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Using Radial Basis Functions for Numerical Solving Two-Dimensional Voltrra Linear Functional Integral Equations

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Abstract. This article is an attempt to obtain the numerical solution of functional linear Voltrra two-dimensional integral equations using Radial Basis Function (RBF) interpolation which is based on linear composition of terms. By using RBF in functional integral equation, first a linear system $\Gamma C = G$ will be achieved; then the coefficients vector is defined, and finally the target function will be approximated. In the end, the validity of the method is shown by a number of examples.

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1. Introduction

Functional differential equations are the subject that has attracted many researchers interested in the aspect of numerical or analytical solutions. In this study, linear Voltrra two-dimensional functional integral equations of the second kind are investigated.

RBFs, which use simpler and easy to understand functions are computationally served to approximate both complicated and multi-variable functions. There are some number of advantages to using RBFs:

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Unlike multivariable polynomial interpolation or splines [4], RBFs have a wider range of applicability for scattered data. They result in existence and uniqueness. This is because, on one hand, there are few restrictions on dimensions and, on the other hand, there is high accuracy or fast convergence to the target function.
RBFs do not need any triangulation of the data points, whereas numerical methods like finite element or multivariate spline methods require triangulations [4, 13]. This requirement computationally costly for more than two-dimensions.
In this article, functional linear two-dimensional integral equations of Voltrra type with unknown function $u(x)$.
The set of RBFs, $\{\phi_i\}_{i=1}^{m}$ is shown below:

$$\phi : \mathbb{R}^d \to \mathbb{R},\ \phi_i(x) = \phi(\|x - x_i\|)$$

where $\|.\|$ stand for the Euclidean norm and x_i is the center of RBF.
Gaussian (GA) $\phi(r) = \exp(-\sigma r^2)$, Multiquadric (MQ) $\phi(r) = \sqrt{r^2 + \sigma^2}$ are some well-known functions which produce RBF that is infinitely globally supported differentiable, and depend on a free parameter σ which is a real constant known as the shape parameter of the RBF that must be determined by the gainer and $r = \|x - x_i\|$. More functions are shown in Table 1. In this study, the shape parameter is considered $\beta = 1$.
The concept of radial basis function was introduced by Hardy [16, 17]. Hardy,

Table 1. Well-known functions that generate RBF.

function's Name	Definition
Gaussian (GA)	$\phi(r) = \exp(-\sigma r^2)$
Multi quadric (MQ)	$\phi(r) = \sqrt{r^2 + \sigma^2}$
Inverse multi quadric (IMQ)	$\phi(r) = (r^2 + \sigma^2)^{(-\frac{1}{2})}$
Inverse quadric (IQ)	$\phi(r) = (r^2 + \sigma^2)^{(-1)}$

showed that multi-quadrics RBF is related to a consistent solution of the biharmonic potential problem. Frank [9], celebrated the multi-quadratic method.
Buhmann and Micchelli [8, 10], have shown that RBF is connected to pre-wavelets (wavelets which do not possess orthogonality properties). Kansa [3, 14], proposes a method for solving partial differential equations based on radial basis functions. Firouzdor [20], applied RBF to approximate functional integral equations and Biazar [15], used RBF for the numerical solution of functional integral equations by the variational iteration method. Alipanah and Esmaeili [6], used RBF for the solution of two-dimensional Voltrra integral equation but it's different from ours. Furthermore, our method is more accurate, simpler and involves less computation. See for e.g. [1, 5, 7, 18, 19].

In this article, we consider RBF to approximate the solution of the problem approximations. We consider Voltrra two-dimensional integral equations of the second kind as the following formula[6]:

$$g(X) = u(X) + A(X)u(h(X)) + \lambda \int_a^{x_1} \int_c^{x_2} k(X,T)u(T)DT, \tag{1}$$

where $A(X)$, $h(X)$ and $g(X)$ are analytical known functions and $u(h(X))$ and $k(x,t)$ are given continuous that defined, respectively on $D = [a,b] \times [c,d]$ and $u(X)$ is unknown on D. In this paper DT means that dt_1dt_2. In order to approximate the target function $u(X)$, we employ RBF interpolation in distinct grids from a definite domain. To do so, a linear component is considered as follows:

$$u(X) \approx \sum_{i=0}^{N} c_i \phi_i(X), \tag{2}$$

where $\phi_i(X)$ can be selected from one of the basic functions which was referred to above, depending on the kind of the desired target function. Considering Gaussian function, for instance, would give us

$$\phi_i(X) = e^{-\|X-X_i\|^2} \tag{3}$$

and similarly, it is true for $u(h(X)) \approx \sum_{i=0}^{N} c_i \phi_i(h(X))$ where $h : [a, b] \to [a, b]$ is a known function. In the next section,we approximate the solution of functional linear Voltrra two-dimensional integral equations is be described and in Section 3, the efficiency of the method is shown by two examples. Section 4, is devoted to some concluding remarks. Let in Eq.(1) $T = [t_1, t_2], X = [x_1, x_2] \in \mathbb{R}^2$.

2. Main idea

In dimensional Euclidean space $\mathbb{R}^n$, we suppose distinct points as $(x_1, x_2, \ldots, x_n)$ where in the function to be approximated is known and real scalars $(g(x_1), g(x_2), \ldots, g(x_n))$ which are given values at the points and a continuous function $s : \mathbb{R}^d \to \mathbb{R}$ is considered to construct so that $s(x_j) = g(x_j)$ for $j = 1, 2, \ldots, n$. This technique is based on choosing a continuous function such as $\phi : \mathbb{R}^d \to \mathbb{R}$ and a norm $\|.\|$ in $\mathbb{R}^d$, then we can write

$$s(X) = \sum_{j=1}^{n} c_j \phi(\|X - X_j\|), \tag{4}$$

where c_j's are unknown scalars for $j = 0, 1, \ldots, n$ which should be defined so that s approximates g in point x_j for $j = 1, 2, \ldots, m$. By interpolation conditions, a linear system will be defined as $\Gamma C = G$, where $\Gamma \in \mathbb{R}^{2\times 2}$ is called an interpolation matrix, and given by

$$\Gamma_{ij} = \phi(\|X_i - X_j\|), \tag{5}$$

and also $C = (c_1, c_1, \ldots, c_n)^T$ and $G = (g_1, g_1, \ldots, g_n)^T$.
The interpolation matrix is non-singular because it is a positive definite matrix. This will generate the coefficients c_j.

2.1 *Convergence of the RBF*

To analyze the convergence of RBF, imagine that the data points are on equispaced grids in $\mathbb{R}^d$ and the spacing is showed by h, when $h \to 0$. Indeed, we have infinite the uniform grids of spacing h. Considering s as an approximate of function g by RBF, uniform difference between s and g is closed to zero at the same rate as some power of h.

Definition 2.1 (Native space [11]) The native space that is conditionally positive definite on Ω concerning p-unisolvent subset is defined by

$$N_{\phi(\Omega)} := R(F_{\phi(\Omega)}) + p.$$

Space is equipped with a semi-inner product via

$$(f, g)_{N_{\phi(\Omega)}} = (R^{-1}(f - T_p f), R^{-1}(g - T_p g))\phi,$$

where $T_p : C(\Omega) \to p$ and $T_p(f) = \sum_{k=1}^{Q} f(\xi k) p_k$.

On a domain $\Omega \in \mathbb{R}^d$ The concept of convergence of RBF interpolation is based on functions on native spaces $N_{\phi(\Omega)}$. For some radial basis functions including Gaussians and inverse quadratics [11], which are strictly positive definite basis functions, we will build the native space of conditionally positive basis functions as the completion of the pre-Hilbert space. Take the following linear space into consideration:

$$F_{\phi(\Omega)} = \Big\{ \sum_{j=1}^{N} \alpha_j \phi(\|. - x_j\|), \alpha \in \mathbb{R}^N, x_j \in \Omega, j = 1, ..., N \Big\}. \tag{6}$$

Defining the inner product, $F_{\phi(\Omega)}$ becomes a pre-Hilbert space

$$\Big(\sum_{j=1}^{N} \alpha_j \phi(\|. - x_j\|), \sum_{k=1}^{N} \beta_k \phi(\|. - x_j\|))_{\phi(\Omega)} := (\sum_{j=0}^{N} \sum_{k=1}^{N} \alpha_j \beta_k \phi(\|x_j - x_k\|) \Big). \tag{7}$$

In RBF applications fill distance of h is considered as follows:

$$h_N := \sup \min_{x \in \Omega} \|x - x_j\|_2, \qquad j = 1, 2, \ldots, N,$$

so for sufficiently small h_N and data x_j, the following relations have resulted for inverse quadratics

$$\|g - s_{f,N}\|_{L_\infty(\Omega)} \leqslant e^{(\frac{-c}{h_N})} \|f\|_{N_{\phi(\Omega)}},$$

and Gaussians

$$\|g - s_{f,N}\|_{L_\infty(\Omega)} \leqslant e^{(\frac{-c \log h_N}{h_N})} \|f\|_{N_{\phi(\Omega)}}. \tag{8}$$

Concept of convergence for RBF approximation is defined for functions over domain $\Omega \in \mathbb{R}^d$ that are on native spaces $N_{\phi(\Omega)}$.

Theorem 2.2 *Imagine that g is an analytical function in an open region that encompasses the strip $|Im(z)| \leqslant \frac{1}{2\epsilon}$ in its interior. Then for all node distributions $\{x_j\}$, $j = 1, 2, \ldots, N$ with fill distance $h_N = O(\frac{1}{N})$ on the unit interval, the error in the inverse quadratic RBF interpolation would be:*

$$\|g - s_{f,N}\|_{L_\infty([-1,1])} \leqslant e^{(-cN)}, N \to \infty.$$

It is proved in [15].

2.2 *RBF for functional linear two-dimensional integral equations of the second kind*

The functional linear Voltrra two-dimensional integral equations of the second kind are shown below:

$$g(X) = u(X) + A(X)u(h(X)) + \lambda \int_a^{x_1} \int_c^{x_2} k(X,T)u(T)DT, \quad X \in D. \tag{9}$$

RBF interpolation has been used to approximate $u(x)$ as the solution of functional linear Voltrra two-dimensional integral equations of the second kind. To do so, the linear component of functions ϕ_i is replaced in $u(x)$ as the follows:

$$u(X) \approx \sum_{i=0}^{N} c_i \phi_i(X). \tag{10}$$

So

$$u(h(X)) \approx \sum_{i=0}^{N} c_i \phi_i(h(X)), \tag{11}$$

where $h : [a,b] \times [c,d] \to [a,b] \times [c,d]$ is a known function, we have:

$$\int_a^{x_1} \int_c^{x_2} k(X,T)u(T)DT \approx \int_a^{x_1} \int_c^{x_2} k(X,T) \sum_{i=0}^{N} c_i \phi_i(T)DT = \sum_{i=0}^{N} c_i \varphi_i(X), \tag{12}$$

where

$$\varphi_i(X) = \int_a^{x_1} \int_c^{x_2} k(X,T)\phi_i(T)dt. \tag{13}$$

Replacing Eqs.(10), (11) and (12) into Eq.(9), the following equivalence is obtained:

$$g(X) = \sum_{i=0}^{N} c_i \phi_i(X) + A(X) \sum_{i=0}^{N} c_i \phi_i(h(X)) + \lambda \sum_{i=0}^{N} c_i \varphi_i(X). \tag{14}$$

Let $X = X_j$ for $j = 1, 2, \ldots, N$ then

$$g_i = \sum_{i=0}^{N} c_i \phi_{ij} + A_j \sum_{i=0}^{N} c_i \tilde{\Phi}_{ij} + \lambda \sum_{i=0}^{N} c_i \varphi_{ij}. \tag{15}$$

where

$$\begin{gathered} g_j = g(X_j), \quad A_j = A(X_j), \\ \phi_{ij} = \phi_i(h(X_j)), \quad \tilde{\Phi}_{ij} = \Phi_i(X_j), \\ \varphi_{ij} = \varphi_i(X_j) = \int_a^{x_1} \int_c^{x_2} k(X_j,T)\Phi_i(t)DT. \end{gathered}$$

The matrix from for Eq.(15) is $\Gamma C = G$, where $G = (g_1, g_1, \ldots, g_n)^T$ and $C = (c_1, c_1, \ldots, c_n)^T$. So we have

$$\begin{pmatrix} \Gamma_{1,1} & \Gamma_{1,2} & \ldots & \Gamma_{1,N} \\ \Gamma_{2,1} & \Gamma_{2,2} & \ldots & \Gamma_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{N,1} & \Gamma_{N,2} & \ldots & \Gamma_{N,N} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{pmatrix}, \tag{16}$$

where

$$\Gamma_{ij} = \phi_{ij} + A_j \tilde{\Phi}_{ij} + \lambda \varphi_{ij}. \tag{17}$$

Finally, the functional linear two-dimensional integral equation is approximated by a system of N linear equations.
The ability and performance of the current method were investigated by using a test problem. To explain the accuracy and the efficiency of the suggested method in the next section, we are investigated, furthermore, the robustness and performance of the method are employed with two examples.

3. Numerical examples

To show the efficiency of the suggested method, the following examples are presented. The results have been provided by Matematica.

Example 3.1 Consider the following Voltrra two-dimensional integral equation;

$$g(X) = u(X) + A(X)u(h(X)) + \lambda \int_a^{x_1} \int_c^{x_2} k(X,T)u(T)DT,$$

where

$$A(X) = 1, \quad h(X) = \{x_1, x_2\}, \quad k(X,T) = e^{((x_1+x_2)-(t_1+t_2))},$$
$$g(x) = e^{((x_1+x_2)+(t_1+t_2))} x_1 x_2,$$

and

$$a = 0, \quad b = 1.1, \quad c = 0, \quad d = 1.1, \quad \lambda = 1,$$

where $u(X) = e^{(x_1+x_2)}$, is the exact solution.
We suppose two case mesh and mesh-less points. Consider that $X, T \in \mathbb{R}^2$, According to Eq.(17), we have the following relation;

$$\Gamma_{ij} = \phi_{ij} + A_j \tilde{\Phi}_{ij} + \lambda \varphi_{ij},$$

where

$$g_j = g(X_j), A_j = A(X_j), \phi_{ij} = \phi_i(X_j), \tilde{\Phi}_{ij} = \phi_i(h(X_j)),$$

$$\varphi_{ij} = \varphi_i(x_j) = \int_0^{1.1} \int_0^{1.1} e^{(x_{1j} - x_{2j}) - (t_1+t_2)} \phi_i(t) dt.$$

We would obtain C from $\Gamma C = G$ was mentioned above. In order to, $u(X) \approx \sum_{i=0}^{N} c_i\phi_i(x)$ is defined. In Figure 1, shown graph mesh and meshless points for the Voltrra two-dimensional integral equation. Error plot $3D$ for the mesh and meshless points shown in Figure 2. We have an error Contour plot for mesh and meshless points in Figure 3. Figure 4, shown the error List Contour plot for the mesh and meshless points. As it is obvious all Figures 1-4, the same because the solution of Voltrra two-dimensional integral equation are the identical $u(x) = e^{x_1+x_2}$. Also Table 2, shows errors for Ferdholm two-dimensional integral equation in mesh and meshless points.

Table 2. Accuracy of RBF to Voltrra two-dimensional linear functional integral equation for mesh and meshless points.

N	An error of Mesh points	An error of Meshless points
9	0.0937767	0.0971223
25	0.00229234	0.00359776
49	0.0000561179	0.0000633058
100	0.0000360878	0.0000219636

Example 3.2 We would have in Eq.(1)

$$g(X) = u(X) + A(X)u(h(X)) + \lambda \int_a^{x_1}\int_c^{x_2} k(X,T)u(T)DT,$$

where

$$A(X) = e^{-(x_1+x_2)}, \quad h(X) = \{\tfrac{x_1}{2}, \tfrac{x_2}{2}\}, \quad k(X,T) = e^{((x_1+x_2)-(t_1+t_2))},$$
$$g(X) = e^{-(x_1+x_2)} + 2.21e^{(x_1+x_2)},$$

and

$$a = 0, \quad b = 1.1, \quad c = 0, \quad d = 1.1, \quad \lambda = 1,$$

where $u(X) = (x_1 + x_2)^2$, is the exact solution.
We suppose two case mesh and mesh less points. Consider that $x, t \in \mathbb{R}^2$, According to Eq.(17), we have the following relation;

$$\Gamma_{ij} = \phi_{ij} + A_j\tilde{\Phi}_{ij} + \lambda\varphi_{ij},$$

where

$$g_j = g(X_j), A_j = A(X_j), \phi_{ij} = \phi_i(X_j), \tilde{\Phi}_{ij} = \phi_i(h(X_j)),$$

$$\varphi_{ij} = \varphi_i(X_j) = \int_0^{1.1}\int_0^{1.1} e^{(x_{1j}-x_{2j})-(t_1+t_2)}\phi_i(T)DT.$$

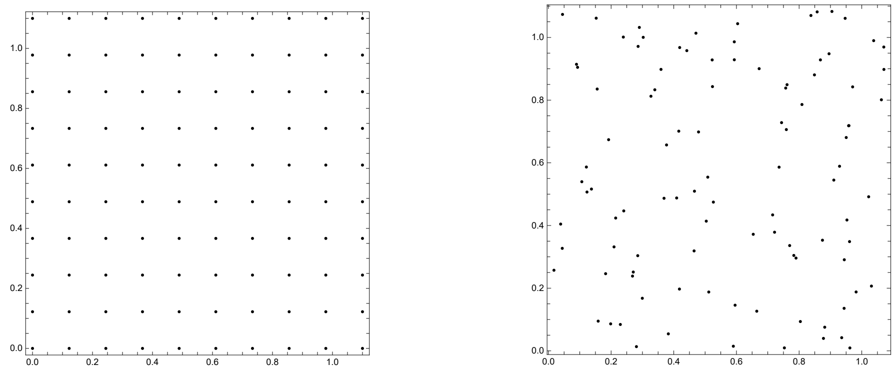

Figure 1. Left fig. shown mesh points and right fig. shown meshless points, there are 100 points interpolation.

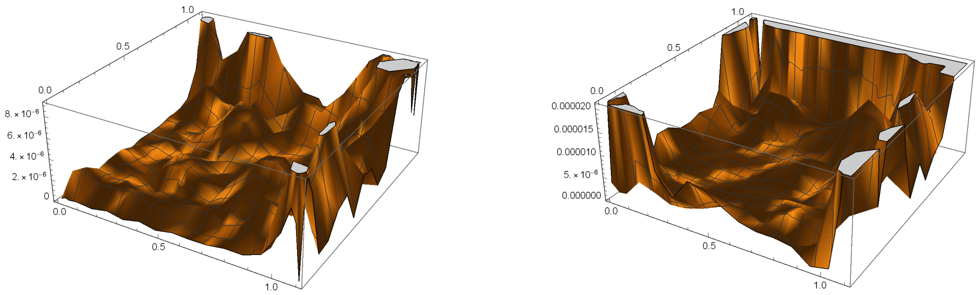

Figure 2. Left fig. shown plot 3D error for mesh points and right fig. shown plot $3D$ error for meshless points. There are 100 points for interpolation.

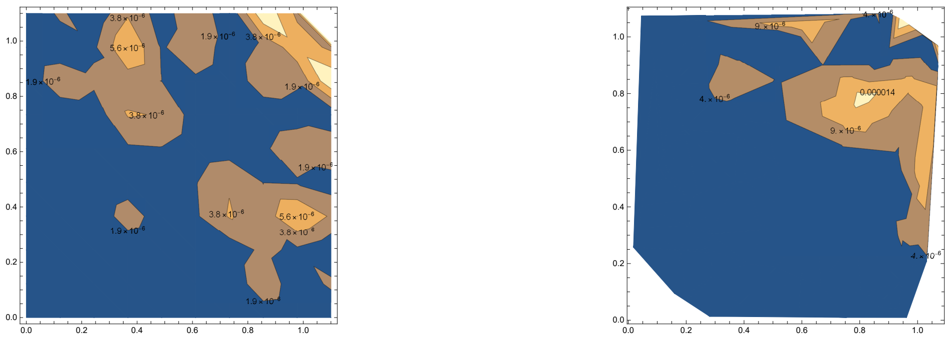

Figure 3. Left fig. shown Error List Contour plot for the mesh points and right fig. shown Error List Contour plot for the meshless points.

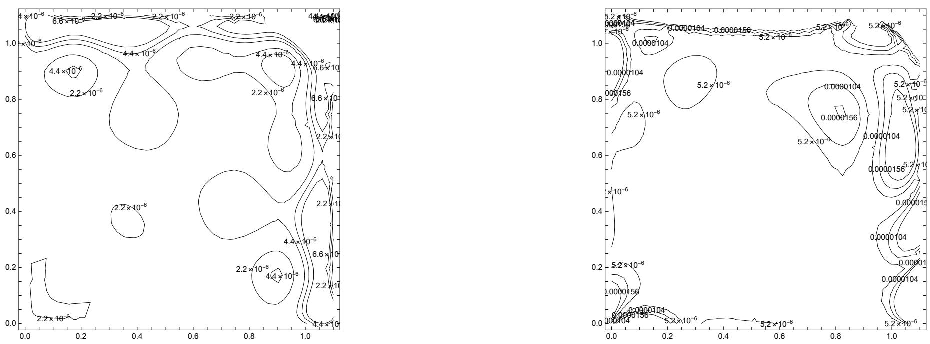

Figure 4. Left fig. shown Error Contour plot for the mesh points and right figure shown Error Contour plot for the meshless.

We would get C from which $\Gamma C = G$ was mentioned above. In order to, $u(X) \approx \sum_{i=0}^{N} c_i \phi_i(x)$ is defined. In Figure 5 shown error graph mesh and meshless points for the Voltrra two-dimensional linear functional integral equation. Error plot $3D$ for the mesh and meshless points shown in Figure 6. We have an error Contour plot for mesh and meshless points in Figure 7. Figure 8, shown the error List Contour plot for the mesh and meshless points. As it is obvious all Figures 5-8, the same because the solution of Voltrra two-dimensional integral equation are the identical $u(x) = (x_1 + x_2)^2$. Also Table 3, shows errors for Ferdholm two-dimensional integral equation in mesh and meshless points.

Table 3. Accuracy of RBF to Voltrra functional integral equation for mesh and meshless points.

N	An error of Mesh points	An error of Meshless points
9	0.0947598	0.0600808
25	0.00246095	0.00289682
49	0.0000882828	0.000269197
100	0.0000747001	0.0000885497

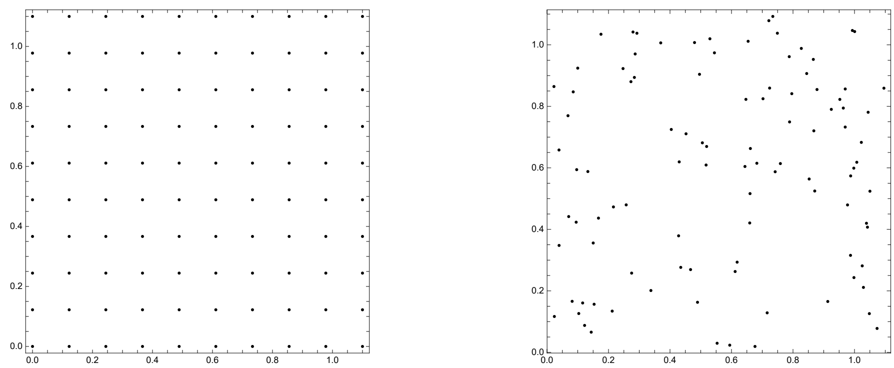

Figure 5. Left fig. shown mesh points and right fig. shown meshless points, there are 100 points interpolation.

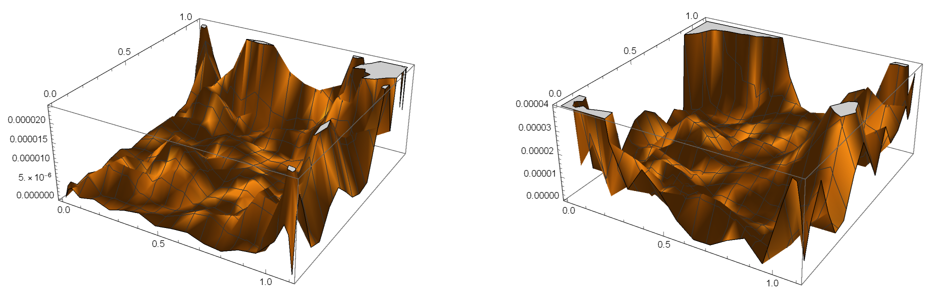

Figure 6. Left fig. shown plot 3D for mesh points and right fig. shown plot $3D$ error for meshless points. There are 100 points for interpolation.

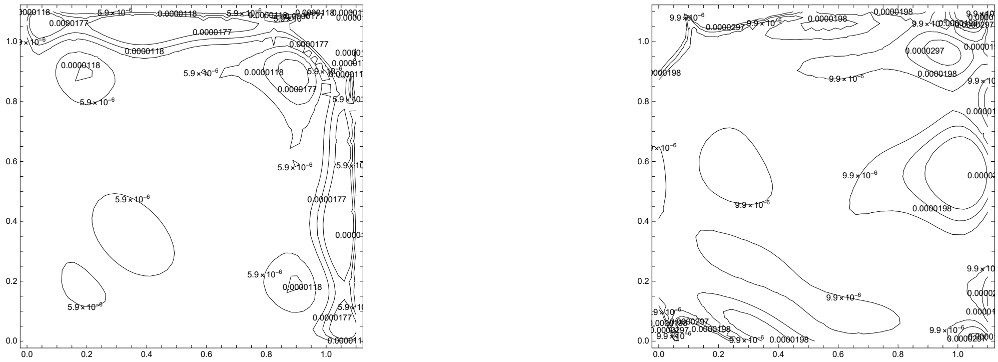

Figure 7. Left fig. shown Error Contour plot for the mesh points and right fig. shown Error Contour plot for the meshless points.

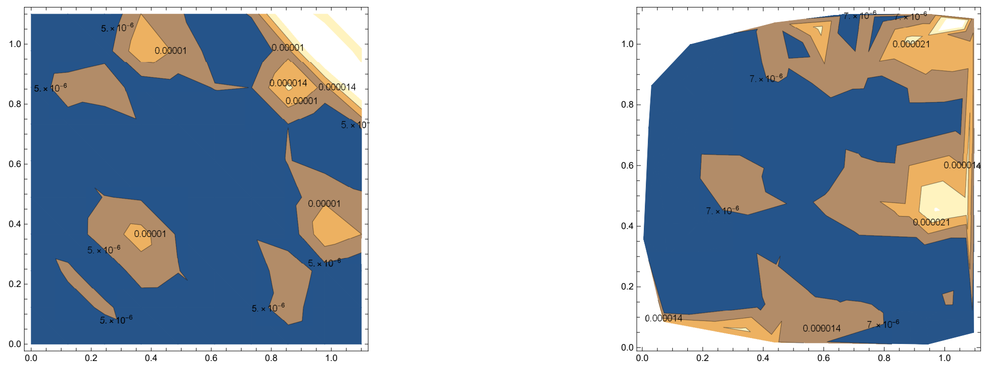

Figure 8. Left fig. shown Error Contour plot for the mesh points and right figure shown Error Contour plot for the meshless.

4. Conclusion

A number of approaches to functional Voltrra integral equations have so far been suggested. These approaches have, however, been computationally expensive. In this article, was an attempt to investigated the application of interpolation by RBF to solve the Voltrra two-dimensional linear functional integral equations. This technique is very simple and needs fewer computations. Furthermore, two examples were presented to prove the efficiency of the suggested method and the results showed that by applying RBF, approximates will show higher accuracy even in a few distinct points. Other advantages of this approach a less costly computation and ease of setting up the equations. We used mesh and meshless points in this example to show our proposed method can be used in both spaces.

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