{22}$$

and

$$R^* = \frac{\mathbb{1}^t diag(\gamma) I^*}{\mu_r}.$$

Plugging (22) in the first equation of (21) yields

$$-\left[\frac{\sigma e_n \eta^t}{\mu_v + \sigma} + \tilde{A}_S - \langle \beta \mid I^* \rangle diag(\alpha)\right] S^* = \Lambda$$

i.e.,

$$-\left[_S - \langle \beta \mid I^* \rangle diag(\alpha)\right] S^* = \Lambda \tag{23}$$

$_S$ is a stable compartmental matrix. Hence, for any $I \geqslant 0$, the matrix $_S - \langle \beta \mid I \rangle diag(\alpha)$ is also a stable compartmental matrix. From Equation (23), it then follows that

$$S^* = -\left[_S - \langle \beta \mid I^* \rangle diag(\alpha)\right]^{-1} \Lambda = -M(\langle \beta \mid I^* \rangle)^{-1} \Lambda, \tag{24}$$

where M is the stable Metzler matrix, depending linearly on the real variable $x \geqslant 0$, defined as

$$M(x) =_S - x diag(\alpha) =_S - diag(x\alpha).$$

Furthermore, A_I is a Metzler matrix. The third equation of (21) then involves

$$I^* = \langle \beta \mid I^* \rangle \left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha) S^*. \tag{25}$$

From (24) and (25), we deduce that finding $\langle \beta \mid I^* \rangle$ is sufficient to determine S^* and I^*.

Substituting (25) into $\langle \beta \mid I^* \rangle$, we obtain

$$\langle \beta \mid I^* \rangle = \langle \beta \mid I^* \rangle \left\langle \beta \mid \left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha) S^* \right\rangle.$$

As a result, we find

$$\langle \beta \mid I^* \rangle = 0 \quad \text{or} \quad \left\langle \beta \mid \left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha) S^* \right\rangle = 1.$$

If $\langle \beta \mid I^* \rangle = 0$, then $I^* = 0$. From which it follows that $R^* = 0,\ S^* = S_0^*$ and $V = V_0^*$. In this case, we obtain the disease free equilibrium.

Suppose $\left\langle \beta \mid \left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha) S^* \right\rangle = 1$. Substituting (24) into this relation yields

$$\left\langle \beta \mid \left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha) \left(-M(\langle \beta \mid I^* \rangle)^{-1}\right) \Lambda \right\rangle = 1.$$

Which means that $\langle \beta \mid I^* \rangle$ is solution of the equation $H(x) = 1,$ where H is the function of the real variable $x \geqslant 0$ defined by

$$H(x) = \left\langle \beta \mid \left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha) \left(-M(x)^{-1}\right) \Lambda \right\rangle.$$

$H(x)$ is a strictly decreasing function. Indeed, $M(x)' = -diag(\alpha)$. Thus, the derivative of H is defined by

$$H(x)' = \left\langle \beta \mid \left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha) M(x)^{-1} \left(-diag(\alpha)\right) M(x)^{-1} \Lambda \right\rangle.$$

By Lemma 3.2, we have $-M(x)^{-1}\Lambda \gg 0$. In addition, $\alpha \gg 0$, therefore

$$diag(\alpha) \left(-M(x)^{-1}\right) \Lambda \gg 0$$

which in turn, by the fact that $M(x)$ is a stable Metzler matrix, involves

$$-M(x)^{-1} diag(\alpha) \left(-M(x)^{-1}\right) \Lambda \gg 0.$$

Accordingly,

$$diag(\alpha) \left[-M(x)^{-1} diag(\alpha) \left(-M(x)^{-1}\right) \Lambda\right] \gg 0,$$

i.e.,

$$-diag(\alpha) M(x)^{-1} \left(-diag(\alpha)\right) M(x)^{-1} \Lambda \gg 0.$$

By Lemma 3.3, we have

$$-\left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha)M(x)^{-1}\left(-diag(\alpha)\right)M(x)^{-1}\Lambda \gg 0.$$

Since $\beta \geqslant 0$, it follows that

$$H(x)' < 0.$$

The function $H(x)$ satisfies $\lim\limits_{x\to+\infty} H(x) = 0$. In fact it is easy to see that the determinant $|M(x)|$ is a n degree polynomial in x, and that any cofactor of $M(x)$ is a polynomial of which the degree is strictly less than n. In addition, we have $H(0) = \mathcal{R}_0$. Consequently, the equation $H(x) = 1$ has a unique positive solution if and only if $\mathcal{R}_0 > 1$.

Furthermore, from Equation (24) we have by Lemma 3.2 that $S^* \gg 0$. Also, since $\alpha \gg 0$ we have by Lemma 3.3 that $\left(-\tilde{A}_I^{-1}\right) Pdiag(\alpha)S \gg 0$. Hence, it follows from Equation (25) that $I^* \gg 0$. Therefore, the equilibrium found is an endemic equilibrium.

On the other side, we have $S^* = -M(\langle\beta \mid I^*\rangle)^{-1}\Lambda$ and $S_0^* = -M(0)^{-1}\Lambda$. But

$$\begin{aligned}\frac{d(-M(x)^{-1})}{dx} &= M(x)^{-1}(-diag(\alpha))M(x)^{-1}\\ &= -\left(-M(x)^{-1}\right)\left(diag(\alpha)\right)\left(-M(x)^{-1}\right).\end{aligned}$$

Since $-M(x)^{-1} \geqslant 0$ and $diag(\alpha) \geqslant 0$, it follows that

$$\frac{d(-M(x)^{-1})}{dx} \leqslant 0,$$

Thus $S^* \leqslant S_0^*$.

Besides, it follows from Equation (22) that

$$V^* \leqslant \frac{\eta^t S_0^*}{\mu_v + \sigma} = V_0^*.$$

Therefore, the Endemic equilibrium found is in Ω.

In sight of achieving the proof, we see from the preceding analysis that if $\mathcal{R}_0 = 1$, then the equation $H(x) = 1$ has a unique solution which is obtained for $x = 0$, i.e., $\langle\beta \mid I^*\rangle = 0$. In this case, we find again the disease free equilibrium. ■

5. Global dynamics of Model (1)

5.1 *Disease free equilibrium*

By a straightforward computation, we find that the disease free equilibrium of Model (1) is X_0^* such as $X_0^* = (S_0^*, V_0^*, I_0^*, 0)$, with $S_0^* = (S_{01}^*, S_{02}^*, \cdots, S_{0n}^*)$, $I_0^* =$

$(0, 0, \cdots, 0)$,

$$S_{01}^* = \frac{\Lambda_1}{a_1},$$

$$S_{0i}^* = \frac{\Lambda_1 \prod_{l=1}^{i-1} \theta_l + \sum_{k=2}^{i-1} \Lambda_k \prod_{l=1}^{k-1} a_l \prod_{l=k}^{i-1} \theta_l + \Lambda_i \prod_{l=1}^{i-1} a_l}{\prod_{l=1}^{i} a_l}, \qquad i = 2, \cdots, n-1,$$

$$S_{0n}^* = \frac{a_n (\mu_v + \sigma)}{\mu_v \pi + \mu_v \mu_n + \sigma \mu_n} \left(\frac{\Lambda_1 \prod_{l=1}^{n-1} \theta_l + \sum_{k=2}^{n-1} \Lambda_k \prod_{l=1}^{k-1} a_l \prod_{l=k}^{n-1} \theta_l + \Lambda_i \prod_{l=1}^{n-1} a_l}{\prod_{l=1}^{n} a_l} \right).$$

and

$$V_0^* = \frac{\pi a_n}{\mu_v \pi + \mu_v \mu_n + \sigma \mu_n} \left(\frac{\Lambda_1 \prod_{l=1}^{n-1} \theta_l + \sum_{k=2}^{n-1} \Lambda_k \prod_{l=1}^{k-1} a_l \prod_{l=k}^{n-1} \theta_l + \Lambda_i \prod_{l=1}^{n-1} a_l}{\prod_{l=1}^{n} a_l} \right).$$

Moreover, it follows from Theorem 4.1 that the disease free equilibrium X_0^* of Model (1) is globally asymptotically stable on the nonnegative orthant $\mathbb{R}_+^{n+m+1}$ when $\mathcal{R}_0 \geqslant 1$.

5.2 *Endemic equilibrium*

Suppose $\mathcal{R}_0 > 1$, then by Proposition 4.2, there exist a unique endemic equilibrium X^* for Model (1).

Theorem 5.1. *Assume* $\mathcal{R}_0 > 1$.

* *In the case where* $\theta_i = 0, \ i = 1, \cdots, n-1,$ *the endemic equilibrium* X^* *is globally asymptotically stable on the nonnegative orthant* $\mathbb{R}_+^{n+m+1}$.

* *In the opposite cases, if* $\mu_i S_i^* + \beta_{i1} S_i^* I_1^* \geqslant \Lambda_i$, *for all* $i, \ i = 2, \cdots, n$, *then the endemic equilibrium* X^* *is globally asymptotically stable on the nonnegative orthant* $\mathbb{R}_+^{n+m+1}$.

Proof For the proof of Theorem 5.1, see 6. ■

Remark 5.1.

- *The stability condition of the endemic equilibrium is fulfilled when* $\Lambda_i = 0$ *for all* $i, \ i = 2, \cdots, n$. *That is to say Model (1) is a stage progression susceptibility model and all the recruitment is in the first stage. Therefore, when in addition* $\pi = \sigma = 0$, *Theorem 5.1 generalizes analogous theorem in [9] in the absence of vertical transmission; also the one in [13].*
- *In [5], the incidence of the disease is modeled by a general function. For the special case of the mass-action law, Theorem 5.1 extends its analogous one. Just consider that* $n = 1$ *and remove the class* V.
- *When we remove the stage progression susceptibility* ($\theta_i = 0, \ i = 1, \cdots, n-1$) *and the class of vaccinated persons* V, *Theorem 3.1 in [2] becomes a special case of Theorem 5.1, the corresponding theorems in [4, 6, 7] too.*

6. Summary

For many diseases, infectivity or susceptibility can evolve. Cases of HIV infection and HBV infection are illuminating examples. In this paper, we have addressed this issue by proposing a general model encompassing those of [2, 6, 9, 18]. The model includes the difference in susceptibility, the difference in infectivity and the loss of vaccine-induced immunity. We have used mass action incidence to model the transmission. Furthermore, we have consider different death rates and a constant influx distributed into the susceptible classes. From a biological point of view, we have considered two relevant hypothesis.

We have derived an explicit formula of the basic reproduction ratio $\mathcal{R}_0$. Furthermore, we have proven the global stability of the disease free equilibrium when the basic reproduction ratio $\mathcal{R}_0 \leqslant 1$ and the existence of a unique endemic equilibrium when $\mathcal{R}_0 > 1$. As regards to the stability of the endemic equilibrium when it exists, we have established that it is globally asymptotically stable in the case of a stage progression susceptibility model with recruitment only in the first stage, generalising by the way the analogous results in [9] in the absence of vertical transmission. Also in the case where there is no flow between susceptible individuals with no constraint on recruitment, extending by the way analogous results in [2, 5] and therefore in [4, 6, 7]. In the general case of a differential stage progression susceptibility class of models, we have provided some conditions under which it is globally asymptotically stable.

Acknowledgements

The authors would like to greatly thank the independent reviewer for his highly relevant remarks and suggestions that have immensely improved the contents.

References

[1] R. M. Anderson, B. Anderson and R. M. May, Infectious Diseases of Humans: Dynamics and Control, Oxford University Press, (1992).
[2] B. Bonzi, A. A. Fall, A. Iggidr and G. Sallet, Stability of differential susceptibility and infectivity epidemic models, Journal of Mathematical Biology, **62 (1)** (2011) 39–64.
[3] O. Diekmann, J. A. P. Heesterbeek and J. A. J. Metz, On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations, Journal of Mathematical Biology, **28 (4)** (1990) 365–382.
[4] H. Guo and M. Y. Li, Global dynamics of a staged progression model for infectious diseases, Mathematical Biosciences and Engineering, **3 (3)** (2006) 513.
[5] H. Guo and M. Y. Li, Global dynamics of a staged-progression model with amelioration for infectious diseases, Journal of Biological Dynamics, **2 (2)** (2008) 154–168.
[6] J. M. Hyman and J. Li, Differential susceptibility epidemic models, Journal of Mathematical Biology, **50 (6)** (2005) 626–644.
[7] A. Iggidr, J. Mbang, G. Sallet and J. J. Tewa, Multi-compartment models, Discrete and Continuous Dynamical Systems-Series S, **2007 (Special)** (2007) 506–519.
[8] A. V. Kamyad, R. Akbari, A. A. Heydari and A. Heydari, Mathematical modeling of transmission dynamics and optimal control of vaccination and treatment for hepatitis b virus, Computational and Mathematical Methods in Medicine, **2014** (2014), doi:10.1155/2014/475451.
[9] J. P. II Kouenkam, J. Mbang and Y. Emvudu, Global dynamics of a model of hepatitis b virus infection in a sub-saharan african rural area, International Journal of Biomathematic, **13 (6)** (2020) 2050054, doi:10.1142/S1793524520500540.
[10] C. M. Kribs-Zaleta and J. X. Velasco-Hernandez, A simple vaccination model with multiple endemic states, Mathematical Biosciences, **164 (2)** (2000) 183–201.
[11] J. P. La Salle, The Stability of Dynamical Systems, SIAM, (1976).
[12] J. Li, Y. Yang and Y. Zhou, Global stability of an epidemic model with latent stage and vaccination, Nonlinear Analysis: Real World Applications, **12 (4)** (2011) 2163–2173.
[13] X. Liu, Y. Takeuchi and S. Iwami, Svir epidemic models with vaccination strategies, Journal of Theoretical Biology, **253 (1)** (2008) 1–11.
[14] M. L. Mann Manyombe, J. Mbang, J. Lubuma and B. Tsanou, Global dynamics of a vaccination model for infectious diseases with asymptomatic carriers, Mathematical Biosciences and Engineering, **13** (2016) 813–840.
[15] P. Van den Driessche and J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Mathematical Biosciences, **180 (1)** (2002) 29–48.
[16] R. S. Varga, Matrix Iterative Analysis, Springer Science & Business Media, (2009).
[17] W. Walter, Ordinary Differential Equations, Springer, (1998).
[18] L. Zou, W. Zhang and S. Ruan, Modeling the transmission dynamics and control of hepatitis b virus in china, Journal of Theoretical Biology, **262 (2)** (2010) 330–338.

Appendix A. Some useful results

The first lemma is obvious to establish, that is why we have omitted their proofs.

Lemma 6.1. *Let* (u_n) *and* (v_n) *be two sequences of real numbers.*
For all $n \geqslant 2$*, we have*

$$\sum_{j=2}^{n}\sum_{l=j}^{n} u_l(1-v_j) = \sum_{j=2}^{n} u_j \left(j-1-\sum_{l=2}^{j} v_l\right). \tag{26}$$

For all $n \geqslant 5$*, we have*

$$\sum_{j=2}^{n-3}\sum_{l=n-2}^{n} u_l(1-v_j) = \sum_{j=n-2}^{n} u_j \left(n-4-\sum_{l=2}^{n-3} v_l\right). \tag{27}$$

For all $n \geqslant 2$*, we have*

$$\sum_{i=1}^{n-1}\sum_{k=i+1}^{n} u_k v_i = \sum_{i=2}^{n} u_i \sum_{k=1}^{i-1} v_k. \tag{28}$$

Lemma 6.2. *Let be the following positive constants:*

$$\begin{cases} A_1 = 1; \\ A_2 = \dfrac{b_1 - \sum_{i=1}^{n} \beta_{i1} S_i^*}{k_1}; \\ A_j = \dfrac{\sum_{i=1}^{n}\sum_{l=j}^{m} \beta_{il} S_i^* I_l^* + \delta_j I_j^* A_{j-1}}{k_{j-1} I_{j-1}^*}, \quad j = 3, \cdots, m-2; \\ A_p = \dfrac{(\delta_p b_f + k_p \delta_f')A_c + b_f \sum_{i=1}^{n} \beta_{ip} S_i^* + k_p \sum_{i=1}^{n} \beta_{if} S_i^*}{b_p b_f - k_p \delta_f}; \\ A_f = \dfrac{(\delta_f' b_p + \delta_p \delta_f)A_c + \delta_f \sum_{i=1}^{n} \beta_{ip} S_i^* + b_p \sum_{i=1}^{n} \beta_{if} S_i^*}{b_p b_f - k_p \delta_f}. \end{cases} \tag{29}$$

The constants $A_j,\ j=1,\cdots,m,$ *satisfy the following relations*

$$\begin{cases} A_1 - 1 = 0, \\ k_1A_2 - b_1A_1 + \displaystyle\sum_{i=1}^{n} \beta_{i1}S_i^* = 0, \\ k_jA_{j+1} + \delta_jA_{j-1} - b_jA_j + \displaystyle\sum_{i=1}^{n} \beta_{ij}S_i^* = 0, \quad j = 2,\cdots,m-3, \\ k_cA_p + k_c'A_f + \delta_cA_{m-3} - b_cA_c + \displaystyle\sum_{i=1}^{n} \beta_{ic}S_i^* = 0, \\ k_pA_f + \delta_pA_c - b_pA_p + \displaystyle\sum_{i=1}^{n} \beta_{ip}S_i^* = 0, \\ \delta_fA_p + \delta_f'A_c - b_fA_f + \displaystyle\sum_{i=1}^{n} \beta_{if}S_i^* = 0. \end{cases} \tag{30}$$

Proof The following equilibrium equations, where $\lambda_i^* = \sum_{j=1}^{m} \beta_{ij}I_j^*$, holds :

$$\begin{cases} \Lambda_1 - (\mu_1+\theta_1)S_1^* - \lambda_1^*S_1^* = 0, \\ \Lambda_i - (\mu_i+\theta_i)S_i^* - \lambda_i^*S_i^* + \theta_{i-1}S_{i-1}^* = 0, & i = 2,\cdots,n-1 \\ \Lambda_n + \sigma V^* - (\mu_n+\pi)S_n^* - \lambda_n^*S_n^* + \theta_{n-1}S_{n-1}^* = 0, \\ \pi S_n^* - (\mu_v+\sigma)V^* = 0, \\ \displaystyle\sum_{i=1}^{n} \lambda_i^*S_i^* + \delta_2I_2^* - b_1I_1^* = 0, \\ k_{j-1}I_{j-1}^* + \delta_{j+1}I_{j+1}^* - b_jI_j^* = 0, & j = 2,\cdots,m-3 \\ k_{m-3}I_{m-3}^* + \delta_pI_p^* + \delta_f'I_f^* - b_cI_c^* = 0, \\ k_cI_c^* + \delta_fI_f^* - b_pI_p^* = 0, \\ k_c'I_c^* + k_pI_p^* - b_fI_f^* = 0. \end{cases} \tag{31}$$

Let us first show that the constant A_2 is positive. Indeed, it follows from the expression of A_2 that

$$k_1A_2I_1^* = b_1I_1^* - \sum_{i=1}^{n} \beta_{i1}S_i^*I_1^*.$$

Keeping in mind the expression of λ_i^*, one deduces from the fifth equation in System (31) that

$$b_1I_1^* - \sum_{i=1}^{n} \beta_{i1}S_i^*I_1^* = \sum_{i=1}^{n}\sum_{j=2}^{m} \beta_{ij}S_i^*I_j^* + \delta_2I_2^* > 0.$$

Hence, $k_1A_2I_1^* > 0$, which in turn implies that $A_2 > 0$.

Now, the first, the second, the fifth and the sixth relations in (30) are easy to establish. Hence, we will show only the third and the fourth relations in (30).

In order to prove the third relation in (30), we first consider the case $j = 2$. Using the expression of A_3 and A_1 as given in (29), we have

$$\begin{aligned}
&k_2 A_3 + \delta_2 A_1 - b_2 A_2 + \sum_{i=1}^{n} \beta_{i2} S_i^* \\
&= \frac{1}{I_2^*} \left(\sum_{i=1}^{n} \sum_{l=3}^{m} \beta_{il} S_i^* I_l^* + A_2 \left(\delta_3 I_3^* - b_2 I_2^* \right) + \delta_2 I_2^* + \sum_{i=1}^{n} \beta_{i2} S_i^* I_2^* \right).
\end{aligned}$$

It follows from the sixth equation in System (31) that

$$\delta_3 I_3^* - b_2 I_2^* = -k_1 I_1^*.$$

Then,

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{l=3}^{m} \beta_{il} S_i^* I_l^* + A_2 \left(\delta_3 I_3^* - b_2 I_2^* \right) + \delta_2 I_2^* + \sum_{i=1}^{n} \beta_{i2} S_i^* I_2^* \\
&= \sum_{i=1}^{n} \sum_{l=2}^{m} \beta_{il} S_i^* I_l^* - k_1 A_2 I_1^* + \delta_2 I_2^*.
\end{aligned}$$

Keeping the fifth equation in System (31) in mind, plugging A_2 into the right hand side of this relation yields

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{l=3}^{m} \beta_{il} S_i^* I_l^* + A_2 \left(\delta_3 I_3^* - b_2 I_2^* \right) + \delta_2 I_2^* + \sum_{i=1}^{n} \beta_{i2} S_i^* I_2^* \\
&= \sum_{i=1}^{n} \sum_{l=2}^{m} \beta_{il} S_i^* I_l^* - b_1 I_1^* + \sum_{i=1}^{n} \beta_{i1} S_i^* I_1^* + \delta_2 I_2^* \\
&= \sum_{i=1}^{n} \sum_{l=1}^{m} \beta_{il} S_i^* I_l^* - b_1 I_1^* + \delta_2 I_2^* \\
&= 0.
\end{aligned}$$

Therefore,

$$k_2 A_3 + \delta_2 A_1 - b_2 A_2 + \sum_{i=1}^{n} \beta_{i2} S_i^* = 0.$$

Hence, the third relation in (30) holds for $j = 2$.

Now, we have to establish the third relation in (30) for $j,\ \ 3 \leqslant j \leqslant m - 3$. Substituting A_{j+1} by its expression as given in (29) into the left hand side of the third relation in (30), we have

$$\begin{aligned}
&k_j A_{j+1} + \delta_j A_{j-1} - b_j A_j + \sum_{i=1}^{n} \beta_{ij} S_i^* \\
&= \frac{1}{I_j^*} \left(\sum_{i=1}^{n} \sum_{l=j+1}^{m} \beta_{il} S_i^* I_l^* + \delta_j I_j^* A_{j-1} + A_j \left(\delta_{j+1} I_{j+1}^* - b_j I_j^* \right) + \sum_{i=1}^{n} \beta_{ij} S_i^* I_j^* \right).
\end{aligned}$$

From the sixth equation in System (31), we deduce that

$$\delta_{j+1}I^*_{j+1} - b_jI^*_j = -k_{j-1}I^*_{j-1}.$$

Then,

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{l=j+1}^{m}\beta_{il}S^*_iI^*_l + \delta_jI^*_jA_{j-1} + A_j\left(\delta_{j+1}I^*_{j+1} - b_jI^*_j\right) + \sum_{i=1}^{n}\beta_{ij}S^*_iI^*_j\\ &= \sum_{i=1}^{n}\sum_{l=j}^{m}\beta_{il}S^*_iI^*_l + \delta_jI^*_jA_{j-1} - k_{j-1}I^*_{j-1}A_j.\end{aligned}$$

Taking into account the fifth equation in System (31), substituting the expression of A_j into the right hand side of the above relation, we easily obtain

$$\sum_{i=1}^{n}\sum_{l=j+1}^{m}\beta_{il}S^*_iI^*_l + \delta_jI^*_jA_{j-1} + A_j\left(\delta_{j+1}I^*_{j+1} - b_jI^*_j\right) + \sum_{i=1}^{n}\beta_{ij}S^*_iI^*_j = 0.$$

Therefore,

$$k_jA_{j+1} + \delta_jA_{j-1} - b_jA_j + \sum_{i=1}^{n}\beta_{ij}S^*_i = 0.$$

Hence, we have established the third relation in (30) for $j = 2, \ldots, m-3$.

We next show that the fourth relation in (30) holds true. Combining the height with the ninth equations in System (31), we obviously get:

$$\begin{cases} I^*_p = \dfrac{I^*_c\left(k_cb_f + k'_c\delta_f\right)}{b_pb_f - k_p\delta_f} \\ I^*_f = \dfrac{I^*_c\left(k_ck_p + k'_cb_p\right)}{b_pb_f - k_p\delta_f} \end{cases} \tag{32}$$

Recall $m-2 = c$, thus by the third relation in (30), we have

$$A_c = \frac{\sum_{i=1}^{n}\beta_{ic}S^*_iI^*_c + \sum_{i=1}^{n}\beta_{ip}S^*_iI^*_p + \sum_{i=1}^{n}\beta_{if}S^*_iI^*_f + \delta_cI^*_cA_{m-3}}{k_{m-3}I^*_{m-3}}. \tag{33}$$

From the seventh equation in System (31), we obtain

$$k_{m-3}I^*_{m-3} = -\delta_pI^*_p - \delta'_fI^*_f + b_cI^*_c. \tag{34}$$

On the one hand, substituting the expressions of I^*_p and I^*_f as given in (32) into (34), we get

$$k_{m-3}I^*_{m-3} = \frac{\mathcal{A}_1I^*_c}{b_pb_f - k_p\delta_f}, \tag{35}$$

where

$$\mathcal{A}_1 = b_c\left(b_pb_f - k_p\delta_f\right) - \delta_p\left(k_cb_f + k'_c\delta_f\right) - \delta'_f\left(k_ck_p + k'_cb_p\right)$$

On the other hand, using the same expressions of I_p^* and I_f^*, we have

$$\sum_{i=1}^{n} \beta_{ic} S_i^* I_c^* + \sum_{i=1}^{n} \beta_{ip} S_i^* I_p^* + \sum_{i=1}^{n} \beta_{if} S_i^* I_f^* = \frac{\mathcal{A}_2 I_c^*}{b_p b_f - k_p \delta_f} \tag{36}$$

where

$$\mathcal{A}_2 = (b_p b_f - k_p \delta_f) \sum_{i=1}^{n} \beta_{ic} S_i^* + \left(k_c b_f + k_c' \delta_f\right) \sum_{i=1}^{n} \beta_{ip} S_i^* + \left(k_c k_p + k_c' b_p\right) \sum_{i=1}^{n} \beta_{if} S_i^*.$$

Substituting (35) and (36) into (33), we get by an easy computation

$$A_c = \frac{\mathcal{A}_2 + \delta_c A_{m-3} \left(b_p b_f - k_p \delta_f\right)}{\mathcal{A}_1}. \tag{37}$$

Plugging A_p and A_f as defined in the System (29) into the fourth equation in (30), we easily have

$$k_c A_p + k_c' A_f + \delta_c A_{m-3} - b_c A_c + \sum_{i=1}^{n} \beta_{ic} S_i^* = \frac{\mathcal{A}_2 + \delta_c A_{m-3} \left(b_p b_f - k_p \delta_f\right) - \mathcal{A}_1 A_c}{b_p b_f - k_p \delta_f}. \tag{38}$$

Substituting (37) into relation (38), we obtain

$$k_c A_p + k_c' A_f + \delta_c A_{m-3} - b_c A_c + \sum_{i=1}^{n} \beta_{ic} S_i^* = 0.$$

This achieves the proof. ■

Appendix B. Proof of Theorem 5.1

To prove the global stability of the endemic equilibrium, we make use of a Lyapunov function V such as

$$V(t) = \sum_{i=1}^{n} \left(S_i - S_i^* \ln(S_i)\right) + B\left(V - V^* \ln(V)\right) + \sum_{j=1}^{m} A_j \left(I_j - I_j^* \ln(I_j)\right)$$

where $B = \dfrac{\sigma}{\mu_v + \sigma}$ and $A_j,\ j = 1, \cdots, m$ are positive constants defined in (29).

The derivative of $V(t)$ along the trajectories of System (1) is given by

$$\dot{V}(t) = \sum_{i=1}^{n} \left(1 - \frac{S_i^*}{S_i}\right) \dot{S}_i + B\left(1 - \frac{V^*}{V}\right) \dot{V} + \sum_{j=1}^{m} A_j \left(1 - \frac{I_j^*}{I_j}\right) \dot{I}_j.$$

Using the first, the second and the third equations of System (31), we can rewrite

$\dot{V}(t)$ as

$$\begin{aligned}
&\dot{V}(t)\\
&=\sum_{i=1}^{n}\left[a_iS_i^*\left(1-\frac{S_i}{S_i^*}\right)+\lambda_i^*S_i^*-\lambda_iS_i\right]\left(1-\frac{S_i^*}{S_i}\right)\\
&+\sum_{i=2}^{n}\theta_{i-1}\left(S_{i-1}-S_{i-1}^*\right)\left(1-\frac{S_i^*}{S_i}\right)+\sigma\left(V-V^*\right)\left(1-\frac{S_n^*}{S_n}\right)\\
&+B\pi S_n\left(1-\frac{V^*}{V}\right)+B(\mu_v+\sigma)(V^*-V)+A_1\sum_{i=1}^{n}\lambda_iS_i\left(1-\frac{I_1^*}{I_1}\right)\\
&+\sum_{j=1}^{m-3}\delta_{j+1}A_jI_{j+1}\left(1-\frac{I_j^*}{I_j}\right)+\sum_{j=2}^{m-3}k_{j-1}A_jI_{j-1}\left(1-\frac{I_j^*}{I_j}\right)\\
&-\sum_{j=1}^{m-3}b_jA_jI_j\left(1-\frac{I_j^*}{I_j}\right)+A_c\left(k_{m-3}I_{m-3}+\delta'_fI_f+\delta_pI_p-b_cI_c\right)\left(1-\frac{I_c^*}{I_c}\right)\\
&+A_p\left(k_cI_c+\delta_fI_f-b_pI_p\right)\left(1-\frac{I_p^*}{I_p}\right)+A_f\left(k'_cI_c+k_pI_p-b_fI_f\right)\left(1-\frac{I_f^*}{I_f}\right).
\end{aligned}$$

To simplify the notation in what follows, we will denote by:

$v=\frac{V}{V^*};\ y_c=\frac{I_c}{I_c^*};\ y_p=\frac{I_p}{I_p^*};\ y_f=\frac{I_f}{I_f^*};\ y_j=\frac{I_j}{I_j^*},\ 1\leqslant j\leqslant m-3;\ x_i=\frac{S_i}{S_i^*},\ 1\leqslant i\leqslant n.$

For convenience, define $U_k,\ 1\leqslant k\leqslant n$ and $W_k,\ 1\leqslant k\leqslant n-1$ respectively by

$$U_k=\left(1-\frac{S_k}{S_k^*}\right)\left(1-\frac{S_k^*}{S_k}\right)=2-x_k-\frac{1}{x_k} \tag{39}$$

and

$$W_k=x_k+\frac{1}{x_{k+1}}-\frac{x_k}{x_{k+1}}-1. \tag{40}$$

Accordingly, we have

$$\begin{aligned}
&a_nS_n^*U_n+\sigma\left(V-V^*\right)\left(1-\frac{S_n^*}{S_n}\right)+B\pi S_n\left(1-\frac{V^*}{V}\right)+B(\mu_v+\sigma)(V^*-V)\\
=&a_nS_n^*U_n-\sigma V\frac{S_n^*}{S_n}-\sigma V^*\left(1-\frac{S_n^*}{S_n}\right)+B\pi S_n\left(1-\frac{V^*}{V}\right)+B(\mu_v+\sigma)V^*\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+V[\sigma-(\mu_v+\sigma)B]\\
=&a_nS_n^*U_n-\sigma V^*\frac{v}{x_n}-\sigma V^*\left(1-\frac{1}{x_n}\right)+B\pi S_n^*x_n\left(1-\frac{1}{v}\right)+B(\mu_v+\sigma)V^*\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+V[\sigma-(\mu_v+\sigma)B].
\end{aligned}$$

The fourth equation in System (31) implies

$$V^* = \frac{\pi S_n^*}{\mu_v + \sigma}. \tag{41}$$

Then, using the expression of B and Relation (41), we get

$$\begin{aligned}
&a_n S_n^* U_n + \sigma\left(V - V^*\right)\left(1 - \frac{S_n^*}{S_n}\right) + B\pi S_n\left(1 - \frac{V^*}{V}\right) + B(\mu_v + \sigma)(V^* - V)\\
=&a_n S_n^* U_n - \sigma V^* \frac{v}{x_n} - \sigma V^*\left(1 - \frac{1}{x_n}\right) + \sigma V^* x_n\left(1 - \frac{1}{v}\right) + \sigma V^*\\
=&a_n S_n^* U_n + \sigma V^*\left(\frac{1}{x_n} + x_n - \frac{v}{x_n} - \frac{x_n}{v}\right)\\
=&a_n S_n^* U_n - \sigma V^* U_n + \sigma V^* U_v.
\end{aligned}$$

where

$$U_v = 2 - \frac{v}{x_n} - \frac{x_n}{v}. \tag{42}$$

Substituting $a_n = \mu_n + \pi$ and Relation (41) in the previous sum, we obtain,

$$\begin{aligned}
a_n S_n^* U_n - \sigma V \frac{S_n^*}{S_n} - \sigma V^*\left(1 - \frac{S_n^*}{S_n}\right) + B\pi S_n\left(1 - \frac{V^*}{V}\right) + B(\mu_v + \sigma)V^*&\\
= S_n^* U_n\left(\frac{\mu_n\sigma + a_n\mu_v}{\mu_v + \sigma}\right) + \sigma V^* U_v.&
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum_{i=2}^{n}\theta_{i-1}\left(S_{i-1} - S_{i-1}^*\right)\left(1 - \frac{S_i^*}{S_i}\right) &= \sum_{i=2}^{n}\theta_{i-1}S_{i-1}^*\left(x_{i-1} + \frac{1}{x_i} - \frac{x_{i-1}}{x_i} - 1\right)\\
&= \sum_{i=1}^{n-1}\theta_i S_i^* W_i,
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^{m-3}\delta_{j+1}A_j I_{j+1}\left(1 - \frac{I_j^*}{I_j}\right) &= \sum_{j=2}^{m-2}\delta_j A_{j-1} I_j\left(1 - \frac{I_{j-1}^*}{I_{j-1}}\right)\\
&= \sum_{j=2}^{m-2}\delta_j A_{j-1} I_j - \sum_{j=2}^{m-2}\delta_j A_{j-1} I_j^* \frac{y_j}{y_{j-1}},
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=2}^{m-3} k_{j-1}A_jI_{j-1}\left(1-\frac{I_j^*}{I_j}\right)-\sum_{j=1}^{m-3} b_jA_jI_j\left(1-\frac{I_j^*}{I_j}\right)\\
&=\sum_{j=1}^{m-4} k_jA_{j+1}I_j\left(1-\frac{1}{y_{j+1}}\right)-\sum_{j=1}^{m-3} b_jA_jI_j\left(1-\frac{1}{y_j}\right)\\
&=\sum_{j=1}^{m-4}\left(k_jA_{j+1}-b_jA_j\right)I_j-\sum_{j=1}^{m-4} k_jA_{j+1}I_j^*\frac{y_j}{y_{j+1}}+\sum_{j=1}^{m-3} b_jA_jI_j^*-b_{m-3}A_{m-3}I_{m-3}.
\end{aligned}$$

Substituting the four above relations into the previous expression of $\dot{V}(t)$, expanding, then rearranging some terms, we obtain

$$\begin{aligned}
\dot{V}(t)=&K_1(t)+\sigma V^*U_v+\sum_{i=1}^{n}\lambda_i^*S_i^*\left(1-\frac{1}{x_i}\right)+(A_1-1)\sum_{i=1}^{n}\lambda_iS_i+I_1\left(k_1A_2-b_1A_1\right)\\
&-A_1\frac{I_1^*}{I_1}\sum_{i=1}^{n}\lambda_iS_i+\sum_{j=2}^{m-4}I_j\left(k_jA_{j+1}+\delta_jA_{j-1}-b_jA_j\right)-\sum_{j=1}^{m-4}k_jA_{j+1}I_j^*\frac{y_j}{y_{j+1}}\\
&+\sum_{j=1}^{m-3}b_jA_jI_j^*-\sum_{j=2}^{m-2}\delta_jA_{j-1}I_j^*\frac{y_j}{y_{j-1}}+\delta_{m-3}A_{m-4}I_{m-3}-b_{m-3}A_{m-3}I_{m-3}\\
&+I_c\left(k_cA_p+k_c'A_f+\delta_cA_{m-3}-b_cA_c\right)+I_p\left(k_pA_f+\delta_pA_c-b_pA_p\right)\\
&+I_f\left(\delta_fA_p+\delta_f'A_c-b_fA_f\right)-A_c\left(k_{m-3}I_{m-3}+\delta_f'I_f+\delta_pI_p\right)\frac{1}{y_c}+b_cA_cI_c^*\\
&+b_pA_pI_p^*+b_fA_fI_f^*-A_p\left(k_cI_c+\delta_fI_f\right)\frac{1}{y_p}-A_f\left(k_pI_p+k_c'I_c\right)\frac{1}{y_f},
\end{aligned}$$

where

$$K_1(t)=\sum_{i=1}^{n-1}\theta_iS_i^*W_i+S_n^*U_n\left(\frac{\mu_n\sigma+a_n\mu_v}{\mu_v+\sigma}\right)+\sum_{i=1}^{n-1}a_iS_i^*U_i. \tag{43}$$

Notice that $\sum_{i=1}^{n}\lambda_iS_i^*=\sum_{i=1}^{n}\sum_{j=1}^{m}\beta_{ij}S_i^*I_j$ and recall that $m-2=c$, $m-1=p$ and

$m = f$. Therefore,

$$
\begin{aligned}
&\dot{V}(t) \\
&= K_1(t) + \sigma V^* U_v + \sum_{i=1}^{n} \lambda_i^* S_i^* \left(1 - \frac{1}{x_i}\right) + (A_1 - 1)\sum_{i=1}^{n} \lambda_i S_i + I_1 \Bigg(k_1 A_2 - b_1 A_1 \\
&+ \sum_{i=1}^{n} \beta_{i1} S_i^* \Bigg) - \frac{A_1}{y_1} \sum_{i=1}^{n} \lambda_i S_i + \sum_{j=2}^{m-3} I_j \left(k_j A_{j+1} + \delta_j A_{j-1} - b_j A_j + \sum_{i=1}^{n} \beta_{ij} S_i^* \right) \\
&- \sum_{j=1}^{m-3} k_j A_{j+1} I_j^* \frac{y_j}{y_{j+1}} + \sum_{j=1}^{m-3} b_j A_j I_j^* - \sum_{j=2}^{m-2} \delta_j A_{j-1} I_j^* \frac{y_j}{y_{j-1}} + I_c \Bigg(k_c A_p + k'_c A_f \\
&+ \delta_c A_{m-3} - b_c A_c + \sum_{i=1}^{n} \beta_{ic} S_i^* \Bigg) + I_p \left(k_p A_f + \delta_p A_c - b_p A_p + \sum_{i=1}^{n} \beta_{ip} S_i^* \right) \\
&+ I_f \left(\delta_f A_p + \delta'_f A_c - b_f A_f + \sum_{i=1}^{n} \beta_{if} S_i^* \right) + b_c A_c I_c^* + b_p A_p I_p^* + b_f A_f I_f^* \\
&- A_c \left(\delta'_f I_f^* y_f + \delta_p I_p^* y_p \right) \frac{1}{y_c} - A_p \left(k_c I_c^* y_c + \delta_f I_f^* y_f \right) \frac{1}{y_p} - A_f \left(k_p I_p^* y_p + k'_c I_c^* y_c \right) \frac{1}{y_f}.
\end{aligned}
$$

In addition, we have $\displaystyle\sum_{i=1}^{n} \lambda_i^* S_i^* = \sum_{i=1}^{n}\sum_{j=1}^{m} \beta_{ij} S_i^* I_j^*$ and $\displaystyle\frac{1}{y_1}\sum_{i=1}^{n} \lambda_i S_i = \sum_{i=1}^{n}\sum_{j=1}^{m} \beta_{ij} S_i^* I_j^* \frac{x_i y_j}{y_1}$. Hence, cancelling some terms by substituting relations in (30) into $\dot{V}(t)$, we get

$$
\begin{aligned}
\dot{V}(t) =& K_1(t) + \sigma V^* U_v + \sum_{i=1}^{n}\sum_{j=1}^{m} \beta_{ij} S_i^* I_j^* \left(1 - \frac{1}{x_i}\right) + I_1^* \left(b_1 - k_1 A_2 \frac{y_1}{y_2} \right) \\
&- \sum_{i=1}^{n}\sum_{j=1}^{m} \beta_{ij} S_i^* I_j^* \frac{x_i y_j}{y_1} + \sum_{j=2}^{m-3} I_j^* \left(b_j A_j - k_j A_{j+1} \frac{y_j}{y_{j+1}} - \delta_j A_{j-1} \frac{y_j}{y_{j-1}} \right) \\
&- \delta_c A_{m-3} I_c^* \frac{y_c}{y_{m-3}} + I_c^* \left(b_c A_c - k_c A_p \frac{y_c}{y_p} - k'_c A_f \frac{y_c}{y_f} \right) \\
&+ I_p^* \left(b_p A_p - \delta_p A_c \frac{y_p}{y_c} - k_p A_f \frac{y_p}{y_f} \right) + I_f^* \left(b_f A_f - \delta'_f A_c \frac{y_f}{y_c} - \delta_f A_p \frac{y_f}{y_p} \right).
\end{aligned}
$$

Later on, the third relation in (30) yields

$$
k_j A_{j+1} = b_j A_j - \delta_j A_{j-1} - \sum_{i=1}^{n} \beta_{ij} S_i^*, \quad j = 2, \cdots, m-3.
$$

Thus, for $j = 2, \cdots, m-3$, we have

$$b_jA_j - k_jA_{j+1}\frac{y_j}{y_{j+1}} - \delta_jA_{j-1}\frac{y_j}{y_{j-1}} = b_jA_j\left(1-\frac{y_j}{y_{j+1}}\right) + \delta_jA_{j-1}\left(\frac{y_j}{y_{j+1}} - \frac{y_j}{y_{j-1}}\right) + \sum_{i=1}^{n}\beta_{ij}S_i^*\frac{y_j}{y_{j+1}}.$$

On the other hand, it follows from the fifth equation in System (31) that

$$b_1I_1^* = \delta_2I_2^* + \sum_{i=1}^{n}\sum_{j=1}^{m}\beta_{ij}S_i^*I_j^*,$$

and from the second relation in (30) that

$$k_1A_2 = b_1 - \sum_{i=1}^{n}\beta_{i1}S_i^*.$$

Consequently

$$\begin{aligned}
&I_1^*\left(b_1 - k_1A_2\frac{y_1}{y_2}\right)\\
&= b_1I_1^*\left(1-\frac{y_1}{y_2}\right) + \sum_{i=1}^{n}\beta_{i1}S_i^*I_1^*\frac{y_1}{y_2}\\
&= \delta_2I_2^*\left(1-\frac{y_1}{y_2}\right) + \sum_{i=1}^{n}\sum_{j=1}^{m}\beta_{ij}S_i^*I_j^*\left(1-\frac{y_1}{y_2}\right) + \sum_{i=1}^{n}\beta_{i1}S_i^*I_1^*\frac{y_1}{y_2}.
\end{aligned}$$

Hence, substituting the three above relations into $\dot{V}(t)$, we obtain,

$$\begin{aligned}
&\dot{V}(t)\\
&= K_1(t) + \sigma V^*U_v + \sum_{i=1}^{n}\sum_{j=1}^{m}\beta_{ij}S_i^*I_j^*\left(1-\frac{1}{x_i}\right) + \delta_2I_2^*\left(1-\frac{y_1}{y_2}\right)\\
&+ \sum_{i=1}^{n}\sum_{j=1}^{m}\beta_{ij}S_i^*I_j^*\left(1-\frac{y_1}{y_2}\right) + \sum_{i=1}^{n}\beta_{i1}S_i^*I_1^*\frac{y_1}{y_2} - \sum_{i=1}^{n}\sum_{j=1}^{m}\beta_{ij}S_i^*I_j^*\frac{x_iy_j}{y_1}\\
&+ \sum_{j=2}^{m-3}b_jA_jI_j^*\left(1-\frac{y_j}{y_{j+1}}\right) + \sum_{i=1}^{n}\sum_{j=2}^{m-3}\beta_{ij}S_i^*I_j^*\frac{y_j}{y_{j+1}} + \sum_{j=2}^{m-3}\delta_jA_{j-1}I_j^*\left(\frac{y_j}{y_{j+1}} - \frac{y_j}{y_{j-1}}\right)\\
&- \delta_cA_{m-3}I_c^*\frac{y_c}{y_{m-3}} + I_c^*\left(b_cA_c - k_cA_p\frac{y_c}{y_p} - k_c'A_f\frac{y_c}{y_f}\right) + I_p^*\Bigg(b_pA_p - \delta_pA_c\frac{y_p}{y_c}\\
&- k_pA_f\frac{y_p}{y_f}\Bigg) + I_f^*\left(b_fA_f - \delta_f'A_c\frac{y_f}{y_c} - \delta_fA_p\frac{y_f}{y_p}\right).
\end{aligned}$$

Furthermore, the sixth equation in System (31) implies

$$b_jI_j^* = k_{j-1}I_{j-1}^* + \delta_{j+1}I_{j+1}^*, \quad j = 2, \cdots, m-3.$$

Thus, using the expression of A_j, as given in (29), we obtain

$$b_jA_jI_j^* = k_{j-1}I_{j-1}^*A_j + \delta_{j+1}I_{j+1}^*A_j = \delta_jA_{j-1}I_j^* + \sum_{i=1}^{n}\sum_{l=j}^{m}\beta_{il}I_l^*S_i^* + \delta_{j+1}A_jI_{j+1}^*.$$

As a consequence,

$$\begin{aligned}
&\sum_{j=2}^{m-3} b_jA_jI_j^*\left(1-\frac{y_j}{y_{j+1}}\right)\\
&= \left(\sum_{j=2}^{m-3}\delta_jA_{j-1}I_j^* + \sum_{i=1}^{n}\sum_{j=2}^{m-3}\sum_{l=j}^{m}\beta_{il}I_l^*S_i^*\right)\left(1-\frac{y_j}{y_{j+1}}\right)\\
&\qquad\qquad\qquad + \sum_{j=2}^{m-3}\delta_{j+1}A_jI_{j+1}^*\left(1-\frac{y_j}{y_{j+1}}\right)\\
&= \left(\sum_{j=2}^{m-3}\delta_jA_{j-1}I_j^* + \sum_{i=1}^{n}\sum_{j=2}^{m-3}\sum_{l=j}^{m}\beta_{il}I_l^*S_i^*\right)\left(1-\frac{y_j}{y_{j+1}}\right)\\
&\qquad\qquad\qquad + \sum_{j=3}^{m-2}\delta_jA_{j-1}I_j^*\left(1-\frac{y_{j-1}}{y_j}\right).
\end{aligned}$$

By the relation (26) in Lemma 6.1 and the relation (27) in Lemma 6.1, it is straightforward to establish,

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=2}^{m-3}\sum_{l=j}^{m}\beta_{il}I_l^*S_i^*\left(1-\frac{y_j}{y_{j+1}}\right)\\
&= \sum_{i=1}^{n}\sum_{j=2}^{m-3}\sum_{l=j}^{m-3}\beta_{il}S_i^*I_l^*\left(1-\frac{y_j}{y_{j+1}}\right) + \sum_{i=1}^{n}\sum_{j=2}^{m-3}\sum_{l=m-2}^{m}\beta_{il}S_i^*I_l^*\left(1-\frac{y_j}{y_{j+1}}\right)\\
&= \sum_{i=1}^{n}\sum_{j=2}^{m-3}\beta_{ij}S_i^*I_j^*\left(j-1-\sum_{l=2}^{j}\frac{y_l}{y_{l+1}}\right)\\
&\qquad\qquad + \sum_{i=1}^{n}S_i^*\left(\beta_{ic}I_c^*+\beta_{ip}I_p^*+\beta_{if}I_f^*\right)\left(m-4-\sum_{l=2}^{m-3}\frac{y_l}{y_{l+1}}\right).
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\sum_{j=2}^{m-3} b_j A_j I_j^* \left(1 - \frac{y_j}{y_{j+1}}\right) \\
&= \sum_{i=1}^{n}\sum_{j=2}^{m-3} \beta_{ij} S_i^* I_j^* \left(j - 1 - \sum_{q=2}^{j} \frac{y_q}{y_{q+1}}\right) + \sum_{i=1}^{n} S_i^* \left(\beta_{ic} I_c^* + \beta_{ip} I_p^* + \beta_{if} I_f^*\right) \\
&\times \left(m - 4 - \sum_{l=2}^{m-3} \frac{y_l}{y_{l+1}}\right) + \sum_{j=3}^{m-3} \delta_j A_{j-1} I_j^* \left(2 - \frac{y_j}{y_{j+1}} - \frac{y_{j-1}}{y_j}\right) + \delta_2 I_2^* \left(1 - \frac{y_2}{y_3}\right) \\
&+ \delta_c A_{m-3} I_c^* \left(1 - \frac{y_{m-3}}{y_c}\right).
\end{aligned}$$

Thus, substituting the three above relations into $\dot{V}(t)$, we obtain,

$$\begin{aligned}
&\dot{V}(t) = K_1(t) + K_2(t) + \sigma V^* U_v \\
&+ \sum_{i=1}^{n} \beta_{i1} S_i^* I_1^* \left(2 - x_i - \frac{1}{x_i}\right) \sum_{i=1}^{n}\sum_{j=2}^{m-3} \beta_{ij} S_i^* I_j^* \left(j + 1 - \frac{1}{x_i} - \frac{x_i y_j}{y_1} - \sum_{l=1}^{j-1} \frac{y_l}{y_{l+1}}\right) \\
&+ \sum_{i=1}^{n} \beta_{ic} S_i^* I_c^* \left(m - 2 - \frac{1}{x_i} - \frac{x_i y_c}{y_1} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}}\right) \\
&+ \sum_{i=1}^{n} \beta_{ip} S_i^* I_p^* \left(m - 2 - \frac{1}{x_i} - \frac{x_i y_p}{y_1} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}}\right) \\
&+ \sum_{i=1}^{n} \beta_{if} S_i^* I_f^* \left(m - 2 - \frac{1}{x_i} - \frac{x_i y_f}{y_1} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}}\right) \\
&+ \sum_{j=2}^{m-2} \delta_j A_{j-1} I_j^* \left(2 - \frac{y_{j-1}}{y_j} - \frac{y_j}{y_{j-1}}\right).
\end{aligned}$$

where

$$\begin{aligned}
K_2(t) = &- \delta_c A_{m-3} I_c^* + I_c^* \left(b_c A_c - k_c A_p \frac{y_c}{y_p} - k_c' A_f \frac{y_c}{y_f}\right) + I_p^* \left(b_p A_p - \delta_p A_c \frac{y_p}{y_c} - k_p A_f \frac{y_p}{y_f}\right) \\
&+ I_f^* \left(b_f A_f - \delta_f' A_c \frac{y_f}{y_c} - \delta_f A_p \frac{y_f}{y_p}\right).
\end{aligned}$$

Let us find a suitable expression of $K_2(t)$. From the fourth, the fifth, the sixth and the seventh equations in (30), we deduce respectively that:

$$\begin{cases} b_c A_c = k_c A_p + k'_c A_f + \delta_c A_{m-3} + \sum_{i=1}^{n} \beta_{ic} S_i^*, \\ b_p A_p = k_p A_f + \delta_p A_c + \sum_{i=1}^{n} \beta_{ip} S_i^*, \\ b_f A_f = \delta_f A_p + \delta'_f A_c + \sum_{i=1}^{n} \beta_{if} S_i^*. \end{cases} \tag{44}$$

As a result

$$\begin{aligned} &K_2(t) \\ &= k_p A_f I_p^* \left(1 - \frac{y_p}{y_f}\right) + \delta_p A_c I_p^* \left(2 - \frac{y_p}{y_c}\right) + \delta_f A_p I_f^* \left(1 - \frac{y_f}{y_p}\right) + \delta'_f A_c I_f^* \left(2 - y_f y_c\right) \\ &\quad - k_c A_p I_c^* \frac{y_c}{y_p} - k'_c A_f I_c^* \frac{y_c}{y_f} + \sum_{i=1}^{n} \beta_{ic} S_i^* I_c^* + 2\sum_{i=1}^{n} \beta_{ip} S_i^* I_p^* + 2\sum_{i=1}^{n} \beta_{if} S_i^* I_f^*. \end{aligned}$$

From the height and the ninth equations in System (31) we have respectively:

$$k_p I_p^* = b_f I_f^* - k'_c I_c^* \quad \text{and} \quad \delta_f I_f^* = b_p I_p^* - k_c I_c^*.$$

Then, taking into account the second and the third relations in System (44), we get

$$k_p A_f I_p^* = b_f A_f I_f^* - k'_c A_f I_c^* \quad \text{i.e.,} \quad k_p A_f I_p^* = \delta_f A_p I_f^* + \delta'_f A_c I_f^* - k'_c A_f I_c^* + \sum_{i=1}^{n} \beta_{if} S_i^* I_f^*. \tag{45}$$

and

$$\delta_f A_p I_f^* = b_p A_p I_p^* - k_c A_p I_c^* \quad \text{i.e.,} \quad \delta_f A_p I_f^* = k_p A_f I_p^* + \delta_p A_c I_p^* - k_c A_p I_c^* + \sum_{i=1}^{n} \beta_{ip} S_i^* I_p^*. \tag{46}$$

Substituting (46) into (45), we easily deduce the relation

$$\delta_p A_c I_p^* = k_c A_p I_c^* + k'_c A_f I_c^* - \delta'_f A_c I_f^* - \sum_{i=1}^{n} \beta_{ip} S_i^* I_p^* - \sum_{i=1}^{n} \beta_{if} S_i^* I_f^*. \tag{47}$$

Plugging (45) and (47) into $K_2(t)$, we get the suitable expression of $K_2(t)$ that follows

$$
\begin{aligned}
&K_2(t)\\
&= \delta_f A_p I_f^* \left(2 - \frac{y_p}{y_f} - \frac{y_f}{y_p}\right) + \delta_f' A_c I_f^* \left(1 + \frac{y_p}{y_c} - \frac{y_p}{y_f} - \frac{y_f}{y_c}\right) + k_c A_p I_c^* \left(2 - \frac{y_p}{y_c} - \frac{y_c}{y_p}\right)\\
&\quad + k_c' A_f I_c^* \left(1 - \frac{y_p}{y_c} + \frac{y_p}{y_f} - \frac{y_c}{y_f}\right) + \sum_{i=1}^{n} \beta_{ic} S_i^* I_c^* + \sum_{i=1}^{n} \beta_{ip} S_i^* I_p^* \frac{y_p}{y_c}\\
&\quad + \sum_{i=1}^{n} \beta_{if} S_i^* I_f^* \left(1 + \frac{y_p}{y_c} - \frac{y_p}{y_f}\right)\\
&= \left(\delta_f A_p I_f^* - k_c' A_f I_c^*\right) \left(2 - \frac{y_p}{y_f} - \frac{y_f}{y_p}\right) + \delta_f' A_c I_f^* \left(1 + \frac{y_p}{y_c} - \frac{y_p}{y_f} - \frac{y_f}{y_c}\right) + k_c' A_f I_c^*\\
&\quad \times \left(3 - \frac{y_p}{y_c} - \frac{y_f}{y_p} - \frac{y_c}{y_f}\right) + k_c A_p I_c^* \left(2 - \frac{y_p}{y_c} - \frac{y_c}{y_p}\right) + \sum_{i=1}^{n} \beta_{ic} S_i^* I_c^* + \sum_{i=1}^{n} \beta_{ip} S_i^* I_p^* \frac{y_p}{y_c}\\
&\quad + \sum_{i=1}^{n} \beta_{if} S_i^* I_f^* \left(1 + \frac{y_p}{y_c} - \frac{y_p}{y_f}\right).
\end{aligned}
$$

Replacing $K_2(t)$ by its above expression in $\dot{V}(t)$, factoring and rearranging some terms, we obtain

$$
\begin{aligned}
&\dot{V}(t)\\
&= K_1(t) + \sigma V^* U_v + \sum_{i=1}^{n} \beta_{i1} S_i^* I_1^* \left(2 - x_i - \frac{1}{x_i}\right) + \delta_f' A_c I_f^* \left(1 + \frac{y_p}{y_c} - \frac{y_p}{y_f} - \frac{y_f}{y_c}\right)\\
&\quad + k_c A_p I_c^* \left(2 - \frac{y_p}{y_c} - \frac{y_c}{y_p}\right) + k_c' A_f I_c^* \left(3 - \frac{y_p}{y_c} - \frac{y_f}{y_p} - \frac{y_c}{y_f}\right)\\
&\quad + \left(\delta_f A_p I_f^* - k_c' A_f I_c^*\right) \left(2 - \frac{y_p}{y_f} - \frac{y_f}{y_p}\right) + \sum_{j=2}^{m-2} \delta_j A_{j-1} I_j^* \left(2 - \frac{y_{j-1}}{y_j} - \frac{y_j}{y_{j-1}}\right)\\
&\quad + \sum_{i=1}^{n} \sum_{j=2}^{m-3} \beta_{ij} S_i^* I_j^* \left(j + 1 - \frac{1}{x_i} - \frac{x_i y_j}{y_1} - \sum_{l=1}^{j-1} \frac{y_l}{y_{l+1}}\right) + \sum_{i=1}^{n} \beta_{ip} S_i^* I_p^* \Bigg(m - 2 + \frac{y_p}{y_c}\\
&\quad - \frac{1}{x_i} - \frac{x_i y_p}{y_1} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}}\Bigg) + \sum_{i=1}^{n} \beta_{ic} S_i^* I_c^* \left(m - 1 - \frac{1}{x_i} - \frac{x_i y_c}{y_1} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}}\right)\\
&\quad + \sum_{i=1}^{n} \beta_{if} S_i^* I_f^* \left(m - 1 + \frac{y_p}{y_c} - \frac{y_p}{y_f} - \frac{1}{x_i} - \frac{x_i y_f}{y_1} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}}\right).
\end{aligned}
$$

Recall that $m-2 = c$. Substituting the expressions of I_p^*, I_f^* as given in (32), and the ones of A_p, A_f as defined in (29) into the previous expression of $\dot{V}(t)$, we have

$$\begin{aligned}
&\dot{V}(t)\\
&= K_1(t) + \sigma V^* U_v + \frac{k_c' \delta_f' b_p A_c I_c^*}{b_p b_f - k_p \delta_f}\left(2 - \frac{y_f}{y_c} - \frac{y_c}{y_f}\right) + \frac{k_c \delta_p b_f A_c I_c^*}{b_p b_f - k_p \delta_f}\left(2 - \frac{y_p}{y_c} - \frac{y_c}{y_p}\right)\\
&\quad + \xi\left(2 - \frac{y_p}{y_f} - \frac{y_f}{y_p}\right) + \frac{k_c k_p \delta_f' A_c I_c^*}{b_p b_f - k_p \delta_f}\left(3 - \frac{y_c}{y_p} - \frac{y_f}{y_c} - \frac{y_p}{y_f}\right) + \frac{k_c' \delta_p \delta_f A_c I_c^*}{b_p b_f - k_p \delta_f}\left(3 - \frac{y_p}{y_c}\right.\\
&\quad \left. - \frac{y_f}{y_p} - \frac{y_c}{y_f}\right) + \sum_{j=2}^{m-2} \delta_j A_{j-1} I_j^* \left(2 - \frac{y_{j-1}}{y_j} - \frac{y_j}{y_{j-1}}\right) + \sum_{i=1}^{n} \beta_{i1} S_i^* I_1^* U_i\\
&\quad + \sum_{i=1}^{n}\sum_{j=2}^{m-2} \beta_{ij} S_i^* I_j^* Z_j + \frac{k_c b_f I_c^*}{b_p b_f - k_p \delta_f}\sum_{i=1}^{n} \beta_{ip} S_i^* Z_{p_1} + \frac{k_c' \delta_f I_c^*}{b_p b_f - k_p \delta_f}\sum_{i=1}^{n} \beta_{ip} S_i^* Z_{p_2}\\
&\quad + \frac{k_c' b_p I_c^*}{b_p b_f - k_p \delta_f}\sum_{i=1}^{n} \beta_{if} S_i^* I_f^* Z_{f_1} + \frac{k_c k_p I_c^*}{b_p b_f - k_p \delta_f}\sum_{i=1}^{n} \beta_{if} S_i^* I_f^* Z_{f_2}.
\end{aligned}$$

with

$$\begin{aligned}
Z_j &= j + 1 - \frac{1}{x_i} - \frac{x_i y_j}{y_1} - \sum_{l=1}^{j-1} \frac{y_l}{y_{l+1}}, \quad j = 2, \dots, n-2,\\
Z_{p_1} &= m - \frac{1}{x_i} - \frac{x_i y_p}{y_1} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}} - \frac{y_c}{y_p},\\
Z_{p_2} &= m + 1 - \frac{1}{x_i} - \frac{x_i y_p}{y_1} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}} - \frac{y_c}{y_f} - \frac{y_f}{y_p},\\
Z_{f_1} &= m - \frac{1}{x_i} - \frac{x_i y_f}{y_1} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}} - \frac{y_c}{y_f},\\
Z_{f_2} &= m + 1 - \frac{1}{x_i} - \frac{x_i y_f}{y_1} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}} - \frac{y_c}{y_p} - \frac{y_p}{y_f},
\end{aligned}$$

and

$$\xi = \delta_f A_p I_f^* - k_c' A_f I_c^* + \frac{k_c' \delta_f' b_p A_c I_c^*}{b_p b_f - k_p \delta_f} + \frac{k_c' b_p I_c^*}{b_p b_f - k_p \delta_f}\sum_{i=1}^{n} \beta_{if} S_i^* I_f^*.$$

ξ is a positive constant. Indeed, substituting the expression of I_f^* into System (32), the ones of A_p and A_f as defined in (29) into ξ, we easily get

$$\begin{aligned}
\xi =& \frac{k_p \delta_f \left(k_c b_f + k_c' \delta_f\right) I_c^* \sum_{i=1}^{n} \beta_{ip} S_i^* + k_p \delta_f \left(k_c k_p + k_c' b_p\right) I_c^* \sum_{i=1}^{n} \beta_{if} S_i^*}{\left(b_p b_f - k_p \delta_f\right)^2}\\
&+ \frac{k_p A_c I_c^* \left[k_c \left(\delta_p \delta_f b_f + k_p \left(\delta_f'\right)^2\right) + k_c' \delta_f \left(\delta_p \delta_f + \delta_f' b_p\right)\right]}{\left(b_p b_f - k_p \delta_f\right)^2}.
\end{aligned}$$

In the case where $\theta_i = 0$, for all i, $i = 1, \cdots, n-1$, , we have $a_i = \mu_i$ and the first sum in $K_1(t)$ is zero (see (43)). Then, keeping in mind the above definitions of Z_{p_1}, Z_{p_2}, Z_{f_1}, Z_{f_2} and Z_j, $j = 2, \ldots m-2$, also the one of U_i, $i = 1, \ldots n$, as given in (39) and the one of U_v as given in (42), by the property that the arithmetic mean is greater than or equal to the geometric mean, it is easy to see that $\dot{V}(t) \leqslant 0$. Moreover $\dot{V}(t) = 0$ only if $S_i = S_i^*, 1 \leqslant i \leqslant n$, $V = V^*$ and $\dfrac{I_j}{I_j^*} = \dfrac{I_1}{I_1^*}$, $2 \leqslant j \leqslant m$. Thus the largest invariant set contained in the set $\{(S_i, I_j, V, R) \in \Omega, 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ \dot{V} = 0\}$ is reduced to the endemic equilibrium. Since trajectories of System (1) are bounded, by Lasalle's invariance principle [1, 11], the endemic equilibrium is globally and asymptotically stable on the compact Ω.

Let us continue the proof in the case where parameters θ_i, $i = 1, \cdots, n-1$, are not all zero. Firstly, let us find a suitable expression of $K_1(t)$.

It follows from the n first equations in System (31) that for $i = 1, \cdots, n-1$,

$$\theta_i S_i^* = \frac{\mu_v \pi S_n^*}{\mu_v + \sigma} + \sum_{k=i+1}^{n} \left(\mu_k S_k^* - \Lambda_k + \sum_{j=1}^{m} \beta_{kj} S_k^* I_j^* \right), \qquad i = 1, \cdots, n-1. \quad (48)$$

Indeed, It follows from the third equation in System (31) and relation (41) that,

$$\theta_{n-1} S_{n-1}^* = \frac{\mu_v \pi S_n^*}{\mu_v + \sigma} + \left(\mu_n S_n^* - \Lambda_n + \sum_{j=1}^{m} \beta_{nj} S_n^* I_j^* \right).$$

Hence, Relation (48) is valid for $i = n-1$. Afterwards, suppose that Relation (48) holds for some i, $2 \leqslant i \leqslant n-1$. We deduce from the second equation in System (31) that

$$\theta_{i-1} S_{i-1}^* = -\Lambda_i + \mu_i S_i^* + \theta_i S_i^* + \sum_{j=1}^{m} \beta_{ij} S_i^* I_j^*.$$

By the induction assumption,

$$\begin{aligned}
\theta_{i-1} S_{i-1}^* =& -\Lambda_i + \mu_i S_i^* + \sum_{j=1}^{m} \beta_{ij} S_i^* I_j^* + \frac{\mu_v \pi S_n^*}{\mu_v + \sigma} + \sum_{k=i+1}^{n} \left(\mu_k S_k^* - \Lambda_k + \sum_{j=1}^{m} \beta_{kj} S_k^* I_j^* \right) \\
=& \frac{\mu_v \pi S_n^*}{\mu_v + \sigma} + \sum_{k=i}^{n} \left(\mu_k S_k^* - \Lambda_k + \sum_{j=1}^{m} \beta_{kj} S_k^* I_j^* \right)
\end{aligned}$$

Therefore, (48) is valid for $i-1$. By induction, the relation (48) holds for all i, $1 \leqslant i \leqslant n-1$.

Now, since $a_i = (\mu_i + \theta_i),\ \ i = 1, \dots, n-1,$ we have

$$\begin{aligned}
&\sum_{i=1}^{n-1} a_i S_i^* U_i + \sum_{i=1}^{n-1} \theta_i S_i^* W_i \\
&= \sum_{i=1}^{n-1} \mu_i S_i^* U_i + \sum_{i=1}^{n-1} \theta_i S_i^* (U_i + W_i) \\
&= \sum_{i=1}^{n-1} \mu_i S_i^* U_i + \sum_{i=1}^{n-1} \left(\frac{\mu_v \pi S_n^*}{\mu_v + \sigma} + \sum_{k=i+1}^{n} \left(\mu_k S_k^* - \Lambda_k + \sum_{j=1}^{m} \beta_{kj} S_k^* I_j^* \right) \right) (U_i + W_i).
\end{aligned}$$

Using relation (28) in Lemma6.1, it is obvious to see that for all sequence of real numbers (u_n), for all $n \geqslant 2$, we have

$$\sum_{i=1}^{n-1} \sum_{k=i+1}^{n} u_k (U_i + W_i) = \sum_{i=2}^{n} u_i \sum_{k=1}^{i-1} (U_k + W_k) = \sum_{i=2}^{n} u_i \left(T_i - U_i \right).$$

where for $i = 2, \dots, n,$

$$T_i = i + 1 - x_i - \frac{1}{x_1} - \sum_{k=1}^{i-1} \frac{x_k}{x_{k+1}}.$$

Therefore, keeping relation (48) in mind and substituting $a_n = \mu_n + \pi$, we have

$$\begin{aligned}
&K_1(t) \\
&= \sum_{i=1}^{n-1} \mu_i S_i^* U_i + \mu_n S_n^* + \frac{\mu_v \pi S_n^*}{\mu_v + \sigma} U_n + \frac{\mu_v \pi S_n^*}{\mu_v + \sigma} \left(T_n - U_n \right) \\
&\qquad + \sum_{i=2}^{n} \left(\mu_i S_i^* - \Lambda_i + \sum_{j=1}^{m} \beta_{ij} S_i^* I_j^* \right) \left(T_i - U_i \right)) \\
&= \sum_{i=1}^{n} \mu_i S_i^* U_i + \frac{\mu_v \pi S_n^*}{\mu_v + \sigma} T_n + \sum_{i=2}^{n} \left(\mu_i S_i^* - \Lambda_i + \sum_{j=1}^{m} \beta_{ij} S_i^* I_j^* \right) \left(T_i - U_i \right) \\
&= \mu_1 S_1^* U_1 + \frac{\mu_v \pi S_n^*}{\mu_v + \sigma} T_n + \sum_{i=2}^{n} \left(T_i - U_i \right) \sum_{j=2}^{m} \beta_{ij} S_i^* I_j^* - \sum_{i=2}^{n} \beta_{i1} S_i^* I_1 U_i + \sum_{i=2}^{n} \Lambda_i U_i \\
&\qquad + \sum_{i=2}^{n} \left(\mu_i S_i^* + \beta_{i1} S_i^* I_1^* - \Lambda_i \right) T_i.
\end{aligned}$$

Notice that,

$$\sum_{j=2}^{m} \beta_{ij} S_i^* I_j^* = \sum_{j=2}^{m-2} \beta_{ij} S_i^* I_j + \beta_{ip} S_i^* I_p + \beta_{if} S_i^* I_f.$$

Keeping all this in mind, substituting the expressions of I_p^* and I_f^* as given in System (32) into the last expression of $\dot{V}(t)$, we obtain

$$
\begin{aligned}
&\dot{V}(t)\\
&= \sigma V^* U_v + \left(\mu_1 S_1^* + \beta_{11} S_1^* I_1^*\right) U_1 + \sum_{i=2}^{n} \Lambda_i U_i + \frac{\mu_v \pi S_n^*}{\mu_v + \sigma} T_n + \sum_{i=2}^{n} (\mu_i S_i^* + \beta_{i1} S_i^* I_1^*\\
&\quad - \Lambda_i) T_i + \frac{k_c' \delta_f' b_p A_c I_c^*}{b_p b_f - k_p \delta_f} \left(2 - \frac{y_f}{y_c} - \frac{y_c}{y_f}\right) + \frac{k_c \delta_p b_f A_c I_c^*}{b_p b_f - k_p \delta_f} \left(2 - \frac{y_p}{y_c} - \frac{y_c}{y_p}\right)\\
&\quad + \xi \left(2 - \frac{y_p}{y_f} - \frac{y_f}{y_p}\right) + \frac{k_c k_p \delta_f' A_c I_c^*}{b_p b_f - k_p \delta_f} \left(3 - \frac{y_c}{y_p} - \frac{y_f}{y_c} - \frac{y_p}{y_f}\right) + \frac{k_c' \delta_p \delta_f A_c I_c^*}{b_p b_f - k_p \delta_f}\\
&\quad \times \left(3 - \frac{y_p}{y_c} - \frac{y_f}{y_p} - \frac{y_c}{y_f}\right) + \sum_{j=2}^{m-2} \delta_j A_{j-1} I_j^* \left(2 - \frac{y_{j-1}}{y_j} - \frac{y_j}{y_{j-1}}\right) + \sum_{j=2}^{m-2} \beta_{1j} S_1^* I_j^* Z_1\\
&\quad + \sum_{i=2}^{n} \sum_{j=2}^{m-2} \beta_{ij} S_i^* I_j^* \widetilde{Z}_{ij} + \frac{k_c b_f I_c^*}{b_p b_f - k_p \delta_f} \sum_{i=1}^{n} \beta_{ip} S_i^* \widetilde{Z}_{p_1} + \frac{k_c' \delta_f I_c^*}{b_p b_f - k_p \delta_f} \sum_{i=1}^{n} \beta_{ip} S_i^* \widetilde{Z}_{p_2}\\
&\quad + \frac{k_c' b_p I_c^*}{b_p b_f - k_p \delta_f} \sum_{i=1}^{n} \beta_{if} S_i^* I_f^* \widetilde{Z}_{f_1} + \frac{k_c k_p I_c^*}{b_p b_f - k_p \delta_f} \sum_{i=1}^{n} \beta_{if} S_i^* I_f^* \widetilde{Z}_{f_2},
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{Z}_{ij} &= i + j - \frac{1}{x_1} - \frac{x_i y_j}{y_1} - \sum_{k=1}^{i-1} \frac{x_k}{x_{k+1}} - \sum_{l=1}^{j-1} \frac{y_l}{y_{l+1}}, \quad i = 2, \ldots, n, \quad j = 2, \ldots, m-2,\\
\widetilde{Z}_{p_1} &= i + m - 1 - \frac{1}{x_1} - \frac{y_c}{y_p} - \frac{x_i y_p}{y_1} - \sum_{k=1}^{i-1} \frac{x_k}{x_{k+1}} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}},\\
\widetilde{Z}_{p_2} &= i + m - \frac{1}{x_1} - \frac{x_i y_p}{y_1} - \frac{y_c}{y_f} - \frac{y_f}{y_p} - \sum_{k=1}^{i-1} \frac{x_k}{x_{k+1}} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}},\\
\widetilde{Z}_{f_1} &= i + m - 1 - \frac{1}{x_1} - \frac{x_i y_f}{y_1} - \frac{y_c}{y_f} - \sum_{k=1}^{i-1} \frac{x_k}{x_{k+1}} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}},\\
\widetilde{Z}_{f_2} &= i + m - \frac{1}{x_1} - \frac{x_i y_f}{y_1} - \frac{y_c}{y_p} - \frac{y_p}{y_f} - \sum_{k=1}^{i-1} \frac{x_k}{x_{k+1}} - \sum_{l=1}^{m-3} \frac{y_l}{y_{l+1}}.
\end{aligned}
$$

Hence, if $\mu_i S_i^* + \beta_{i1} S_i^* I_1^* \geqslant \Lambda_i$, for all $i,\ i = 2, \cdots, n$, as in the previous case, by the arithmetic geometric inequality, it is easy to see that $\dot{V}(t) \leqslant 0$, and that the equality holds only for $S_i = S_i^*,\ i = 1, \cdots, n$, $V = V^*$ and $\dfrac{I_j}{I_j^*} = \dfrac{I_1}{I_1^*},\ \ j = 2, \cdots, m$

Therefore in this case where parameters $\theta_i,\ i = 1, \cdots, n-1$, are not all zero, we conclude as previously that the endemic equilibrium is globally asymptotically stable on the nonnegative orthant $\mathbb{R}_+^{n+m+1}$ when $\mu_i S_i^* + \beta_{i1} S_i^* I_1^* \geqslant \Lambda_i$, for all $i,\ i = 2, \cdots, n$. This achieves the proof of Theorem 5.1. ■