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On D-Efficiency of Reduced Models for Central Composite Experimental Designs within a Split-Plot Structure

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Abstract. Choosing a response surface design to fit certain kinds of models is a difficult task. Extensive research comprising a collection of efficient second-order response surface designs from which a researcher may choose to best fit his/her needs has been conducted, which are based solely on a widely-accepted assumption of a completely randomized error structure of statistically-designed experiments. However, this assumption is not feasible in industrial experiments, which are often split-plot in nature and for which randomization of some factors have to be restricted due to certain constraints. The performance of such experimental designs depends strongly on the relative magnitude (d) of the whole-plot and sub-plot error variances. This work focuses on reduced second-order models having one, two, or all of their quadratic and/or interaction terms removed from the full models of some chosen candidate split-plot central composite designs (CCDs). It investigates the effects of model reduction on efficiency of these designs by computing the relative D-efficiencies for the formulated reduced models with respect to their corresponding full designs and assessing the efficiency losses under specific values of d. The study revealed a significant loss of D-efficiency in these designs, which depend strongly on the removed term(s) and increases, across all values of d, as the number of whole-plot factors increases.

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Keywords: Relative *D*-efficiency loss; Split-plot design; Design optimality criteria; Reduced models.

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1. Introduction

Experiments are performed by researchers in industry, manufacturing, engineering and physical sciences so as to uncover and model relationships between design (input) and output variables of a process and to identify optimal operating conditions for a system under study. The techniques of response surface methodology (RSM) are often employed in such experimental situations. Often when individual parameters in the proposed fitted model of a designed experiment are tested, some terms may not be significant. In such situations, the experimenter will decide to use a reduced model containing only the significant terms from the fitted original model. In RSM research and applications, statistically-designed experiments have a widely-accepted assumption of a completely randomized error structure. Under this assumption, extensive research comprising a collection of efficient second-order designs from which a researcher may choose to best fit his/her needs, has been conducted. Some authors have studied the design-selection

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problem when the proposed approximating model is an *underparameterized* approximation of the true response surface, of which the low-order polynomial is commonly used when a higher-order polynomial is a better approximating function (see, for example [2, 3, 9]). With regard to the design-problem, many authors have studied the mean squared error (MSE) and have used the integrated mean squared error (IMSE) as a design comparison criterion, see [4, 10, 14], for details on this. $\text{MSE} = V + B$, where $B = \frac{N\Omega}{\sigma^2}\int_R [E(\hat{y}(x) - \eta(x)]^2\, dx$ is the systematic (squared) bias resulting from underestimation of the true response surface with the fitted low-order model and $\text{V} = \frac{N\Omega}{\sigma^2}\int_R Var[\hat{y}(x)]\, dx$ is the prediction variance with $\Omega^{-1} = \int_R dx$ Borkowski and Elsie [1] studied the robustness against many classes of model misspecification for which the proposed approximating model is an overparameterized approximation of the true response surface. The authors presented the D, G, A, and V efficiency plots of their misspecified models against their corresponding numbers of parameters using the number of squared terms in the model as the plotting symbol. They showed that design optimality criteria can be sensitive to deviations from the full second-order response surface model, and that the CCD is robust with respect to the set of reduced models as well as across the four optimality criteria considered. Chomtee and Borkowski [6] used D- and G- optimality criteria to compare reduced models of seven response surface designs in a spherical region using only D- and G-optimality criteria to evaluate them. Their results suggest that replication affects different criteria in different ways. That is, what improves one criterion may be detrimental to another.

All the previous studies outlined above were based on the assumption of a completely randomized error structure. However, this assumption may not be feasible in industrial experiments, which are often split-plot in nature and consist of two sets of factors, those with levels that are difficult to change or control (termed hard-to-change (HTC) factors) due to time or cost constraints, and factors with levels that are easy to control (easy-to-change (ETC) factors). See [12, 13, 15], etc., for further details. For response surface designs with split-plot structure, design performance depends strongly on relative magnitude or ratio (d) of the whole-plot and subplot variance components through the variance-covariance matrix V, and so also is the design selection based on optimal properties. The four commonly-used optimality criteria for assessing the performance of an experimental design are A-, D-, G-, and V- optimality criteria with the following respective goals (See [5, 11]):

$$D\text{-criterion maximizes } |M| = |X'V^{-1}X|, \text{ or equivalently, minimizes } |(X'V^{-1}X)^{-1}|,$$

$$A\text{-criterion minimizes } \text{trace}(X'V^{-1}X)^{-1},$$

$$G \to \min_{\zeta}\left[\max_{\zeta} Nf(z,x)'(X'V^{-1}X)^{-1}f(z,x)\right],$$

$$V \to \min_{\xi}\left\{\frac{N}{K}\int_R f(z,x)'(X'V^{-1}X)^{-1}f(z,x)dz\,dx\right\},$$

where X is the design matrix, x is any point in the design region R, N is the design size, and $f(z,x) = \left[f_1(z,x) \dots f_p(z,x)\right]$ is a vector of p real-valued functions based on the p model parameters while $K = \int_R dzdx$ is the volume of the region. The variance - covariance matrix for the observation vector y is

$$\text{Var}(y) = V = \sigma_\epsilon^2 I_n + \sigma_\gamma^2 ZZ' = \sigma_\epsilon^2(I_n + dZZ'),$$

where $d = \frac{\sigma_\gamma^2}{\sigma_\epsilon^2}$ gives the ratio of the two variance components. The matrix $\boldsymbol{ZZ}$' is a block diagonal matrix with diagonal matrices of J_{n1}, J_{n2}, …, J_{nz}, where J_{ni} is an $n_i \times n_i$ matrix of 1's and n_i is the number of observations in the ith whole-plot. Under the assumption of normal errors, the maximum likelihood estimate (MLE) of the parameters of this model is obtained through the generalized least squares (GLS) estimation equation $\hat{\beta} = (X'V^{-1}X)^{-1}X'V^{-1}y$ with variance $\mathrm{Var}(\hat{\beta}) = (X'V^{-1}X)^{-1}$.
For a k-factor split-plot CCD with given numbers of whole-plot and subplot factors, the fitted second-order model is

$$E(y) = \beta_0 + \sum_{i=1}^{w}\beta_i z_i + \sum_{i=1}^{w-1}\sum_{j=i+1}^{w}\beta_{ij}z_i z_j + \sum_{i=1}^{w}\beta_{ii}z_i^2 + \sum_{i=1}^{k}\theta_i x_i + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\theta_{ij}x_i x_j + \sum_{i=1}^{w}\sum_{j=1}^{k}\gamma_{ij}z_i x_j + \sum_{i=1}^{k}\theta_{ii}x_i^2, \quad (1)$$

where y is the response variable, z is the whole-plot factor, x is the subplot factor, the $\beta's$ are the regression coefficients at the whole-plot level, $\theta's$ and $\gamma's$ are the regression coefficients at the subplot levels ([15]).

Using the above-outlined optimality criteria, Chukwu and Yakubu [7] studied the robustness against many classes of model misspecification in split-plot response surface designs under three different ratios ($d = 0.5$, 1, and 2) of the variance components, for which the proposed approximating model is an overparameterized approximation of the true response surface. For each of the variance component ratios (d) considered, the authors presented the D, G, A, and V efficiency plots of the misspecified models against their corresponding numbers of parameters using the number of squared terms in the model as the plotting symbol. They observed that, unlike [1] for completely randomized response surface designs, the optimality criteria for these designs strongly depend on the ratios (d) of the variance components and are more sensitive to changes in the pure whole plot squared terms than the subplot squared terms. This work, however, used a different approach to study the robustness against classes of model misspecification in response surface designs with restricted randomization. The contribution of each term or combination of terms in the full (proposed) second-order model in (1) to design efficiency is better investigated by studying the efficiency of the misspecified (reduced) models relative to that of the full model. In such a study, loss in efficiency of the design in question due to each misspecified model can clearly be quantified. This work therefore investigates the effect of model misspecification on relative D-efficiency of split-plot response surface designs under specific ratios (d) of the whole plot and subplot variance components.

2. Materials and methods

The designs considered include the k = three, four, and five-factor split-plot central composite designs (CCD's). The three-factor split-plot CCD consists of one whole plot and two subplot factors (here denoted as D12) with 24 points (4 center points); the four-factor split-plot CCD consists of two whole plot and two subplot factors (denoted as D22) with 40 points (4 center points), while the five-factor split-plot CCD consists of three whole plot and two subplot factors (denoted as D32) with 64 points (4 center points). Thus, these candidate designs have different numbers of whole plot factors but the same number of subplot factors and center point replications as given in Table 1 below.

Table 1. Candidate designs.

Design	w	s	k	N
1	1	2	3	24
2	2	2	4	40
3	3	2	5	64

w = number of whole plot factors, s = number of subplot factors, k = total number of factors and N = number of design points

The misspecified (reduced) models are then formed from each design by removing terms from the corresponding proposed full second-order models in (1) such that:

1. if a model contains a square (z_i^2 or x_l^2) term, then it must contain the corresponding linear (z_i or x_l) term.
2. if a model contains the interaction ($z_i z_j$, $z_i x_l$, or $x_l x_m$) term, then it must contain the corresponding z_i, x_l and/or z_j term.

The reduced models were formed from the full second-order models of the candidate split-plot CCDs in Table 1 by removing one term (whole-plot/sub-plot interaction or squared term), two terms (whole-plot/sub-plot interactions or squared terms), all interaction and squared terms (pure linear model), all squared terms (linear model and interaction terms), and all interaction terms (linear model and squared terms) from the full models of the given candidate designs. The formulated reduced models are given in Tables 2, 3, and 4, respectively. Table 2 contains the 9 models considered for a one whole-plot variable/two sub-plot variables CCD; Tables 3 and 4 contain, each, the 11 models considered for a two whole-plot variables/two sub-plot variables CCD and a three whole-plot variables/two sub-plot variables CCD, respectively. Columns indicate the number of model parameters (p) and the number of design variables (dv) present in the model. The 1s and 0s in the L, Q, and C columns indicate, respectively, the presence or absence of that term in the reduced model, while l, q, and c indicates, respectively, the number of pure linear terms, the number of pure quadratic terms, and the number of cross product terms in the reduced model.

We denote the determinant of the information matrix of a reduced design by $|X'V^{-1}X|_r$ and that of the full design by $|X'V^{-1}X|_f$, then the relative D-efficiency of the reduced design compared to the full design is given as

$$RE_D = \frac{|X'V^{-1}X|_r}{|X'V^{-1}X|_f} \qquad (2)$$

For each of the candidate designs, the D-criterion values were first computed for the proposed full model and for the formulated misspecified (reduced) models under specific ratios (d) of the whole plot and subplot error variances, where we have chosen d = 0.5, 1.0, 1.5, 2.0, and 2.5, to indicate the situations that the whole plot error variance is half, same, one and a half, two, and, two and a half, times the subplot error variance, respectively. The D-criterion values were computed using Maple software (Maple15). Next, the effect of model misspecification on the design was quantified by computing the relative D-efficiencies for the given reduced models in Tables 2 to 4 with respect to their corresponding full designs under the given values of d. A relative D-efficiency less than one indicates that the design fitted with the full model is better than the one fitted with the reduced model in terms of D-optimality. This also implies that the design has been adversely affected by the role change of the full model. Losses in D-efficiency of these designs (due to the misspecification) were investigated.

Table 2. Reduced models ($\boldsymbol{K = 3},\ \boldsymbol{w = 1},\ \boldsymbol{s = 2}$).

Model	$\boldsymbol{P}$	$\boldsymbol{dv}$	$\boldsymbol{L}$	$\boldsymbol{Q}$	$\boldsymbol{C}$	$\boldsymbol{l, q, c}$
1	9	3	(1,1,1)	(1,1,1)	(1,0,1)	(3,3,2)
2	9	3	(1,1,1)	(1,0,1)	(1,1,1)	(3,2,3)
3	9	3	(1,1,1)	(0,1,1)	(1,1,1)	(3,2,3)
4	8	3	(1,1,1)	(1,1,1)	(0,1,0)	(3,3,1)
5	8	3	(1,1,1)	(1,0,0)	(1,1,1)	(3,1,3)
6	8	3	(1,1,1)	(0,0,1)	(1,1,1)	(3,1,3)
7	4	3	(1,1,1)	(0,0,0)	(0,0,0)	(3,0,0)
8	7	3	(1,1,1)	(0,0,0)	(1,1,1)	(3,0,3)
9	7	3	(1,1,1)	(1,1,1)	(0,0,0)	(3,3,0)

NOTE: $\boldsymbol{P}$ = no. of model parameters; $\boldsymbol{dv}$ = no. of design variables appearing in the reduced model.
Model terms: $\boldsymbol{L} = (\boldsymbol{z_1}, \boldsymbol{x_1}, \boldsymbol{x_2})$; $\boldsymbol{Q} = \left(\boldsymbol{z_1^2}, \boldsymbol{x_1^2}, \boldsymbol{x_2^2}\right)$; $\boldsymbol{C} = (\boldsymbol{z_1 x_1}, \boldsymbol{z_1 x_2}, \boldsymbol{x_1 x_2})$

Table 3. Reduced Models ($\boldsymbol{K = 4},\ \boldsymbol{w = 2},\ \boldsymbol{s = 2}$).

Model	$\boldsymbol{P}$	$\boldsymbol{dv}$	$\boldsymbol{L}$	$\boldsymbol{Q}$	$\boldsymbol{C}$	$\boldsymbol{l, q, c}$
1	14	4	(1,1,1,1)	(1,1,1,1)	(1,1,1,1,1,0)	(4,4,5)
2	14	4	(1,1,1,1)	(1,1,1,1)	(0,1,1,1,1,1)	(4,4,5)
3	14	4	(1,1,1,1)	(1,1,0,1)	(1,1,1,1,1,1)	(4,3,6)
4	14	4	(1,1,1,1)	(0,1,1,1)	(1,1,1,1,1,1)	(4,3,6)
5	13	4	(1,1,1,1)	(1,1,1,1)	(0,1,1,1,1,0)	(4,4,4)
6	13	4	(1,1,1,1)	(1,1,0,0)	(1,1,1,1,1,1)	(4,2,6)
7	13	4	(1,1,1,1)	(0,0,1,1)	(1,1,1,1,1,1)	(4,2,6)
8	13	4	(1,1,1,1)	(0,1,0,1)	(1,1,1,1,1,1)	(4,2,6)
9	5	4	(1,1,1,1)	(0,0,0,0)	(0,0,0,0,0,0)	(4,0,0)
10	11	4	(1,1,1,1)	(0,0,0,0)	(1,1,1,1,1,1)	(4,0,6)
11	9	4	(1,1,1,1)	(1,1,1,1)	(0,0,0,0,0,0)	(4,4,0)

NOTE: $\boldsymbol{P}$ = no. of model parameters; $\boldsymbol{dv}$ = no. of design variables appearing in the reduced model.
Model terms: $\boldsymbol{L} = (\boldsymbol{z_1}, \boldsymbol{z_2}, \boldsymbol{x_1}, \boldsymbol{x_2})$; $\boldsymbol{Q} = \left(\boldsymbol{z_1^2}, \boldsymbol{z_2^2}, \boldsymbol{x_1^2}, \boldsymbol{x_2^2}\right)$; $\boldsymbol{C} =$
$(\boldsymbol{z_1 z_2}, \boldsymbol{z_1 x_1}, \boldsymbol{z_1 x_2},$
$\boldsymbol{z_2 x_1}, \boldsymbol{z_2 x_2}, \boldsymbol{x_1 x_2})$

3. Results and discussion

In this section, we compare the loss in relative *D*-efficiency for the sets of reduced models in Tables 1 to 3 for the respective candidate split-plot CCDs in Table 1 for the given ratios of *d*.

3.1 *Removing one term from the full model*

The resulting relative D-efficiencies are displayed in Figures 1(a), 1(b), 1(c), and 1(d), respectively when one term, (whole-plot/sub-plot interaction or squared terms) are removed from the full models of these designs.

Removing a subplot interaction term: The resulting relative D-efficiencies when a sub-plot interaction (cross product) term is removed from the full model of each of these

Table 4. Reduced Models ($\boldsymbol{K=5,\ w=3,\ s=2}$).

Model	$\boldsymbol{P}$	$\boldsymbol{dv}$	$\boldsymbol{L}$	$\boldsymbol{Q}$	$\boldsymbol{C}$	$\boldsymbol{l,q,c}$
1	20	5	(1,1,1,1,1)	(1,1,1,1,1)	(1,1,1,1,1,1,1,1,1,0)	(5,5,9)
2	20	5	(1,1,1,1,1)	(1,1,1,1,1)	(0,1,1,1,1,1,1,1,1,1)	(5,5,9)
3	20	5	(1,1,1,1,1)	(1,1,1,1,0)	(1,1,1,1,1,1,1,1,1,1)	(5,4,10)
4	20	5	(1,1,1,1,1)	(0,1,1,1,1)	(1,1,1,1,1,1,1,1,1,1)	(5,4,10)
5	19	5	(1,1,1,1,1)	(1,1,1,1,1)	(0,1,1,1,1,1,1,1,1,0)	(5,5,8)
6	19	5	(1,1,1,1,1)	(1,1,1,0,0)	(1,1,1,1,1,1,1,1,1,1)	(5,3,10)
7	19	5	(1,1,1,1,1)	(0,0,1,1,1)	(1,1,1,1,1,1,1,1,1,1)	(5,3,10)
8	19	5	(1,1,1,1,1)	(0,1,1,1,0)	(1,1,1,1,1,1,1,1,1,1)	(5,3,10)
9	6	5	(1,1,1,1,1)	(0,0,0,0,0)	(0,0,0,0,0,0,0,0,0,0)	(5,0,0)
10	16	5	(1,1,1,1,1)	(0,0,0,0,0)	(1,1,1,1,1,1,1,1,1,1)	(5,0,10)
11	11	5	(1,1,1,1,1)	(1,1,1,1,1)	(0,0,0,0,0,0,0,0,0,0)	(5,5,0)

NOTE: $\boldsymbol{P}$ = no. of model parameters; $\boldsymbol{dv}$ = no. of design variables appearing in the reduced model.
Model terms: $\boldsymbol{L=(z_1,z_2,z_3,x_1,x_2)}$; $\boldsymbol{Q=\left(z_1^2,z_2^2,z_3^2,x_1^2,x_2^2\right)}$; $\boldsymbol{C=}$ $\boldsymbol{(z_1z_2,z_1z_3,z_2z_3,z_1x_1,}$ $\boldsymbol{z_1x_2,z_2x_1,z_2x_2,z_3x_1,z_3x_2,x_1x_2)}$

designs are displayed in Figure 1(a). The plots show a significant loss in D-efficiency of the designs across all ratios (d) of the error variance components. These efficiency losses were also observed to be periodic, as each loss repeats itself over and over across all values of d, with a period length. The D32 CCD recorded the highest loss in D-efficiency, followed by the D22 CCD while the D12 CCD recorded the lowest efficiency loss across all values of d. The D-efficiency loss in the D12 CCD ranges from 75% to 87.5% with a period length of 12.5%; the efficiency loss in the D22 CCD ranges from 87.5% to 93.75% with a period length of 6.25%, which is half the period length of the loss in the D12 CCD. The loss in the D32 CCD ranges from 93.75% to 96.875% with a period length of 3.125%, which is also half the period length of the loss in the D22 CCD.

Removing a subplot or a whole plot squared term: Figure 1(b) and 1(c) show, respectively, plots of the resulting relative D-efficiencies when a subplot and a whole plot squared term is removed from the full model of these designs. Each of the figures shows a significant loss in D-efficiency, which fluctuates but maintains a decreasing trend as values of d rises. The D32 CCD turns out to be the most adversely affected in each case by this alteration of its full model, as it recorded the highest efficiency loss across all values of d. Figure 1(b) shows that when a subplot squared term is removed, the lowest efficiency losses for the D12, D22, and D32 CCDs were 12.48%, 40.28%, and 64.21% respectively at d = 2.5, while their highest efficiency losses were observed as 75.62%, 84.72%, and 91.32%, respectively, at d = 1.0. Figure 2(c) shows that when a whole-plot squared term is removed, the lowest efficiency losses for the D12, D22, and D32 CCDs were 10.03%, 46.27%, and 67.63% respectively at d = 2.5, while their highest efficiency losses were observed as 79.55%, 86.94%, and 92.64%, respectively, at d = 1.0. This shows that the smaller the number of whole-plot variables in these designs, the more the improvement in D-efficiency, provided the number of the sub-plot factors is fixed. These results agree perfectly with those of Goos and Vandebroek [8] for full split-plot CCDs, which revealed that the largest D-efficiency improvements can be realized as the number of whole-plot variables decreases.

Removing a whole-plot interaction term: Figure 1(d) shows plots of the resulting relative D-efficiency when a pure whole-plot interaction term is removed from the full model of the D22 and D32 CCDs. From this figure, a significant loss in D-efficiency is observed in each design and the losses fluctuate but maintain a decreasing trend as d increases. The figure shows that the D22 CCD gains efficiency over the D32 CCD as it maintains a lesser loss in D-efficiency across all values of d.

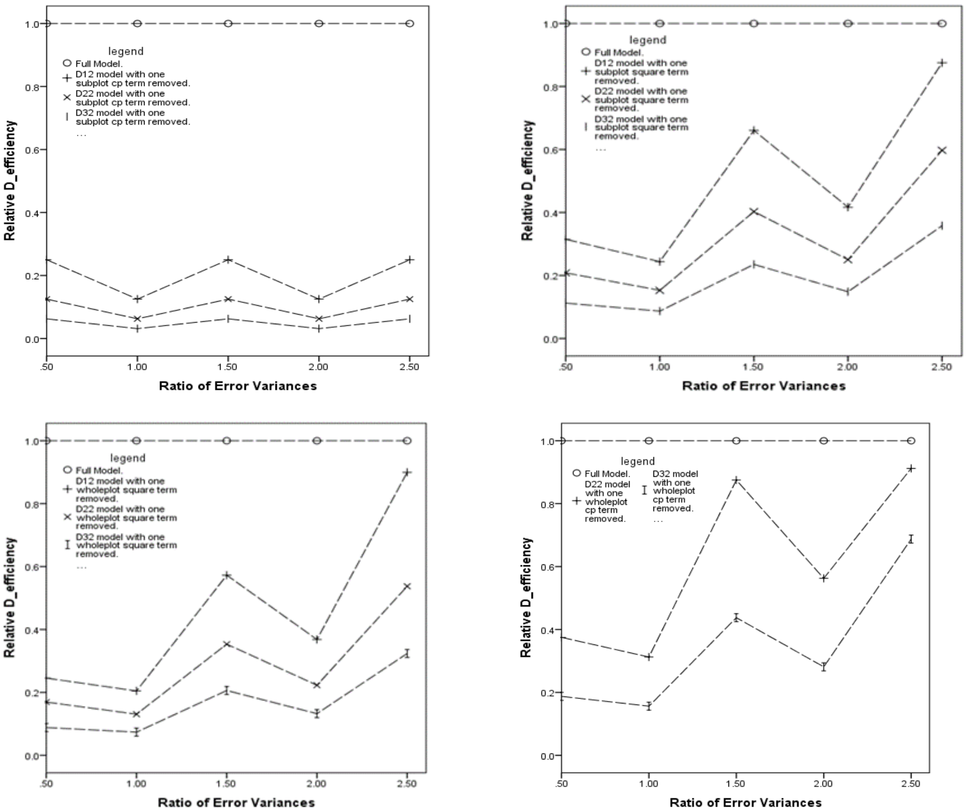


Figure 1. Plots of reduced model relative efficiencies for the one whole-plot variable/two sub-plot variables, two whole-plot variables/two sub-plot variables, and three whole-plot variables/two sub-plot variables CCDs. Plots (a), (b), (c), and (d) contain relative D-efficiencies for the three designs, respectively, when a sub-plot interaction, a sub-plot squared, a whole-plot squared, and a whole-plot interaction terms are removed.

3.2 *Removing two terms from the full model*

The resulting relative D-efficiencies are displayed in Figures 2(a), 2(b), 2(c), and 2(d), respectively when two terms (whole-plot/sub-plot interaction or squared terms) are removed from the full models of the candidate CCDs.

Removing two sub-plot squared terms: The resulting relative D-efficiencies when two sub-plot squared terms are removed from the full model of each of these designs are displayed in Figure 2(a). The plots show a significant loss in D-efficiency of the designs across all ratios (d) of the error variance components. From the figure, the lowest D-efficiency losses for the D12, D22, and D32 CCDs were respectively observed to be 78.88%, 92.93%, and 97.30% at d = 2.5, while their highest efficiency losses were 97.60%, 99.10%, and 99.70%, respectively at d = 1.0. Thus, the D32 CCD recorded the highest loss in D-efficiency, followed by the D22 CCD while the D12 CCD recorded the lowest efficiency loss across all values of d.

Removing a whole-plot and a sub-plot squared terms: Figure 2(b) shows plots of the resulting relative D-efficiencies when a whole-plot and a sub-plot squared terms are removed from the full model of these designs. From this figure, we observed that the D12 CCD surpasses the D22 and the D32 CCDs across all values of d, especially at d = 1.5 and 2.5. The D32 suffers the highest efficiency loss at all values of d, followed by the D22 CCD. Each of these designs recorded their lowest and highest D-efficiency losses of 38.98%, 79.25%, 92.20%, and 96.54%, 98.73%, 99.56%, respectively, at d = 2.5 and 1.0 as can directly be seen from the figure.

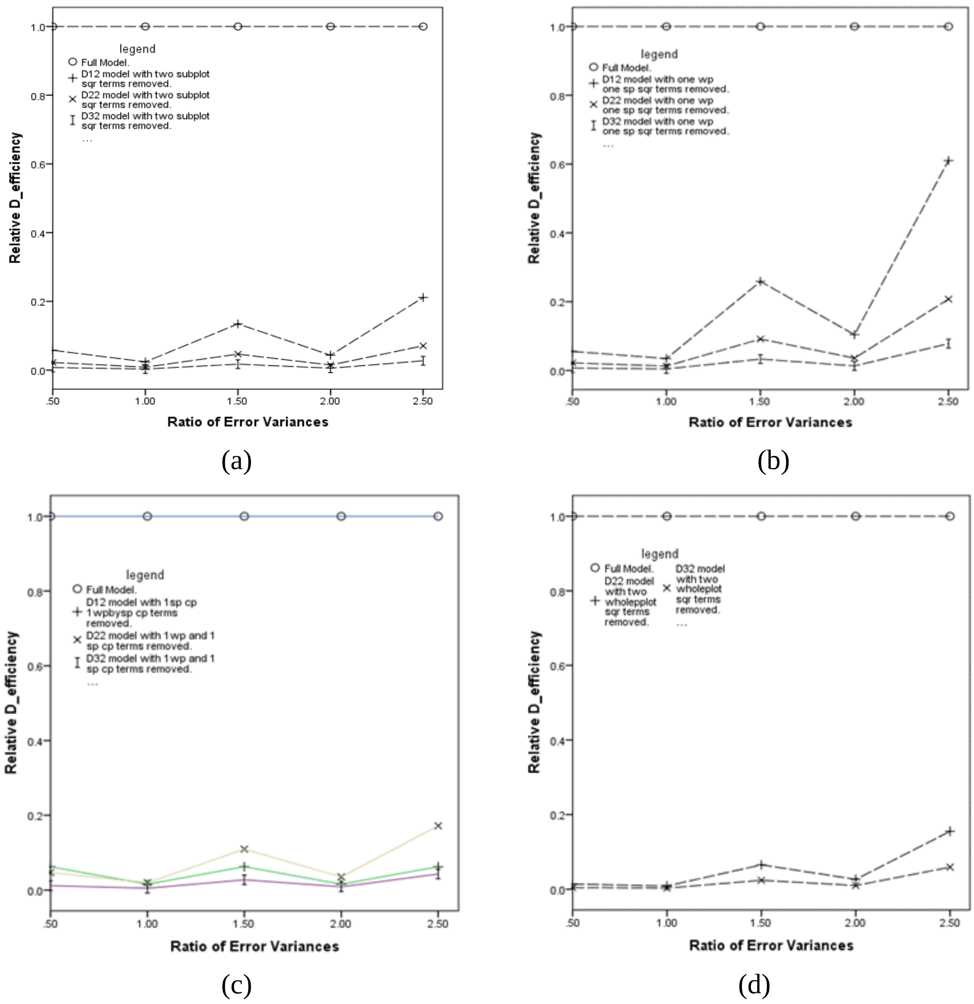


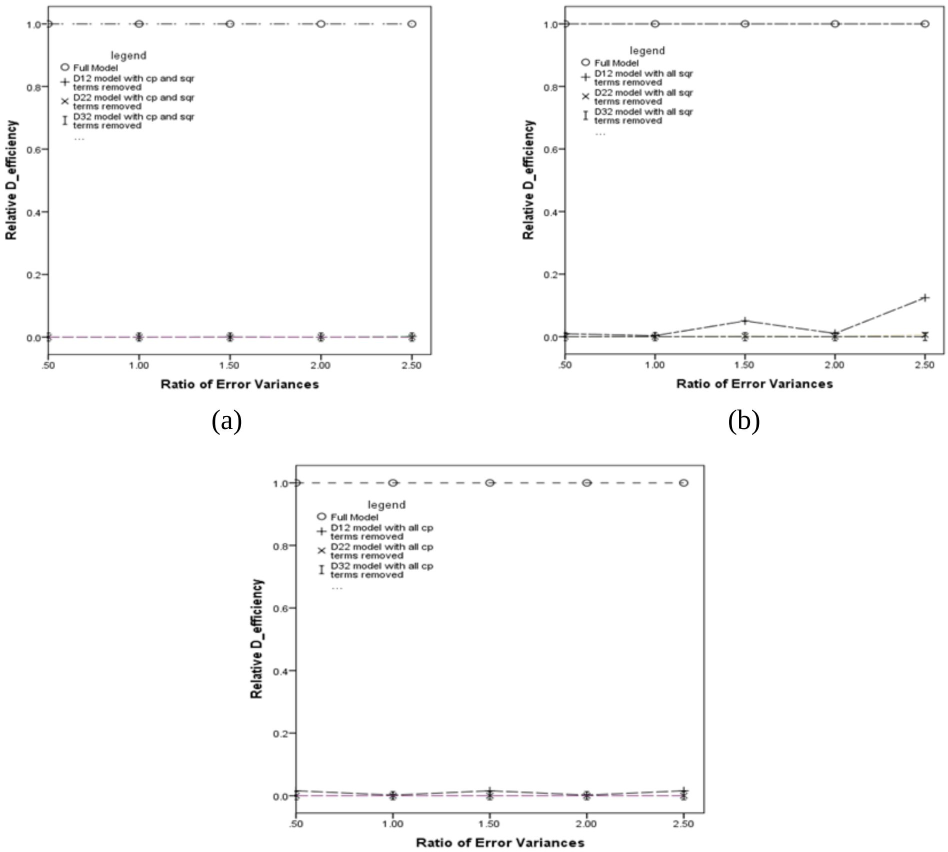
Figure 2. Plots of reduced model relative efficiencies for the one whole-plot variable/two sub-plot variables, two whole-plot variables/two sub-plot variables, and three whole-plot variables/two sub-plot variables CCDs. Plots (a), (b), (c), and (d) contain relative D-efficiencies for the three designs, respectively, when two sub-plot squared, a whole-plot and a sub-plot squared, a whole-plot and a sub-plot interaction, and a two whole-plot squared terms are removed.

Removing a whole-plot and a sub-plot interaction term: Figure 2(c) shows plots of the resulting relative D-efficiencies when a pure sub-plot and a whole-plot by sub-plot interaction terms are removed from the D12 CCD full model, and when a whole-plot and sub-plot interaction terms are removed from the full models of the D22 and D32 CCDs. From this figure we observed that the D32 suffers the highest efficiency loss across all values of d; at d = 0.5, the D12 surpasses the D22 in D-efficiency but as d rises, the D22 takes the lead with lowest loss in D-efficiency. The loss in D12 CCD is periodic and ranges from 93.75% to 98.44%, while each of the D22 and D32 CCDs recorded their lowest and highest relative D-efficiency losses at d = 2.5 and 1.0, respectively.

Removing two whole-plot squared terms: Figure 2(d) shows plots of the resulting relative D-efficiencies when two whole-plot squared terms are removed from the full model of the D22 and the D32 CCDs. From this figure we observed that the D22 slightly surpasses the D32 in D-efficiency at d = 0.5, 1.0, and 2.0, while at d = 1.5 and 2.5, a significant difference in the efficiency losses could be observed with the D22 maintaining the lead. Each of the CCDs recorded their lowest and highest D-efficiency losses of 84.48%, 94.10%, and, 99.1%, 99.7%, respectively at d = 2.5 and 1.0.

3.3 *Removing all whole-plot/sub-plot interaction and/or squared terms from the full model*

The resulting relative D-efficiencies are displayed in Figures 3(a), 3(b), and 3(c), respectively, for a pure linear model, a linear model with interaction terms, and a linear model with squared terms.



(a) (b)

(c)

Figure 3. Plots of reduced model relative efficiencies for the one whole-plot variable/two sub-plot variables, two whole-plot variables/two sub-plot variables, and three whole-plot variables/two sub-plot variables CCDs. Plots (a), (b), and (c) contain, respectively, relative

D-efficiencies for the pure linear models, linear models with interactions, and linear models with squared terms.

Pure Linear Models. Figure 3(a) shows relative D-efficiency plots for the pure linear models (models without interaction and squared terms) of these designs. It can be rightly observed that the three designs clearly maintain equal loss of relative D-efficiency of almost 100% across all values of d.
Linear Model with Interactions: Figure 3(b) shows the plots of the relative D-efficiencies for these designs when all interactions are added to the linear models. The figure shows that the D22 and the D32 CCDs maintain equal D-efficiency losses of almost 100% across all values of d, while the D12 CCD surpasses them at each value of d with its lowest loss at d = 2.5.
Linear Model with Squared terms: The plots of relative D-efficiencies when squared terms are added to the linear models are given in Figure 3(c). This figure shows that the D22 and the D32 CCDs maintain equal D-efficiency loss of almost 100% while the D12 CCD slightly surpasses them across all values of d.

4. Conclusion

The paper evaluated the effect of model reduction on D-efficiency of central composite designs within a split-plot structure, and thereby revealed the extent of contribution of each of the model terms or combination of the terms to the design efficiency. The losses in D-efficiency of the formulated reduced designs were highly-significant and found to depend strongly on the relative magnitude or ratio (d) of the whole-plot and sub-plot error variances. Significant D-efficiency losses were noticed when a whole-plot (squared/ interaction) term or a sub-plot squared term, or a combination of the whole-plot and sub-plot terms were removed from the full model of these designs. These losses were found to increase, across all values of d, as the number of the whole-plot variables is increased, provided the number of subplot variables are kept fixed. However, when a sub-plot interaction term is removed from the full model, periodic D-efficiency losses were observed across all d values, with a period length, which reduces by half, as the number of whole-plot factors increases. Fitting a first-order model (with all squared and interaction terms removed) resulted in almost 100% loss in D-efficiency while the design with one whole-plot variable outperforms other designs with two or more whole-plot variables as interaction or squared terms are added to the linear model. Thus, provided the subplot variables are kept fixed, the reduced split-plot designs with fewer number of whole-plot variables perform better than the ones with higher number of whole-plot variables.

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