International Journal of
Mathematical Modelling & Computations
Vol. 09, No. 01, Winter 2019, 1- 15

Discrete Time Analysis of Multi-Server Queueing System with Multiple Working Vacations and Reneging of Customers

P. Vijaya Laxmi[a,*] and T. W. Kassahun[b]

[a,b] *Department of Applied Mathematics, Andhra University, Visakhapatnam - 530 003, India.*

Abstract. This paper analyzes a discrete-time $Geo/Geo/c$ queueing system with multiple working vacations and reneging in which customers arrive according to a geometric process. As soon as the system gets empty, the servers go to a working vacations all together. The service times during regular busy period, working vacation period and vacation times are assumed to be geometrically distributed. Customers waiting for service are subject to reneging and the reneging times are also assumed to be geometrically distributed. The explicit expressions for the steady-state probabilities are obtained recursively from the difference equations that represent the model. Closed form expressions of the system size are also derived both during regular busy period and during WV. In addition, we obtain some other performance measures and a cost model is formulated to determine the optimal service rate during working vacation.

Received: 25 December 2018, Revised: 30 January 2019, Accepted: 25 March 2019.

Keywords: Discrete time queue; Multi-server; Working vacations; Reneging; Cost analysis.

AMS Subject Classification: 60K25, 68M20, 90B22.

Index to information contained in this paper

1 **Introduction**
2 **Model description**
3 **Analysis of the model**
4 **Performance measures and cost model**
5 **Numerical results**
6 **Conclusions**

1. Introduction

In classical vacation queues, the server completely stops service during the vacation period. However, there are numerous situations where the server remains active and provide service with lower rate during the vacation period. Such a vacation policy is

*Corresponding author. Email: vijaya_iit2003@yahoo.co.in

http://ijm2c.iauctb.ac.ir

known as working vacation (WV). Servi and Finn [11] introduced this class of semi-vacation policy. They studied an $M/M/1$ queue with multiple working vacations. Wu and Takagi [18] generalized Servi and Finn's [11] $M/M/1/WV$ queue to an $M/G/1/WV$ queue. Banik et al. [3] studied a $GI/M/1/N$ WV queue with limited waiting space. Liu et al. [9] derived the stochastic decomposition results in an $M/M/1$ queue with WV. An $M/G/1$ queue with exponential WV was analyzed by Li et al. [8].

Considerable interest in discrete-time queues has arisen because of their potential applications in slotted digital computer, communication system, etc. The early studies of discrete-time queues are found in Meisling [10], Hunter[4], etc. The study of discrete-time multi-server queues was conducted Artalejo and Lopez- Herrero [2]. Li and Tian [5] presented the analysis of discrete-time $Geo/Geo/1$ queue with single working vacation. The detailed analysis of discrete-time $Geo/G/1$ queues with variety of vacation policies are given by Takagi [12]. Tian and Zhang [14] presented a discrete-time $GI/Geo/1$ queue with multiple vacations. Alfa[1] analyzed a class of discrete-time vacation models with non-exhaustive service in which inter-arrival times, service times, vacation times and operational times follow phase type distribution. Li and Tian [6] considered a discrete-time $GI/Geo/1$ queue with WV and vacation interruption. Li et al. [7] considered a discrete-time $GI/Geo/1$ queue with MWV under EAS and LAS schemes. Tian et al. [13] analyzed the discrete-time $Geom/Geom/1$ queue with multiple working vacations under $LAS-DA$. Vijaya Laxmi and Jyothsna [16] analyzed a discrete-time finite buffer renewal input queue with multiple working vacations where services are performed in batches of maximum size b. Vijaya Laxmi et al. [17] presented a discrete-time single server finite buffer queue with single working vacation wherein the customers decide either to join the queue or balk. Recently, a retrial queue with working vacation for the batch arrival $Geo^X/Geo/1$ queue has been analyzed by Upadhyaya [15] by considering the general early arrival system.

2. Model description

Let us consider a discrete-time $Geo/Geo/c$ queueing system with $MWVs$ under $LAS-DA$ (late arrival system, delayed access) policy, where the time is divided into constant length intervals (called slots) and the probability of an arrival and a departure occurring simultaneously is not zero. A potential arrival occurs in the interval (t^-, t) and potential departure occurs in the interval (t, t^+) for $t = 0, 1, 2, \cdots$. The DA means that an arrival finding a server free starts its service time in the next slot. In order to formulate the model, we assume the following:

Inter-arrival times of two successive arrivals are independent and identically distributed (i.i.d) random variables and follow geometric distribution with probability mass function $(p.m.f)$:

$$P\{A = m\} = \lambda\bar{\lambda}^{m-1}, \ \ m \geq 1, \ \ 0 < \lambda < 1.$$

In MWV policy, servers are allowed to go back to working vacation if the system remains empty at the end of a vacation. Suppose the beginning and ending of vacation occur at the epoch which is similar to t in shape. The service is provided with rate μ and at the end of a service if there is no customer in the system, servers begin synchronous WV, *i.e.*, they remain dormant between the service completion epoch in (t, t^+) and the next arrival epoch in $((t+1)^-, t+1)$. If some customers arrive in $((t+1)^-, t+1)$, the dormant period will last until the beginning of epoch

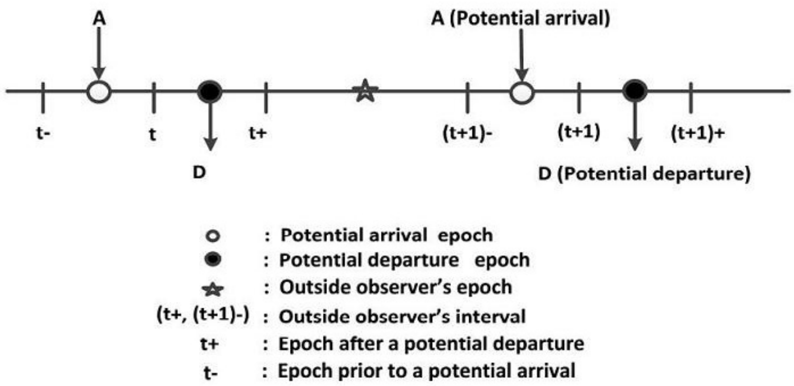

Figure 1. various time epochs in $LAS - DA$.

of service in $(t+1,(t+1)^+)$; otherwise, the servers continues to be dormant until the first vacation is completed and either proceed to the next WV or return to the regular busy period based on the availability of customers in the system. During WV, if customer arrives, the servers provide service with a rate η.

The service times during regular busy period (S_b) and service times during working vacation period (S_{WV}) are independently and geometrically distributed with probability mass functions $(p.m.fs)$ given respectively as:

$$P\{S_b = m\} = \mu\bar{\mu}^{m-1}, \;\; m \geq 1, \;\; 0 < \mu < 1,$$

and

$$P\{S_{WV} = m\} = \eta\bar{\eta}^{m-1}, \;\; m \geq 1, \;\; 0 < \eta < 1.$$

This means that for any busy server the probability that it will finish the undergoing service in the next slot is μ and η respectively.
The average service rate during regular busy period when there are n customers in the system is given by

$$\mu_n = \begin{cases} 1-\overline{\mu}^n, \, 0 \leqslant n \leqslant c \\ 1-\overline{\mu}^c, \;\; n \geqslant c+1 \end{cases}$$

The average service rate during working vacation period when there are n customers in the system is given by

$$\eta_n = \begin{cases} 1-\overline{\eta}^n, \, 0 \leqslant n \leqslant c \\ 1-\overline{\eta}^c, \;\; n \geqslant c+1 \end{cases}$$

The servers begin a working vacation of random length V at the epoch when the queue becomes empty. The vacation times are independently and geometrically distributed with probability mass function $(p.m.f)$:

$$P\{V = m\} = \theta\bar{\theta}^{m-1}, \;\; m \geq 1, \;\; 0 < \theta < 1,$$

After joining the queue each customer will wait a certain length of time T for service to begin. If it has not begun by then, he will get impatient and leave the queue without getting service. The impatience time T is geometrically distributed with probability mass function $(p.m.f)$

$$P\{T=m\}=\alpha\bar{\alpha}^{m-1},\ \ m\geq 1,\ \ 0<\alpha<1,$$

where α is the rate of reneging. The average rate of reneging when there are n customers in the system is given by

$$\alpha_n=\begin{cases}0, & 0\leqslant n\leqslant c\\ 1-\overline{\alpha}^{n-c}, & n\geqslant c+1.\end{cases}$$

In the sequel, for any real number $x\in[0,1]$, we denote $\bar{x}=1-x$.

If servers instantly find a customer at WV completion, they return to regular busy period; otherwise, they continue to take WVs. The arrival times, service times during regular busy period and working vacation period, the vacation times and the reneging times are assumed to be mutually independent. In addition, the service discipline is taken to be first in first out (FIFO).

3. Analysis of the model

In this section, we analyze the queueing model with the objective of obtaining the steady state probabilities when the servers are at different states.

Let $L(t^+)$ denote the number of customers in the system at time t^+ and $J(t^+)$ denote the status of the server at time t^+, which is defined as follows:

$$J(t^+)=\begin{cases}0, & \text{the servers are on working vacation at time } t^+\\ 1, & \text{the servers are on regular busy period at time } t^+.\end{cases}$$

The process $\{(L,\ J)\}$ defines a discrete-time Markov process with state space $\Omega=\{(n,j):n\geq 0, j=0,1\}$.

Let $P_{n,j}=P\{L=n,\ J=j\},\ n\geq 0,\ j=\{0,1\}$ denote the steady-state probabilities of the process $\{(L,\ J)\}$. The set of balance equations are written as:

$$\begin{aligned}
P_{0,0}&=\overline{\lambda}P_{0,0}+(\overline{\lambda}(\eta_1\overline{\alpha}_1+\overline{\eta}_1\alpha_1)+\lambda\eta_1\alpha_1)P_{1,0}+\overline{\lambda}\eta_2\alpha_2P_{2,0}\\
&\quad+(\overline{\lambda}(\mu_1\overline{\alpha}_1+\overline{\mu}_1\alpha_1)+\lambda\mu_1\alpha_1)P_{1,1}+\overline{\lambda}\mu_2\alpha_2P_{2,1}, &(1)\\
P_{n,0}&=\overline{\theta}[\lambda\overline{\eta}_{n-1}\overline{\alpha}_{n-1}P_{n-1,0}+(\overline{\lambda}\overline{\eta}_n\overline{\alpha}_n+\lambda(\eta_n\overline{\alpha}_n+\overline{\eta}_n\alpha_n))P_{n,0}\\
&\quad+(\overline{\lambda}(\eta_{n+1}\overline{\alpha}_{n+1}+\overline{\eta}_{n+1}\alpha_{n+1})+\lambda\eta_{n+1}\alpha_{n+1})P_{n+1,0}\\
&\quad+\overline{\lambda}\eta_{n+2}\alpha_{n+2}P_{n+2,0}],\quad 1\leqslant n\leqslant N-3, &(2)\\
P_{N-2,0}&=\overline{\theta}[\lambda\overline{\eta}_{N-3}\overline{\alpha}_{N-3}P_{N-3,0}+(\overline{\lambda}\overline{\eta}_{N-2}\overline{\alpha}_{N-2}+\lambda(\eta_{N-2}\overline{\alpha}_{N-2}+\overline{\eta}_{N-2}\alpha_{N-2}))P_{N-2,0}\\
&\quad+(\overline{\lambda}(\eta_{N-1}\overline{\alpha}_{N-1}+\overline{\eta}_{N-1}\alpha_{N-1})+\lambda\eta_{N-1}\alpha_{N-1})P_{N-1,0}\\
&\quad+\eta_N\alpha_NP_{N,0}], &(3)\\
P_{N-1,0}&=\overline{\theta}[\lambda\overline{\eta}_{N-2}\overline{\alpha}_{N-2}P_{N-2,0}+(\overline{\lambda}\overline{\eta}_{N-1}\overline{\alpha}_{N-1}+\lambda(\eta_{N-1}\overline{\alpha}_{N-1}+\overline{\eta}_{N-1}\alpha_{N-1}))P_{N-1,0}\\
&\quad+(\eta_N\overline{\alpha}_N+\overline{\eta}_N\alpha_N)P_{N,0}], &(4)\\
P_{N,0}&=\overline{\theta}[\lambda\overline{\eta}_{N-1}\overline{\alpha}_{N-1}P_{N-1,0}+\overline{\eta}_N\overline{\alpha}_NP_{N,0}], &(5)\\
P_{1,1}&=(\overline{\lambda}\overline{\mu}_1\overline{\alpha}_1+\lambda(\mu_1\overline{\alpha}_1+\overline{\mu}_1\alpha_1))P_{1,1}+(\overline{\lambda}(\mu_2\overline{\alpha}_2+\overline{\mu}_2\alpha_2)+\lambda\mu_2\alpha_2)P_{2,1}\\
&\quad+\overline{\lambda}\mu_3\alpha_3P_{3,1}+\theta[\lambda P_{0,0}+(\overline{\lambda}\overline{\eta}_1\overline{\alpha}_1+\lambda(\eta_1\overline{\alpha}_1+\overline{\eta}_1\alpha_1))P_{1,0}\\
&\quad+(\overline{\lambda}(\eta_2\overline{\alpha}_2+\overline{\eta}_2\alpha_2)+\lambda\eta_2\alpha_2)P_{2,0}+\overline{\lambda}\eta_3\alpha_3P_{3,0}], &(6)
\end{aligned}$$

$$\begin{aligned}
P_{n,1} &= \lambda\overline{\mu}_{n-1}\overline{\alpha}_{n-1}P_{n-1,1} + (\overline{\lambda}\overline{\mu}_n\overline{\alpha}_n + \lambda(\mu_n\overline{\alpha}_n + \overline{\mu}_n\alpha_n))P_{n,1} \\
&+(\overline{\lambda}(\mu_{n+1}\overline{\alpha}_{n+1} + \overline{\mu}_{n+1}\alpha_{n+1}) + \lambda\mu_{n+1}\alpha_{n+1})P_{n+1,1} + \overline{\lambda}\mu_{n+2}\alpha_{n+2}P_{n+2,1} \\
&+\theta[\lambda\overline{\eta}_{n-1}\overline{\alpha}_{n-1}P_{n-1,0} + (\overline{\lambda}\overline{\eta}_n\overline{\alpha}_n + \lambda(\eta_n\overline{\alpha}_n + \overline{\eta}_n\alpha_n))P_{n,0} \\
&+(\overline{\lambda}(\eta_{n+1}\overline{\alpha}_{n+1} + \overline{\eta}_{n+1}\alpha_{n+1}) + \lambda\eta_{n+1}\alpha_{n+1})P_{n+1,0} \\
&+\overline{\lambda}\eta_{n+2}\alpha_{n+2}P_{n+2,0}], \qquad 2 \leqslant n \leqslant N-3, && (7) \\
P_{N-2,1} &= \lambda\overline{\mu}_{N-3}\overline{\alpha}_{N-3}P_{N-3,1} + (\overline{\lambda}\overline{\mu}_{N-2}\overline{\alpha}_{N-2} + \lambda(\mu_{N-2}\overline{\alpha}_{N-2} + \overline{\mu}_{N-2}\alpha_{N-2}))P_{N-2,1} \\
&+(\overline{\lambda}(\mu_{N-1}\overline{\alpha}_{N-1} + \overline{\mu}_{N-1}\alpha_{N-1}) + \lambda\mu_{N-1}\alpha_{N-1})P_{N-1,1} + \mu_N\alpha_N P_{N,1} \\
&+\theta[\lambda\overline{\eta}_{N-3}\overline{\alpha}_{N-3}P_{N-3,0} + (\overline{\lambda}\overline{\eta}_{N-2}\overline{\alpha}_{N-2} + \lambda(\eta_{N-2}\overline{\alpha}_{N-2} + \overline{\eta}_{N-2}\alpha_{N-2}))P_{N-2,0} \\
&+(\overline{\lambda}(\eta_{N-1}\overline{\alpha}_{N-1} + \overline{\eta}_{N-1}\alpha_{N-1}) + \lambda\eta_{N-1}\alpha_{n+1})P_{N-1,0} \\
&+\eta_N\alpha_N P_{N,0}], && (8) \\
P_{N-1,1} &= \lambda\overline{\mu}_{N-2}\overline{\alpha}_{N-2}P_{N-2,1} + (\overline{\lambda}\overline{\mu}_{N-1}\overline{\alpha}_{N-1} + \lambda(\mu_{N-1}\overline{\alpha}_{N-1} + \overline{\mu}_{N-1}\alpha_{N-1}))P_{N-1,1} \\
&+(\mu_N\overline{\alpha}_N + \overline{\mu}_N\alpha_N)P_{N,1} \\
&+\theta[\lambda\overline{\eta}_{N-2}\overline{\alpha}_{N-2}P_{N-2,0} + (\overline{\lambda}\overline{\eta}_{N-1}\overline{\alpha}_{N-1} + \lambda(\eta_{N-1}\overline{\alpha}_{N-1} + \overline{\eta}_{N-1}\alpha_{N-1}))P_{N-1,0} \\
&+(\eta_N\overline{\alpha}_N + \overline{\eta}_N\alpha_N)P_{N,0}], && (9) \\
P_{N,1} &= \lambda\overline{\mu}_{N-1}\overline{\alpha}_{N-1}P_{N-1,1} + \overline{\mu}_N\overline{\alpha}_N P_{N,1} \\
&+\theta[\lambda\overline{\eta}_{N-1}\overline{\alpha}_{N-1}P_{N-1,0} + \overline{\eta}_N\overline{\alpha}_N P_{N,0}], && (10)
\end{aligned}$$

where we assume that the buffer space N is large compared to c. The normalizing condition is given by

$$\sum_{n=0}^{N} P_{n,0} + \sum_{n=1}^{N} P_{n,1} = 1. \qquad (11)$$

The state transition diagram of the model is presented in Figure 2. In the Figure, the subscriptions used are as given below:

(1) Transitions due to arrival of a customer during WV

$$av\_n = \overline{\theta}\lambda\overline{\eta}_n\overline{\alpha}_n, \qquad n = 0, 1, 2, \ldots, N-1.$$

(2) Transitions due to arrival of a customer during regular busy period

$$ab\_n = \lambda\overline{\mu}_n\overline{\alpha}_n, \qquad n = 1, 2, \ldots, N-1.$$

(3) Transitions due to departure of a customer during WV

$$\begin{aligned}
dv_1 &= \overline{\lambda}(\eta_1\overline{\alpha}_1 + \overline{\eta}_1\alpha_1) + \lambda\eta_1\alpha_1, \\
dv_n &= \overline{\theta}[\overline{\lambda}(\eta_n\overline{\alpha}_n + \overline{\eta}_n\alpha_n) + \lambda\eta_n\alpha_n], \qquad n = 2, 3, \ldots, N-1, \\
dv_N &= \overline{\theta}(\eta_N\overline{\alpha}_N + \overline{\eta}_N\alpha_N),
\end{aligned}$$

Due to departure of two customers at a time due to simultaneous service and reneging

$$ddv\_n = \overline{\theta\lambda}\eta_n\alpha_n, \qquad n = c+1, c+2, , \ldots, N-1,$$
$$ddv\_N = \overline{\theta}\eta_N\alpha_N.$$

(4) Transitions due to departure of a customer during regular busy period

$$db\_n = \overline{\lambda}(\mu_n\overline{\alpha}_n + \overline{\mu}_n\alpha_n) + \lambda\mu_n\alpha_n, \qquad n = 2, \ldots, N-1,$$
$$db\_N = \mu_N\overline{\alpha}_N + \overline{\mu}_N\alpha_N.$$

Due to departure of two customers at a time due to simultaneous service and reneging

$$ddb\_n = \overline{\lambda}\mu_n\alpha_n, \qquad n = c+1, c+2, , \ldots, N-1,$$
$$ddb\_N = \mu_N\alpha_N.$$

(5) Transition from state on to the same state (loop) during WV

$$sv\_0 = \overline{\lambda},$$
$$sv\_n = \overline{\theta}[\lambda(\eta_n\overline{\alpha}_n + \overline{\eta}_n\alpha_n) + \overline{\lambda}\overline{\eta}_n\overline{\alpha}_n], \qquad n = 1, 2, \ldots, N-1,$$
$$sv\_N = \overline{\theta}\overline{\eta}_N\overline{\alpha}_N.$$

(6) Transition from state on to the same state (loop) during regular busy period

$$sb\_n = \lambda(\mu_n\overline{\alpha}_n + \overline{\mu}_n\alpha_n) + \overline{\lambda}\overline{\mu}_n\overline{\alpha}_n, \qquad n = 1, 2, \ldots, N-1,$$
$$sb\_N = \overline{\mu}_N\overline{\alpha}_N.$$

(7) Vacation completion: Transition from WV with i customers to regular busy period with j customers denoted by vi, j

$$vi, i-1 = \theta[\overline{\lambda}(\eta_i\overline{\alpha}_i + \overline{\eta}_i\alpha_i) + \lambda\eta_i\alpha_i],$$
$$vi, i-2 = \theta\overline{\lambda}\eta_i\alpha_i.$$
$$vi, i = \theta[\lambda(\eta_i\overline{\alpha}_i + \overline{\eta}_i\alpha_i) + \overline{\lambda}\overline{\eta}_i\overline{\alpha}_i],$$
$$vi, i+1 = \theta\lambda\overline{\eta}_i\overline{\alpha}_i.$$

To obtain the steady state probabilities $P_{n,0},\ 0 \leqslant n \leqslant N$ and $P_{n,1},\ 1 \leqslant n \leqslant N$, we solve the equations (1) - (10).
Solving equations (2) - (5) recursively, we find

$$P_{n,0} = a_n P_{N,0}, \quad 0 \leqslant n \leqslant N, \tag{12}$$

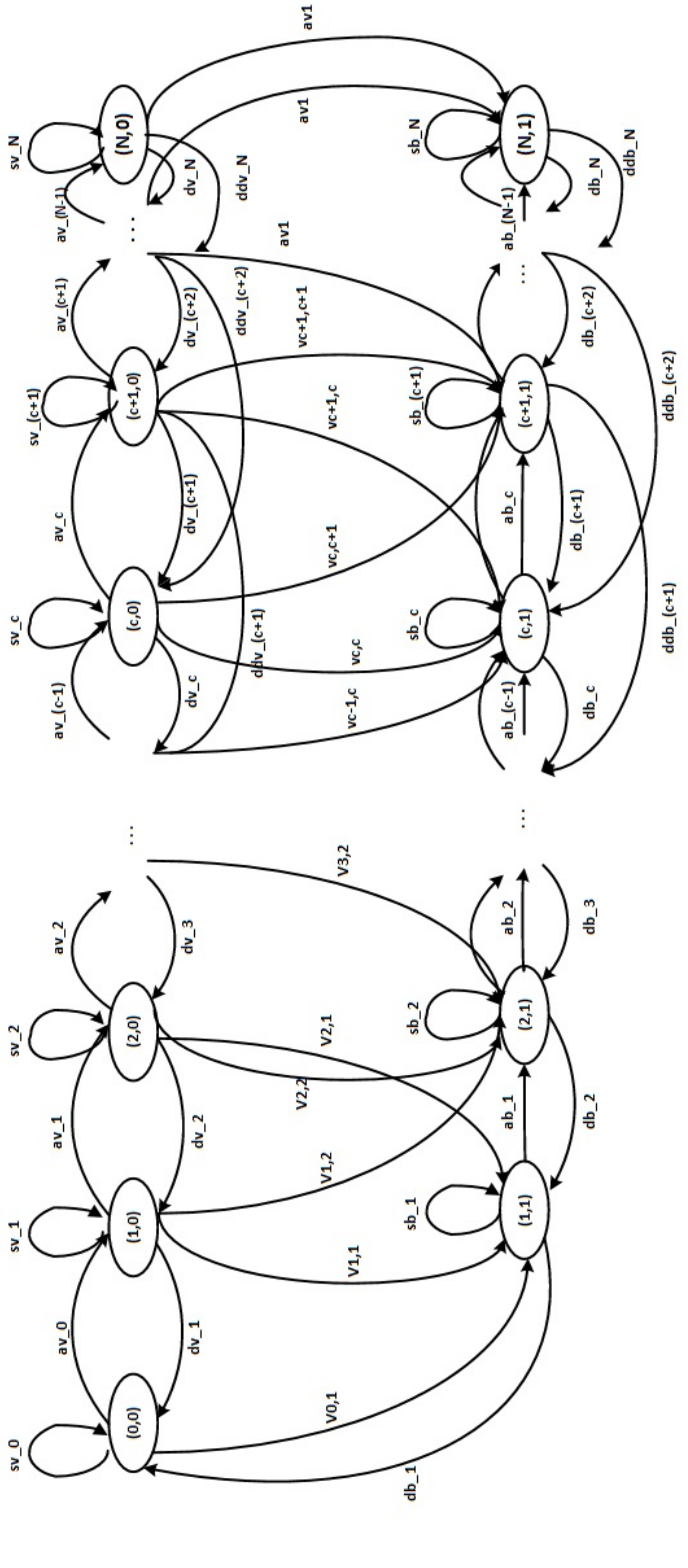

Figure 2. The state transition diagram of the model.

where

$$
\begin{aligned}
a_N &= 1, && (13)\\
a_{N-1} &= \frac{1-\overline{\theta}\overline{\eta}_N\overline{\alpha}_N}{\overline{\theta}\lambda\overline{\eta}_{N-1}\overline{\alpha}_{N-1}} && (14)\\
a_{N-2} &= \{(1-\overline{\theta}(\overline{\lambda}\overline{\eta}_{N-1}\overline{\alpha}_{N-1}+\lambda(\eta_{N-1}\overline{\alpha}_{N-1}+\overline{\eta}_{N-1}\alpha_{N-1})))a_{N-1}\\
&\quad -\overline{\theta}(\eta_N\overline{\alpha}_N+\overline{\eta}_N\alpha_N)\}(\overline{\theta}\lambda\overline{\eta}_{N-2}\overline{\alpha}_{N-2})^{-1} && (15)\\
a_{N-3} &= [(1-\overline{\theta}(\overline{\lambda}\overline{\eta}_{N-2}\overline{\alpha}_{N-2}+\lambda(\eta_{N-2}\overline{\alpha}_{N-2}+\overline{\eta}_{N-2}\alpha_{N-2})))a_{N-2}\\
&\quad -\overline{\theta}(\overline{\lambda}(\eta_{N-1}\overline{\alpha}_{N-1}+\overline{\eta}_{N-1}\alpha_{N-1})+\lambda\eta_{N-1}\alpha_{N-1})a_{N-1}\\
&\quad -\overline{\theta}\eta_N\alpha_N](\overline{\theta}\lambda\overline{\eta}_{N-3}\overline{\alpha}_{N-3})^{-1} && (16)\\
a_n &= [(1-\overline{\theta}(\overline{\lambda}\overline{\eta}_{n+1}\overline{\alpha}_{n+1}+\lambda(\eta_{n+1}\overline{\alpha}_{n+1}+\overline{\eta}_{n+1}\alpha_{n+1})))a_{n+1}\\
&\quad -\overline{\theta}(\overline{\lambda}(\eta_{n+2}\overline{\alpha}_{n+2}+\overline{\eta}_{n+2}\alpha_{n+2})+\lambda\eta_{n+2}\alpha_{n+2})a_{n+2}\\
&\quad -\overline{\theta\lambda}\eta_{n+3}\alpha_{n+3}a_{n+3}](\overline{\theta}\lambda\overline{\eta}_n\overline{\alpha}_n)^{-1}, \quad 1\leqslant n\leqslant N-4 && (17)\\
a_0 &= [(1-\overline{\theta}(\overline{\lambda}\overline{\eta}_1+\lambda\eta_1))a_1-\overline{\theta}(\overline{\lambda}(\eta_2\overline{\alpha}_2+\overline{\eta}_2\alpha_2)+\lambda\eta_2\alpha_2)a_2\\
&\quad -\overline{\theta\lambda}\eta_3\alpha_3a_3](\overline{\theta}\lambda)^{-1}, && (18)
\end{aligned}
$$

Substituting equation (12) in equations (6) - (10), yields

$$P_{n,1} = \beta_n P_{N,1} + \gamma_n P_{N,0}, \quad 1\leqslant n\leqslant N, \tag{19}$$

where

$$
\begin{aligned}
\beta_N &= 1, \quad \gamma_N = 0.\\
\beta_{N-1} &= (1-\overline{\mu}_N\overline{\alpha}_N)(\lambda\overline{\mu}_{N-1}\overline{\alpha}_{N-1})^{-1}\\
\gamma_{N-1} &= -\theta[\lambda\overline{\eta}_{N-1}\overline{\alpha}_{N-1}a_{N-1}+\overline{\eta}_N\overline{\alpha}_N](\lambda\overline{\mu}_{N-1}\overline{\alpha}_{N-1})^{-1}\\
\beta_{N-2} &= \{(1-(\overline{\lambda}\overline{\mu}_{N-1}\overline{\alpha}_{N-1}+\lambda(\mu_{N-1}\overline{\alpha}_{N-1}+\overline{\mu}_{N-1}\alpha_{N-1})))\beta_{N-1}\\
&\quad -(\mu_N\overline{\alpha}_N+\overline{\mu}_N\alpha_N)\}(\lambda\overline{\mu}_{N-2}\overline{\alpha}_{N-2})^{-1},\\
\gamma_{N-2} &= \{(1-(\overline{\lambda}\overline{\mu}_{N-1}\overline{\alpha}_{N-1}+\lambda(\mu_{N-1}\overline{\alpha}_{N-1}+\overline{\mu}_{N-1}\alpha_{N-1})))\gamma_{N-1}\\
&\quad -\theta[\lambda\overline{\eta}_{N-2}\overline{\alpha}_{N-2}a_{N-2}+(\overline{\lambda}\overline{\eta}_{N-1}\overline{\alpha}_{N-1}\\
&\quad +\lambda(\eta_{N-1}\overline{\alpha}_{N-1}+\overline{\eta}_{N-1}\alpha_{N-1}))a_{N-1}+(\eta_N\overline{\alpha}_N+\overline{\eta}_N\alpha_N)]\}\\
&\quad (\lambda\overline{\mu}_{N-2}\overline{\alpha}_{N-2})^{-1},\\
\beta_{N-3} &= \{(1-(\overline{\lambda}\overline{\mu}_{N-2}\overline{\alpha}_{N-2}+\lambda(\mu_{N-2}\overline{\alpha}_{N-2}+\overline{\mu}_{N-2}\alpha_{N-2})))\beta_{N-2}\\
&\quad -(\overline{\lambda}(\mu_{N-1}\overline{\alpha}_{N-1}+\overline{\mu}_{N-1}\alpha_{N-1})+\lambda\mu_{N-1}\alpha_{N-1})\beta_{N-1}-\mu_N\alpha_N\}\\
&\quad (\lambda\overline{\mu}_{N-3}\overline{\alpha}_{N-3})^{-1},
\end{aligned}
$$

$$\begin{aligned}
\gamma_{N-3} = {} & \{(1-(\overline{\lambda}\overline{\mu}_{N-2}\overline{\alpha}_{N-2}+\lambda(\mu_{N-2}\overline{\alpha}_{N-2}+\overline{\mu}_{N-2}\alpha_{N-2})))\gamma_{N-2} \\
& -(\overline{\lambda}(\mu_{N-1}\overline{\alpha}_{N-1}+\overline{\mu}_{N-1}\alpha_{N-1})+\lambda\mu_{N-1}\alpha_{N-1})\gamma_{N-1} \\
& -\theta[\lambda\overline{\eta}_{N-3}\overline{\alpha}_{N-3}a_{N-3}+(\overline{\lambda}\overline{\eta}_{N-2}\overline{\alpha}_{N-2} \\
& +\lambda(\eta_{N-2}\overline{\alpha}_{N-2}+\overline{\eta}_{N-2}\alpha_{N-2}))a_{N-2} \\
& +(\overline{\lambda}(\eta_{N-1}\overline{\alpha}_{N-1}+\overline{\eta}_{N-1}\alpha_{N-1})+\lambda\eta_{N-1}\alpha_{N-1})a_{N-1} \\
& +\eta_N\alpha_N]\}(\lambda\overline{\mu}_{N-3}\overline{\alpha}_{N-3})^{-1}, \\
\beta_{n-1} = {} & \{(1-(\overline{\lambda}\overline{\mu}_n\overline{\alpha}_n+\lambda(\mu_n\overline{\alpha}_n+\overline{\mu}_n\alpha_n))\beta_n \\
& -(\overline{\lambda}(\mu_{n+1}\overline{\alpha}_{n+1}+\overline{\mu}_{n+1}\alpha_{n+1})+\lambda\mu_{n+1}\alpha_{n+1})\beta_{n+1}- \\
& \overline{\lambda}\mu_{n+2}\alpha_{n+2}\beta_{n+2}\}(\lambda\overline{\mu}_{n-1}\overline{\alpha}_{n-1})^{-1}, \quad 2\leqslant n\leqslant N-3, \\
\gamma_{n-1} = {} & \{(1-(\overline{\lambda}\overline{\mu}_n\overline{\alpha}_n+\lambda(\mu_n\overline{\alpha}_n+\overline{\mu}_n\alpha_n))\gamma_n \\
& -(\overline{\lambda}(\mu_{n+1}\overline{\alpha}_{n+1}+\overline{\mu}_{n+1}\alpha_{n+1})+\lambda\mu_{n+1}\alpha_{n+1})\gamma_{n+1}-\overline{\lambda}\mu_{n+2}\alpha_{n+2}\gamma_{n+2} \\
& -\theta[\lambda\overline{\eta}_{n-1}\overline{\alpha}_{n-1}a_{n-1}+(\overline{\lambda}\overline{\eta}_n\overline{\alpha}_n+\lambda(\eta_n\overline{\alpha}_n+\overline{\eta}_n\alpha_n))a_n \\
& +(\overline{\lambda}(\eta_{n+1}\overline{\alpha}_{n+1}+\overline{\eta}_{n+1}\alpha_{n+1})+\lambda\eta_{n+1}\alpha_{n+1})a_{n+1} \\
& +\overline{\lambda}\eta_{n+2}\alpha_{n+2}a_{n+2}]\}(\lambda\overline{\mu}_{n-1}\overline{\alpha}_{n-1})^{-1}, \quad 2\leqslant n\leqslant N-3,
\end{aligned}$$

Substituting equations (12) and (19) in equation (6), we obtain

$$P_{N,1} = \omega P_{N,0}, \tag{20}$$

where $\omega = \frac{D_1}{D_2}$, with

$$\begin{aligned}
D_1 = {} & -\gamma_1+(\overline{\lambda}\overline{\mu}_1\overline{\alpha}_1+\lambda(\mu_1\overline{\alpha}_1+\overline{\mu}_1\alpha_1))\gamma_1+(\overline{\lambda}(\mu_2\overline{\alpha}_2+\overline{\mu}_2\alpha_2)+\lambda\mu_2\alpha_2)\gamma_2 \\
& +\overline{\lambda}\mu_3\alpha_3\gamma_3+\theta[\lambda a_0+(\overline{\lambda}\overline{\eta}_1\overline{\alpha}_1+\lambda(\eta_1\overline{\alpha}_1+\overline{\eta}_1\alpha_1))a_1+(\overline{\lambda}(\eta_2\overline{\alpha}_2+\overline{\eta}_2\alpha_2) \\
& +\lambda\eta_2\alpha_2)a_2+\overline{\lambda}\eta_3\alpha_3a_3], \\
D_2 = {} & \beta_1-(\overline{\lambda}\overline{\mu}_1\overline{\alpha}_1+\lambda(\mu_1\overline{\alpha}_1+\overline{\mu}_1\alpha_1))\beta_1-(\overline{\lambda}(\mu_2\overline{\alpha}_2+\overline{\mu}_2\alpha_2)+\lambda\mu_2\alpha_2)\beta_2-\overline{\lambda}\mu_3\alpha_3\beta_3.
\end{aligned}$$

Finally, $P_{N,0}$ is obtained from the normalization condition

$$\sum_{n=0}^{N} P_{n,0} + \sum_{n=1}^{N} P_{n,1} = 1$$

which is given by

$$P_{N,0} = \left(\sum_{n=0}^{N} a_n + \sum_{n=1}^{N} \beta_n\omega + \gamma_n\right)^{-1}$$

4. Performance measures and cost model

Once the state probabilities at are determined, we can evaluate various performance measures such as the average number of customers in system during the different vacation states and the average waiting time in system, etc. They are given by

4.1 *Performance measures*

- Average system size during WV, $E[L_{WV}]$: The average system size during WV is given by

$$E[L_{WV}] = \sum_{n=1}^{N} nP_{n,0},$$

- Average system size during regular busy period, $E[L_B]$: The average system size during regular busy period is given by

$$E[L_B] = \sum_{n=1}^{N} nP_{n,1},$$

- Average system size, $E[L]$: The average system size is given by

$$E[L] = E[L_{WV}] + E[L_B] = \sum_{n=1}^{N} n(P_{n,0} + P_{n,1}),$$

- Average queue size, $E[L_Q]$: The average queue size is given by

$$E[L_Q] = \sum_{n=c+1}^{N} (n-c)(P_{n,0} + P_{n,1}),$$

  where $\sum_{n=c+1}^{N}(n-c)P_{n,0}$, and $\sum_{n=c+1}^{N}(n-c)P_{n,1}$, are respectively the average queue sizes during WV and regular busy periods.
- Probability of loss due to finite system capacity, P_{loss}: The probability of loss of customers or the blocking probability of the system is given by

$$P_{loss} = P_{N,0} + P_{N,1},$$

- Average waiting time in the system, $E[W]$: The average waiting time of the system (the sojourn time) is given by

$$E[W] = \frac{E[L]}{\lambda_{eff}}, \quad \text{where} \quad \lambda_{eff} = \lambda(1 - P_{loss})$$

- Average reneging rate, $E[R]$: The average reneging rate of the system is given by

$$E[R] = \sum_{n=c+1}^{N} (n-c)\alpha(P_{n,0} + P_{n,1})$$

- Probability that servers are on busy state (P_B) and on WV state (P_{WV}) are given respectively as

$$P_B = \sum_{n=1}^{N} P_{n,1}, \qquad P_{WV} = \sum_{n=0}^{N} P_{n,0}$$

- Probability that the servers are idle during WV, P_{Idle}

$$P_{Idle} = P_{0,0}$$

4.2 *Cost model*

In this subsection, we formulate an expected cost function, in which mean service rate μ is the control variable. For this purpose lets introduce the following cost coefficients:

$C_1 =$ Cost per unit time when the servers are busy;

$C_2 =$ Cost per unit time when the servers are on working vacation period;

$C_3 =$ Cost per unit time when the servers are idles during WV;

$C_4 =$ Cost per unit time when a customer joins the queue and waits for service;

$C_5 =$ Cost per unit time when a customer reneges from the system;

$C_6 =$ Cost per service per unit time during WV period;

$C_7 =$ Cost per service per unit time during busy period;

$C_8 =$ Fixed cost of a server;

Then, let TC be the total expected cost per unit time of the system which is given by:

$$TC = C_1 P_B + C_2 P_{WV} + C_3 P_{Idle} + C_4 E[L_Q] + C_5 E[R] + c(C_6\eta + C_7\mu) + C_8 c; \quad (21)$$

Our objective is to determine the optimal mean service rate μ^* that minimizes the cost function $TC(\mu)$. We employ the quadratic fit search method ($QFSM$) to solve the above optimization problem numerically.

4.3 *Quadratic fit search method*

In this section, we introduce the optimization technique by which the cost function under consideration could be optimized. The quadratic fit search method ($QFSM$) utilizes a 3-point pattern for fitting a quadratic function that has a unique optimum. The objective is to determine the optimal service rate μ that minimizes the total expected cost function. Keeping other parameters fixed, minimizing the cost function TC with respect to the service rate μ is obtained in such a way that given a 3-point pattern, we compute a quadratic function via corresponding functional values that has a unique minimum, x^q, for the given objective function $TC(\mu)$. Quadratic fit uses this approximation to improve the current 3-point pattern by replacing one of its points with optimum x^q. The unique optimum x^q of the quadratic function that agrees with $TC(\mu)$ at the 3-points (x^l, x^m, x^h) is given by

$$x_q \approxeq \frac{1}{2}\left[\frac{F(x^l)((x^m)^2-(x^h)^2)+F(x^m)((x^h)^2-(x^l)^2)+F(x^h)((x^l)^2-(x^m)^2)}{F(x^l)(x^m-x^h)+F(x^m)(x^h-x^l)+F(x^h)(x^l-x^m)}\right].$$

Table 1. Effect of μ and other parameters on some of the performance measures.

μ		$E[L]$	$E[L_Q]$	P_B	$E[W]$	$E[R]$	P_{Idle}	TC
	$\lambda = 0.3$	0.7395	0.0180	0.4169	2.4652	0.0090	0.4296	26.6716
0.5	$\lambda = 0.4$	0.9890	0.0404	0.5487	2.4725	0.0202	0.3015	27.0678
	$\lambda = 0.5$	1.2395	0.0754	0.6713	2.4790	0.0377	0.1997	27.5905
	$\lambda = 0.3$	0.5640	0.0073	0.2799	1.8800	0.0036	0.5305	27.2034
0.8	$\lambda = 0.4$	0.7465	0.0150	0.3822	1.8662	0.0075	0.4128	27.3917
	$\lambda = 0.5$	0.9256	0.0250	0.4889	1.8512	0.0125	0.3105	27.6458
	$\theta = 0.4$	2.0089	0.2835	0.9427	2.5112	0.1417	0.0267	29.9631
0.5	$\theta = 0.6$	1.9991	0.2783	0.9582	2.4989	0.1391	0.0275	29.9751
	$\theta = 0.8$	1.9939	0.2762	0.9664	2.4924	0.1381	0.0280	29.9869
	$\theta = 0.4$	1.4739	0.0525	0.8327	1.8424	0.0262	0.0780	28.6176
0.8	$\theta = 0.6$	1.4112	0.0326	0.8697	1.7640	0.0163	0.0858	28.5928
	$\theta = 0.8$	1.3714	0.0250	0.8898	1.7142	0.0125	0.0919	28.6119
0.5	$\alpha = 0.3$	2.1987	0.4351	0.9506	2.7484	0.1305	0.0230	30.6629
	$\alpha = 0.4$	2.0849	0.3432	0.9460	2.6061	0.1372	0.0251	30.2388
	$\alpha = 0.5$	2.0089	0.2835	0.9427	2.5112	0.1417	0.0267	29.9631
0.8	$\alpha = 0.3$	1.5075	0.0740	0.8362	1.8844	0.0222	0.0764	28.7056
	$\alpha = 0.4$	1.4879	0.0614	0.8342	1.8599	0.0245	0.0774	28.6542
	$\alpha = 0.5$	1.4739	0.0525	0.8327	1.8424	0.0262	0.0780	28.6176
	$\eta = 0.3$	2.0089	0.2835	0.9427	2.5112	0.1417	0.0267	29.9631
0.5	$\eta = 0.4$	1.9995	0.2789	0.9416	2.4994	0.1394	0.0275	30.3222
	$\eta = 0.5$	1.9906	0.2753	0.9404	2.4883	0.1376	0.0284	30.6889
	$\eta = 0.3$	1.4739	0.0525	0.8327	1.8424	0.0262	0.0780	28.6176
0.8	$\eta = 0.4$	1.4469	0.0423	0.8284	1.8086	0.0211	0.0810	28.9212
	$\eta = 0.5$	1.4211	0.0346	0.8236	1.7764	0.0173	0.0843	29.2456
	$c = 2$	2.0089	0.2835	0.9427	2.5112	0.1417	0.0267	29.9631
0.5	$c = 3$	2.3914	0.0840	0.9649	2.9892	0.0420	0.0163	39.6748
	$c = 4$	2.6435	0.0174	0.9708	3.3043	0.0087	0.0136	50.4734
	$c = 2$	1.4739	0.0525	0.8327	1.8424	0.0262	0.0780	28.6176
0.8	$c = 3$	1.5990	0.0068	0.8478	1.9987	0.0034	0.0710	40.0679
	$c = 4$	1.6270	0.0010	0.8497	2.0338	0.0005	0.0701	51.8232
	$N = 10$	2.0089	0.2835	0.9427	2.5112	0.1417	0.0267	29.9631
0.5	$N = 15$	13.8704	11.7827	1.5535	17.3379	5.8913	-0.0085	129.7730
	$N = 20$	16.0001	14.0001	1.4054	20.0001	7.0001	≈ 0	148.1200
	$N = 10$	1.4739	0.0525	0.8327	1.8424	0.0262	0.0780	28.6176
0.8	$N = 15$	10.7172	8.7344	0.8496	13.3965	4.3672	0.0023	102.3210
	$N = 20$	16.0000	14.0000	0.8536	20.0000	7.0000	≈ 0	147.0880

5. Numerical results

To study the parameter impact on the system performance, numerical computations are carried out and a few of those are presented in this section in the form of tables and graphs. This numerical results are obtained using $Mathematica$ 9.0 software. For the purpose of numerical illustration we have assumed the following constants for the system parameters involved in the queueing model under investigation: $c = 2,\ \lambda = 0.8,\ \eta = 0.3,\ \mu = 0.6,\ \theta = 0.4,\ \alpha = 0.5$, and $N = 10$ and the cost parameters are taken as: $C_1 = 5$, $C_2 = 1.5$, $C_3 = 2$, $C_4 = 5$, $C_5 = 7$, $C_6 = 2$, $C_7 = 1.5$ and $C_8 = 10$ unless they are considered as variables or their values are mentioned in the respective tables or figures. In Table 1 we have investigated the effect of various parameters on the system performance measures where the change in each parameter is associated with the effect of the change on the service rate μ. Thus we present the numerical findings as follows:

(1) We observe that increasing the arrival rate λ results in an increase of all the performance measures listed in the table except the idle state during WV, P_{Idle}. This is because due to more arrivals the tendency of the system to come to an idle state decreases. On the other hand, this trend is found to be opposite for the expected waiting time $E[W]$ at $\mu = 0.8$. That is, with the increase in λ, $E[W]$ is found to show a decreasing trend. This can be justified to the fact that the increase in the arrival rate is relatively smaller compared to the increase in the service rate. Thus, a decrease in the waiting time of the system unlike the trend that was observed when $\mu = 0.5$.

(2) The effect of the system performance measures with the change in the value of θ is presented. Since the mean vacation time is the inverse of the are of vacation θ, increasing θ implies decreasing the mean vacation time. This the mean busy time (regular busy period) time increases. Hence P_B increases. This in tern results in the use of higher service rate μ more time in the system. Thus, the system (queue) sizes decrease with the increase in θ. Waiting time $E[W]$ also shows a decreasing trend. With the decrease in the system size, the reneging rate $E[R]$ also decreases. Whereas, P_{Idle} increases since the queue size is getting smaller and smaller.

(3) The effect of increasing the reneging rate α is shown to have a decreasing effect on $E[L]$, $W[L_Q]$, P_B, and $E[W]$. That is, more reneging implies less system (queue) size which leads to shorter waiting time and hence higher probability to go to WV state. That is why P_B decreases. On the other hand, average reneging rate $E[R]$ which is directly proportional to the reneging rate (α) and P_{Idle} increases with the increase in θ.

(4) The effect of increasing the WV service rate η is shown to have a decreasing effect on all performance measures listed in the table except on P_{Idle}. P_{Idle} shows an increasing trend due to the decrease in the queue size which ultimately leads to the idle state of the system. It is also evident from the table that increasing the busy period service rate μ results the same effect on the performance measures. That is, all but P_{Idle} decreases with the increase in μ. The same reasoning that works for η works for μ because basically both are service rates which have the same effect on the system performance measures.

(5) Increasing the number of servers in the system increases the performance measures $E[L]$, $E[W]$ and P_B. That is, customers which were expected to renege from the system will be retained due to availability of more servers, hence the reason for the increases in the system size. In the same line, the waiting time in the system increases. The increases in the system size intern results in the increase of P_B. On the other hand, increasing c decreases the queue size $E[L_Q]$ and the average reneging rate $E[R]$. Since customers in the queue join the service due to the increase in the number of servers, the queue size declines give the same arrival rate and service rates. The average reneging rate $E[R]$ decreases since it is directly proportional to the queue size.

(6) Increasing the system capacity N on the system performance measures is also shown to result in an increase of all the performance measures except P_{Idle}. We can see that increasing the system capacity reduces the blocking probability. Hence results in a significant effect to make an increase in the system (queue) size. Which intern makes the system more busy and results

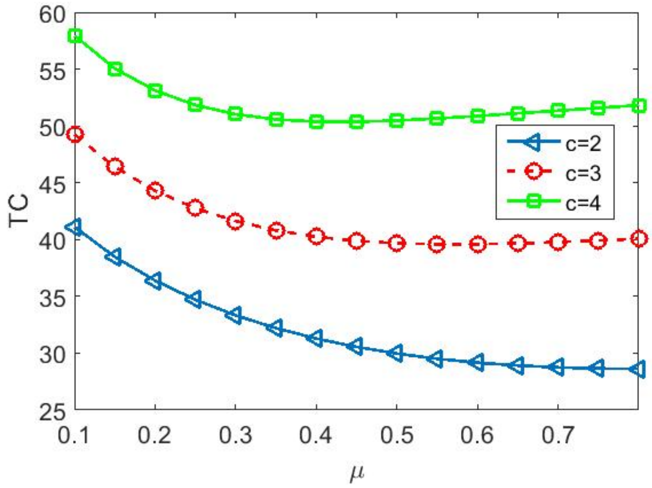

Figure 3. Effect of μ on the TC for different c values.

in more reneging. The sojourn time also increases proportionally. Thus the only thing that decreases is the probability of the system to become idle. Which is obvious because the system is getting busier and hence not likely to come to an idle state.

Table 2. Search for optimum service rate μ for $c = 2,\ \lambda = 0.8,\ N = 10,\ \eta = 0.3,\ \theta = 0.4,\ \alpha = 0.5,\ \epsilon = 10^{-6}$.

μ^l	μ^m	μ^h	$TC(\mu^l)$	$TC(\mu^m)$	$TC(\mu^h)$	μ^q	$TC(\mu^q)$
0.500000	0.650000	0.800000	29.9631	28.9145	28.6176	0.784256	28.6231
0.650000	0.784256	0.800000	28.9145	28.6231	28.6176	0.806548	28.6168
0.784256	0.806548	0.800000	28.6231	28.6168	28.6176	0.809644	28.6167
0.806548	0.809644	0.800000	28.6168	28.6167	28.6176	0.809861	28.6167
0.809644	0.809861	0.800000	28.6167	28.6167	28.6176	0.809883	28.6167
0.809861	0.809883	0.800000	28.6167	28.6167	28.6176	0.809886	28.6167
0.809861	0.809883	0.809886	28.6167	28.6167	28.6167	0.809877	28.6167
0.809877	0.809883	0.809886	28.6167	28.6167	28.6167	0.809878	28.6167
0.809878	0.809883	0.809886	28.6167	28.6167	28.6167	0.809887	28.6167
0.809883	0.809887	0.809886	28.6167	28.6167	28.6167	0.809885	28.6167
0.809885	0.809887	0.809886	28.6167	28.6167	28.6167	0.809887	28.6167

Table 3. Optimum values of μ at different c values.

c	2	3	4	6
μ^*	0.809887	0.571157	0.428403	0.286118
$TC(\mu^*)$	28.616652	39.582174	50.352499	71.627631

The effect of the service rate μ on the total expected cost, TC for different values of c is presented in Figure 3. It can be seen that the optimum service rate μ^* decreases as the number of servers c increases. This is in agreement with what is expected since the service rate can be increased either directly by increasing the service rate itself or indirectly by increasing the number of servers. More servers means nothing but more service rate. The optimum service rate that is computed for $c = 2$ is computed using $QFSM$ and presented using Table 2. After about 11 iterations, the optimum service rate is found to be $\mu^* = 0.809887$ and the corresponding TC is $TC(\mu^*) = 28.6167$. In Table 3, we have given the optimum service rate for different service rates.

6. Conclusions

In this paper, we have studied a $Geo/Geo/c$ queueing system with $MWVs$ and reneging of customers. We have derived the mean system sizes when the servers are in different states. The closed-form expressions of some performance measures are derived and also a cost model is formulated to study the effect of the system parameters on the cost function. Numerical results are presented in the form of graphs. The technique adopted in this paper can be applied to analyze models like $Geo^X/G/c$ queue with multiple working vacations, impatient customer $GI^X/Geo/c$ queue with multiple working vacations, etc.

Acknowledgments

The authors would sincerely thank the Editor and the anonymous referees for their useful comments and suggestions which have helped in improving the paper.

References

[1] A. S. Alfa, Vacation models in discrete-time, Queueing Systems, **44** (2003) 5-30.
[2] J. R Artalejo and M. J. Lopez-Herrero, A discrete-time multi-server retrial queue, Performance Analysis and Simulation, In S. M. Ermakov, V. B. Melas and A. N. Pepelyshev (Eds.), Proceedings of the 5^{th} Workshop on Simulation, St. Petersburg, (2005) 8590.
[3] A. D. Banik, U. C. Gupta and S. S. Pathak, On the $GI/M/1/N$ queue with multiple working vacations-Analytic analysis and computation, Applied Mathematical Modelling, **31** (2007) 1701-1710.
[4] J. Hunter, Mathematical Techniques of Applied Probability Discrete Time Models: Techniques and Applications, Academic Press, New York, **2** (1983).
[5] J. Li and N. Tian, Analysis of the discrete-time *Geo/Geo/*1 queue with single working vacation, Quality Technology & Quantitative Management, **5** (2008) 77-89.
[6] J. Li and N. Tian, The discrete-time $GI/Geo/1$ queue with working vacations and vacation interruption, Applied Mathematics and Computation, **185** (2007) 1-10.
[7] J. Li, N. Tian and W. Liu, Discrete-time $GI/Geo/1$ queue with multiple working vacations, Queueing Systems, **56** (2007) 53-63.
[8] J. H. Li, N. Tian, Z. G. Zhang and H. P. Luh, Analysis of the $M/G/1$ queue with exponentially working vacationsa matrix analytic approach, Queueing Systems, **61** (2009) 139-166.
[9] W. Liu, X. Xu and N. Tian, Stochastic decompositions in the $M/M/1$ queue with working vacations, Operations Research Letters, **35** (2007) 595-600.
[10] T. Meisling, Discrete time queueing theory, Operation Research, **6** (1958) 96-105.
[11] L. D. Servi and S. G. Finn, $M/M/1$ queues with working vacations ($M/M/1/WV$), Performance Evaluation, **50** (2002) 41-52.
[12] H. Takagi, Queueing Analysis: Discrete Time Systems, Elsevier Science Publishers, Amsterdam **3** (1993).
[13] N. Tian, Z. Ma and M. Liu, The discrete-time *Geom/Geom/*1 queue with multiple working vacations, Applied Mathematical Modelling, **32** (2008) 2941-2953.
[14] N. Tian and Z. G. Zhang, The discrete-time $GI/Geo/1$ queue with multiple vacations, Queueing Systems, **40** (2002) 283-294.
[15] S. Upadhyaya, Working vacation policy for a discrete-time *GeoX/Geo/*1 retrial queue, OPSEARCH, **52** (2015) 650-669.
[16] P. Vijaya Laxmi and K. Jyothsna, Finite buffer $GI/Geo/1$ batch servicing queue with multiple working vacations, RAIRO-Operations Research, **48** (2014) 521-543.
[17] P. Vijaya Laxmi, K. Jyothsna and D. Seleshi, Analysis of a discrete-time working vacation queue with balking, OPSEARCH, **52** (2015) 562-581.
[18] D. Wu and H. Takagi, $M/G/1$ queue with multiple working vacations, Performance Evaluation, **63** (2006) 654-681.