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Cascade of Fractional Differential Equations and Generalized Mittag-Leffler Stability

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Abstract. This paper address a new vision for the generalized Mittag-Leffler stability of the fractional differential equations. We mainly focus on a new method, consisting of decomposing a given fractional differential equation into a cascade of many sub-fractional differential equations. And we propose a procedure for analyzing the generalized Mittag-Leffler stability for the given fractional differential equation using the generalized Mittag-Leffler input stability of the sub-fractional differential equations. In other words, we prove a cascade of fractional differential equations, which are generalized Mittag-Leffler input stables and governed by a fractional differential equation, which is generalized Mittag-Leffler stable, is generalized Mittag-Leffler stable. We give Illustrative examples to illustrate our main results. Note in our paper; we use the generalized fractional derivative in Caputo-Liouville sense.

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1. Introduction

In fractional calculus, the stability analysis received many investigations in this last decade. The different types of fractional derivatives have many impacts on the stability analysis notions. For example, when we use a classical derivative, the term "exponential stability" is used, but with Riemann-Liouville derivative, the

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term "Mittag-Leffler stability" is used. In other words, in the first case, the comparison function [31] is in exponential form, and in the second case, the comparison function is a Mittag-Leffler function [31]. The nature of the fractional derivative has a significant impact on the type of stability notions in fractional calculus. But the principle ideas as the attractivity and the convergence do not change. There exist in fractional calculus many type of fractional derivatives: Atangana-Baleanu-Caputo derivative [8], Caputo-Liousville derivative find in [10], Caputo-Fabrizio derivative [9], M-derivative, Riemann-Liousville derivative find in [10] and many other [4, 19, 23, 24, 32]. Recently, a new classification of these above fractional derivatives was done, see detail in [5, 13]. The stability and the convergence are two properties that study the behaviors of the analytical solutions of the fractional differential equations. There exist many notions related to the stability and the convergence of fractional differential equations involving fractional derivatives. We enumerate some of them. We can cite the asymptotic stability [16], exponential stability, and their Lyapunov characterizations [34]. We have the Mittag-Leffler stability introduced in [16]. We can found Many properties related to the stability notions in fractional calculus and there Lyapunov characterizations in [25, 31]. We have the Mittag-Leffler input stability introduced in [29] and which its Lyapunov characterization is given as well in the context of fractional calculus. We have fractional input stability recently introduced in the literature in [27] and its Lyapunov characterization. Many other types of stability notions, which we can find in the literature. For more investigations see in [7, 16, 17, 22].

In [18] the author, proposes the Matignon criterion for the linear fractional differential equations described by Riemann-Liouville derivative. We introduce conditional Mittag-Leffler stability with its characterization in [28]. The author proposes Lyapunov characterization for the so-called conditional asymptotic stability in the context of fractional calculus. In [27], the author introduces the fractional input stability in context of fractional differential equations. We extended the fractional input stability to the Mittag-Leffler input stability in [29]. The stability notions with generalized fractional derivative were considered in [31]. In [25], the author proposes exponential form for Lynunov characterization. The stability analysis for the fractional differential equations can be found in [7, 15–17, 20, 22, 34, 35].

Finding the exact analytical solutions for the fractional differential equations is not trivial. The numerical solutions can not be used in our context to study the stability of the trivial solutions. Another alternative to examine the stability of the solution is to use the energy contained in the fractional differential equation called the Lyapunov function. The Lyapunov direct method is not possible all time due to the fact the Lyapunov function for a fractional differential equation is complicated to be found in high dimension. In our paper, we give a possible alternative to avoid these difficulties. We focus on a new procedure to study the stability of the fractional differential equation in a high dimensional space. The method described in the next section consists of rewriting the given fractional differential equation as a cascade of many other sub-fractional differential equations. We study the stability notion of all sub fractional differential equations, and we provide technic to obtain stability for the given fractional differential equation. In this investigation, all sub-fractional differential equations are supposed to be Mittag-Leffler input stable, and the last is Mittag-Leffler stable, we prove the given fractional differential equation is Mittag-Leffler stable as well. In the methodology, we combine an analytical solution and a Lyapunov function.

In Section 2, we recall the fractional calculus tools. In Section 3, we discuss the Mittag-Leffler input stability and the stability notions used in this paper. In Section 4, we present our main results and discuss them in their applicabilities.

In Section 5, we give illustrative examples of our main results. In Section 6, we provide conclusions and futures directions of this present investigation.

2. Fractional operators

In this section, we recall some definition which interests our present works. Fractional calculus is a complex field in mathematics from which many types of fractional derivatives exist. Presently there exist many types of fractional derivatives with or without singular kernels. We cite the Riemann-Liouville fractional derivative, the Caputo-Liouville fractional derivative, the conformable derivative, the proportional fractional derivatives, the Atangana-Baleanu fractional derivative, the Caputo-Fabrizio fractional derivative, and many others [8, 9, 30]. The discrete versions of the above fractional derivatives exist too, much researches in these directions have been developed, and many results were found. For some discrete versions of the fractional derivatives, see in [1–3]. Many generalizations of the existing fractional derivatives have been investigated in the literature, see in [6, 10–12, 14]. In our paper, we particularly use the generalization done on the Caputo-Liouville fractional derivative and the Riemann-Liouville fractional derivative. We give the following definitions for fractional derivatives and integrals:

Definition 2.1 [6, 10–12, 14] The generalized form for Riemann-Liouville integral of order α with $\rho > 0$ of a continuous function $h : [0, +\infty[\longrightarrow \mathbb{R}$ is represented by the following

$$\left(I^{\alpha,\rho}h\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} h(s)\frac{ds}{s^{1-\rho}}, \tag{1}$$

where the function $\Gamma(...)$ represents the Gamma function, for all $t > 0$, and $0 < \alpha < 1$.

Definition 2.2 [6, 10–12, 14] The generalized form for Riemann-Liouville derivative of order α with $\rho > 0$ of a continuous function $h : [0, +\infty[\longrightarrow \mathbb{R}$ is represented by the following

$$\left(D^{\alpha,\rho}h\right)(t) = \left(I^{1-\alpha,\rho}h\right)(t) = \frac{1}{\Gamma(1-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right) \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{-\alpha} h(s)\frac{ds}{s^{1-\rho}}, \tag{2}$$

where the function $\Gamma(...)$ represents the Gamma function, for all $t > 0$, and $0 < \alpha < 1$.

Definition 2.3 [6, 10–12, 14] The generalized form for Caputo-Liouville fractional derivative of order α with $\rho > 0$ of a continuous function $h : [0, +\infty[\longrightarrow \mathbb{R}$ is represented by the following

$$\left(D_c^{\alpha,\rho}h\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{-\alpha} h'(s)ds, \tag{3}$$

where the function $\Gamma(...)$ represents the Gamma function, for all $t > 0$, and the fractional order satisfies $0 < \alpha < 1$.

The Laplace transform of this fractional derivative will be used in our investigations. We define some of them for more information, refer to [10].

Definition 2.4 [10] The so-called "ρ-Laplace transform" of the Caputo-Liouville generalized fractional derivative of a continuous function $h : [0, +\infty[\longrightarrow \mathbb{R}$ is represented as the form

$$\mathcal{L}_\rho \left\{ \left(D_c^{\alpha,\rho} h \right)(t) \right\} = s^\alpha \mathcal{L}_\rho \left\{ h(t) \right\} - s^{\alpha-1} h(0), \tag{4}$$

where the ρ-Laplace transform of the function $h : [0, +\infty[\longrightarrow \mathbb{R}$ is defined as the form

$$\mathcal{L}_\rho \left\{ h(t) \right\}(s) = \int_0^\infty e^{-s\frac{t^\rho}{\rho}} h(t) \frac{dt}{t^{1-\rho}}. \tag{5}$$

Definition 2.5 [10] The so-called ρ-Laplace transform of the Riemann-Liouville generalized fractional derivative of an function $h : [0, +\infty[\longrightarrow \mathbb{R}$ is defined by the following expression

$$\mathcal{L}_\rho \left\{ \left(D^{\alpha,\rho} h \right)(t) \right\} = s^\alpha \mathcal{L}_\rho \left\{ h(t) \right\} - \left(I^{1-\alpha,\rho} h \right)(0). \tag{6}$$

The Mittag-Leffler function plays an essential role in the representation of the solution of the fractional differential equations. This function is fundamental in fractional calculus. There exist three types of Mittag-Leffler functions: with one, two, and three parameters. We recall these functions in the following definition.

Definition 2.6 [6, 31] The Mittag-Leffler function with two, three parameters are defined in the following forms

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{7}$$

$$E_{\alpha,\beta}^\rho(z) = \sum_{k=0}^\infty \frac{\Gamma(\rho + k) z^k}{\Gamma(\rho)\Gamma(\alpha k + \beta) k!}, \tag{8}$$

$$E_{\alpha,\beta}^{\rho,\kappa}(z) = \sum_{k=0}^\infty \frac{\Gamma(\rho + k\kappa) z^k}{\Gamma(\rho)\Gamma(\alpha k + \beta) k!}, \tag{9}$$

where $\alpha > 0$, $\beta, \kappa, \rho \in \mathbb{R}$ and $z \in \mathbb{C}$. We recover the exponential function when the orders satisfy the relationship $\alpha = \beta = \kappa = \rho = 1$.

3. Mittag-Leffler input stability

In this section, we introduce the comparison functions which are fundamental in our study. Nowadays, these functions are used to study the stability of some fractional differential equations. In the second step, we will also recall and discuss the generalized Mittag-Leffler stability and the generalized Mittag-Leffler input stability recently introduced in the stability notions of fractional differential equations. Let's the comparison functions.

Definition 3.1 [31] The class $\mathcal{PD}$ function represents the set of all continuous functions $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{>0}$ satisfying $\alpha(0) = 0$, and $\alpha(s) > 0$ for all $s > 0$. A class

$\mathcal{K}$ function is an increasing $\mathcal{PD}$ function. The class $\mathcal{K}_\infty$ represents the set of all unbounded $\mathcal{K}$ functions.

Definition 3.2 [31] A continuous function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geqslant 0}$ is said to be of class $\mathcal{KL}$ if $\beta(., t) \in \mathcal{K}$ for any $t \geqslant 0$, and $\beta(s, .)$ is non-increasing and tends to zero as its arguments tend to infinity.

For the stability analysis, we recall some notions fundamental in our studies. The first is the generalized asymptotic stability described in the following definition.

Definition 3.3 [31] The fractional differential equation represented by $D_c^{\alpha,\rho}x = f(t, x)$ is said to be generalized globally asymptotically stable if there exist a class $\mathcal{KL}$ function β such that for any initial condition ξ, the following inequality holds

$$\|x(t, \xi)\| \leqslant \beta(\|\xi\|, t^\rho - t_0^\rho). \tag{10}$$

This definition was motived in by the following reasons: the term "generalized" is due to the form of the used fractional derivative and the form of the β function depends on the fractional time in the form $t^\rho - t_0^\rho$. The trivial solution of the fractional differential equation $D_c^{\alpha,\rho}x = f(t, x)$ is called generalized Mittag-Leffler stable when the function β depends on the Mittag-Leffler function and the initial condition. In other words, we have the following form

$$\beta(\|\xi\|, t^\rho - t_0^\rho) = \left[n(\|\xi\|) E_\alpha \left(\eta \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \right]^b, \tag{11}$$

where the constants satisfy the conditions $b > 0$, $\eta < 0$, and a locally Lipschitz function n satisfying the condition $n(0) = 0$.

The Mittag-Leffler input stability was introduced in [27] with Caputo fractional derivative. In [31], this concept was developed and was called the generalized Mittag-Leffler input stability due to the use of the generalized fractional derivative and the form the fact the term $t^\rho - t_0^\rho$ is into the β function. Let's recall primary the definition of the generalized fractional input stable.

Definition 3.4 [27, 31] The fractional equation represented by $D_c^{\alpha,\rho}x = f(x, u)$ is said to be generalized fractional input stable if, for any input $u \in \mathbb{R}^m$, there exist a class $\mathcal{KL}$ function β and a $\mathcal{K}_\infty$ function γ, such that for any initial condition ξ, its solution satisfies

$$\|x(t, \xi, u)\| \leqslant \beta(\|\xi\|, t^\rho - t_0^\rho) + \gamma(\|u\|_\infty). \tag{12}$$

Note for the same reason when the function β is in the form (11), the fractional differential equation $D_c^{\alpha,\rho}x = f(x, u)$ is called generalized Mittag-Leffler input stable. Two interesting properties exist with this new concept for stability analysis. The first is when the input of the fractional differential equation converge then the state converge as well. In other words, converging-input generates converging-state. The second is when the input is bounded, we notice the state is bounded as well. In other words, bounded-input generates bounded-state. The first property plays an essential role in our present work.

4. Cascades of fractional derivative and stability analysis

In this section, we address a new vision in the stability problem. We mainly study the generalized Mittag-Leffler stability of the fractional differential equation in the triangular form defined by

$$D_c^{\alpha,\rho} x = f(x,t), \tag{13}$$

where $x \in \mathbb{R}^n$. In this section we decompose the Eq. (13) as the following form

$$\begin{cases} D_c^{\alpha,\rho} x_1 = f_1(x_1, x_2, ..., x_n) \\ \quad\quad \vdots = \vdots \\ D_c^{\alpha,\rho} x_{n-1} = f_{n-1}(x_{n-1}, x_n) \\ D_c^{\alpha,\rho} x_n = f_n(x_n) \end{cases} \tag{14}$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. Note that x_j has an impact in its own fractional dynamic and $x_{j+1\to n}$ represents the input for each fractional sub equation considered in Eq. (14). For example, let the sub fractional differential equation defined by

$$D_c^{\alpha,\rho} x_j = f_j(x_j, x_{j+1\to n}). \tag{15}$$

In Eq. (15), the variable $x_{j+1\to n}$ are considered as the input of the fractional differential equation. Eq.(14) is called cascades fractional system or triangular fractional differential equation. Studying the stability of the fractional differential equation (13) using the analytical solution is not trivial in general due to the complexity of the fractional differential equation or the nonlinearity aspect. Furthermore, finding the Lyapunov function for Eq. (13) is also no trivial. An alternative is to use the triangular form of the fractional differential equation represented by E. (14). For the main result of this paper, we make the following theorem.

Theorem 4.1 Let the trivial solution for the fractional differential equation

$$D_c^{\alpha,\rho} x_n = f_n(x_n), \tag{16}$$

generalized Mittag-Leffler stable, and for all $j \in \{1, 2, ..., n-1\}$, the fractional differential equations

$$D_c^{\alpha,\rho} x_j = f_j(x_j, x_{j+1\to n}), \tag{17}$$

are generalized Mittag-Leffler input stable respecting the input $x_{j+1\to n}$. Then the cascade fractional differential equation (14) is generalized Mittag-Leffler stable.

Theorem 1 opens a new methodology in the stability analysis for fractional differential equations. Theorem 1 plays an important rule. Let for example the fractional differential equation described by

$$\begin{cases} D_c^{\alpha,\rho} x_1 = -x_1 + x_2 \\ D_c^{\alpha,\rho} x_2 = -3x_2 \end{cases} \tag{18}$$

It is straightforward to study the stability of Eq. (18). In our case, we suppose we have no way to get the quadratic Lyapunov function for the fractional differential equation (18) described by the Caputo generalized fractional derivative. The alternative is to use our proposed Theorem. We can observe the fractional differential equation $D_c^{\alpha,\rho} x_2 = -3x_2$ with initial condition $x_2(0) = \eta$, which is scalar system is generalized Mittag-Leffler stable. Two methods can be used to prove stability. With an analytical solution. Applying the Laplace transform to both sides of $D_c^{\alpha,\rho}x_2 = -3x_2$ we get

$$s^{\alpha}x_2(s) - s^{\alpha-1}\eta = -3x_2(s),$$
$$x_2(s) = \frac{s^{\alpha-1}\eta}{s^{\alpha}+3}. \tag{19}$$

Applying the inverse of Laplace transform according to the generalized fractional derivative in Caputo sense, the analytical solution of the fractional differential equation $D_c^{\alpha,\rho}x_2 = -3x_2$ is given by

$$x_2(t) = \eta E_{\alpha}\left(-3\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right). \tag{20}$$

According to Definition, it follows the fractional differential equation $D_c^{\alpha,\rho}x_2 = -3x_2$ described by Caputo generalized fractional derivative is Mittag-Leffler stable. The second way is to use the quadratic Lyapunov function defined by $V(x) = \frac{1}{2}x_2^2$. The derivative of the Lyapunov function along the trajectories yields

$$D_c^{\alpha,\rho}V(x) \leqslant -3x_2^2 = -6V(x). \tag{21}$$

Using the Lyapunov characterization, it follows the fractional differential equation $D_c^{\alpha,\rho}x_2 = -3x_2$ is generalized Mittag-Leffler stable. The second step of our analysis is to observe to the fractional differential equation defined by $D_c^{\alpha,\rho}x_1 = -x_1+2x_2$ is generalized Mittag-Leffler input stable. There exist two methodologies as previously done. By analytical solution, we apply the Laplace transform to both sides of $D_c^{\alpha,\rho}x_1 = -x_1 + 2x_2$, we get

$$s^{\alpha}x_1(s) - s^{\alpha-1}\zeta = -x_1(s) + 2x_2(s),$$
$$x_1(s) = \frac{s^{\alpha-1}\zeta}{s^{\alpha}+1} + \frac{2x_2(s)}{s^{\alpha}+1}. \tag{22}$$

Applying the inverse of Laplace transform, we get the analytical solution given by

$$x_1(t) = \zeta E_{\alpha}\left(-\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right) + 2\int_0^t \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right) x_2(s)\frac{ds}{s^{1-\rho}}. \tag{23}$$

We apply the Euclidean norm to both sides of equation (23), we get the relation defined by

$$\begin{aligned}\|x_1(t)\| \leqslant \|\zeta\| &\left\|E_{\alpha}\left(-\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\right\| \\ &+ 2\,\|x_2\| \int_0^t \left\|\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\frac{ds}{s^{1-\rho}}\right\|.\end{aligned} \tag{24}$$

Note that, there exist a constant $M > 0$ [27, 28] such that, the following relationship is helds

$$\int_0^t \left\|\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\frac{ds}{s^{1-\rho}}\right\| \leqslant M. \tag{25}$$

The analytical solution of the fractional differential equation $D_c^{\alpha,\rho}x_1 = -x_1 + 2x_2$ described by the Caputo generalized fractional derivative with exogenous input satisfies the following relationship

$$\|x_1(t)\| \leqslant \|\zeta\| \left\| E_\alpha\left(-\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right\| + \|x_2\| M. \tag{26}$$

From which we conclude the sub-fractional differential equation $D_c^{\alpha,\rho}x_1 = -x_1+2x_2$ is generalized Mittag-Leffler input stable. The Lyapunov direct method can also be used here. The advantage of our Theorem is after proving the generalized Mittag-Leffler stability and the generalized Mittag-Leffler input stability; we can use it to conclude the fractional differential equation (18) is generalized Mittag-Leffler stable. In other words, our Theorem gives a procedure to conclude after studying the stability of all sub-fractional differential equations. Our example exposed in this paper is long; what is the real advantage of our proposed method? Our procedure can have advantages and inconveniences. The advantage comes from the fact that when the fractional differential equation is in high dimensional space, finding the analytical solution of the solution is not trivial, and also finding the Lyapunov function is in general inaccessible. An alternative is to use our decomposition procedure. The inconvenience is the method can be very long by doing decomposition, and the advantage is the Lyapunov function of the fractional differential equation after decomposition become trivial. Let now give the proof of the Theorem.

Proof: In the proof, we combine the convergence of the solution and the stability of the solution. This combination implies global asymptotic stability. We consider the fractional differential equation defined by Eq.(14). Using the assumption, the trivial solution of Eq. (16) is Mittag-Leffler stable, it follows

$$\lim_{t\to+\infty} x_n(t) = 0. \tag{27}$$

. From the assumption the fractional differential equation $D_c^{\alpha,\rho}x_{n-1} = f_{n-1}(x_{n-1}, x_n)$, it satisfies the CICS (converging input converging state) property, see in. We know the input is x_n and is convergent, previously proved. Thus, we have the following identity

$$\lim_{t\to+\infty} x_{n-1}(t) = 0. \tag{28}$$

. We repeat the same reasoning at each step, and we have, for all $j \in \{n-1, n-2, ..., 2, 1\}$

$$\lim_{t\to+\infty} x_i(t) = 0. \tag{29}$$

. Finally, we deduce the convergence of the solution of the fractional differential equation defined by Eq. (14), that is

$$\lim_{t\to+\infty} x(t) = 0. \tag{30}$$

. The second step consists of proving the stability of the fractional differential equation (14). From the Mittag-Leffler stability of the fractional differential equation (16), it follows that

$$\|x_n(t)\| \leqslant \beta_n(\|\xi_n\|, t^\rho) \leqslant \beta_n(\|\xi_n\|, 0) = \xi_n, \tag{31}$$

where the function $\beta_n(\|\xi_n\|, t^\rho) = \xi_n E_\alpha\left(-\lambda_n\left(\frac{t^\rho}{\rho}\right)^\alpha\right)$ and note $E_\alpha(0) = 1$. We use the generalized Mittag-Leffler input stability of the fractional differential defined by $D_c^{\alpha,\rho}x_{n-1} = f_{n-1}(x_{n-1}, x_n)$. From which we get the existence of a class $\mathcal{KL}$ function $\beta_{n-1} = \xi_{n-1}E_\alpha\left(-\lambda_{n-1}\left(\frac{t^\rho}{\rho}\right)^\alpha\right)$ and a class $\mathcal{K}_\infty$ function γ_{n-1}, such that for any initial condition $\xi_{n-1} \in \mathbb{R}$, its solution satisfies

$$
\begin{aligned}
\|x_{n-1}(t)\| &\leqslant \beta_{n-1}(\|\xi_{n-1}\|, t^\rho) + \gamma_{n-1}(\|x_n\|_\infty) \\
&\leqslant \beta_{n-1}(\|\xi_{n-1}\|, 0) + .\gamma(\|\beta_n(\|\xi\|, 0)\|) \\
&\leqslant \xi_{n-1} + \gamma(\xi_n) = M.
\end{aligned} \tag{32}
$$

We repeat the same reasoning at each step, and we have, for all $j \in \{n-1, n-2, ..., 2, 1\}$, we have the existence of a class $\mathcal{KL}$ function $\beta_i = \xi_i E_\alpha\left(-\lambda_i\left(\frac{t^\rho}{\rho}\right)^\alpha\right)$ and a class $\mathcal{K}_\infty$ function γ_i, such that for any initial condition $\xi_{n-1} \in \mathbb{R}$, its solution satisfies

$$
\begin{aligned}
\|x_i(t)\| &\leqslant \beta_i(\|\xi_i\|, t^\rho) + \gamma_i(\|x_{i+1}\|_\infty) \\
&\leqslant \beta_i(\|\xi_i\|, 0) + .\gamma(\|\beta_{i+1}(\|\xi_{i+1}\|, 0)\|) \\
&\leqslant \xi_i + \gamma(\xi_{i+1}) = N_i.
\end{aligned} \tag{33}
$$

Finally, that is equivalent to the existence of ϵ such that the following relation is held $\|x(t)\| \leqslant \epsilon$. Thus the fractional differential equation (14) is stable. For the conclusion, we combine the stability and the asymptotic convergence. Then the fractional differential equation defined by (14) is generalized Mittag-Leffler stable.

5. Illustrative example

Let illustrate our main result in this section. We particularly focus on the application of the Theorem 1. For the application we consider nonlinear fractional differential equation described by the following equation

$$
\begin{cases}
D_c^{\alpha,\rho}x_1 = -x_1 + \sqrt{x_2} \\
D_c^{\alpha,\rho}x_2 = -2x_2
\end{cases} \tag{34}
$$

The determination of the analytical solution for the fractional differential equation (34) is complex and not trivial. The Lyapunov function of Eq. (34) is not trivial too. Studied the Mittag-Leffler stability of this class of fractional differential equations become very high, and no method can be applied. Our Theorem 1 offers a useful alternative to this problem. Let describe the application of our Theorem 1 to study the Mittag-Leffler stability of the trivial solution of the fractional differential equation described by Eq. (34). It is straightforward to see the function $V(x_2) = x_2^2/2$ is a Lyapunov candidate function for the fractional differential equation defined by $D_c^{\alpha,\rho}x_2 = -2x_2$. Thus the derivative of the function V along the trajectories gives

$$
D_c^{\alpha,\rho}V(x) \leqslant -2x_2^2 < 0. \tag{35}
$$

From which the Lyapunov characterization, implies the trivial solution of the fractional differential equation $D_c^{\alpha,\rho}x_2 = -2x_2^2$ is Mittag-Leffler stable. The fractional differential equation defined by $D_c^{\alpha,\rho}x_1 = -x_1 + \sqrt{x_2}$ admits as input x_2. Let the Lypunov candidate function defined by $V(x_2) = x_1^2/2$. The derivative of the function V along the trajectories gives

$$\begin{aligned} D_c^{\alpha,\rho}V(x) &\leqslant -x_1^2 + x_1\sqrt{x_2} \\ &\leqslant -x_1^2 + x_1^2/2 + x_2/2 \\ &\leqslant -x_1^2/2 + x_2/2 \\ &\leqslant -(1-\theta)\frac{x_1^2}{2} - \frac{\theta x_1^2}{2} + \frac{x_2}{2}, \end{aligned} \tag{36}$$

with $\theta \in (0,1)$. We observe when

$$\|x_1\| \geqslant \left[\frac{\|x_2\|}{\theta}\right]^{1/2} \Longrightarrow D_c^{\alpha,\rho}V(x) \leqslant -(1-\theta)\frac{x_1^2}{2}. \tag{37}$$

From Eq. (37) follows the generalized Mittag-Leffler input stability of fractional differential equation defined by $D_c^{\alpha,\rho}x_1 = -x_1 + \sqrt{x_2}$. Combining the generalized Mittag-Leffler input stability of $D_c^{\alpha,\rho}x_1 = -x_1 + \sqrt{x_2}$ respecting the input x_2 and generalized Mittag-Leffler stability of the trivial solution of $D_c^{\alpha,\rho}x_2 = -2x_2$, it follows from Theorem 1, the trivial solution of the fractional differential equation (34) is generalized Mittag-Leffler stable.

6. Conclusion

A new vision in the stability analysis is proposed in this paper. Rewriting a system as a cascade as proposed in this paper can have many advantages in the stability analysis. The method avoids the difficulty of finding the exact Lyapunov function of the given fractional differential equation. Note that the method proposed in this paper has also some inconveniences notably when the dimension of the fractional differential equation is high. The perspective of this work is to see the type of stability generated by a cascade of Mittag-Leffler input stability.

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