International Journal of
Mathematical Modelling & Computations
Vol. 10, No. 04, Fall 2020, 265- 279

On Extension of Generalized Laguerre Polynomials of Two Variable

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Abstract. In this paper, we introduce a new extension of generalized Laguerre polynomials of two variable by using the extended Beta function. Some properties of these extension polynomials such as generating functions, integral representation, recurrence relations and summation formulae are obtained.

Received: 26 May 2020, Revised: 16 October 2020, Accepted: 20 October 2020.

Keywords: Beta function; Generalized Laguerre polynomials; Generating functions; Summation formulae.

AMS Subject Classification: 33B15, 33B20, 33C05, 33C20, 33C45, 33C60.

Index to information contained in this paper

1. Introduction

Chaudhry et al. [4] have introduced the extended Beta function $B(x, y; p)$ as follows:

$$B(x,y;p) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[\frac{-p}{t(1-t)}\right] dt \tag{1}$$

$$(Re(p) \geqslant 0, Re(x) > 0, Re(y) > 0).$$

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It is clearly seem that $B(x, y; 0) = B(x, y)$ where $B(x, y)$ is the classical Beta function defined by [15]:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad (Re(x) > 0, Re(y) > 0), \tag{2}$$

where $\Gamma(x)$ is the classical Gamma function [15].

Afterwards, the extended Beta function (1) are used to extend the Gauss hypergeometric and confluent hypergeometric functions as follows [5]:

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \tag{3}$$

$$(Re(p) \geqslant 0; |z| < 1; Re(c) > Re(b) > 0),$$

$$\Phi_p(b;c;z) = \sum_{n=0}^{\infty} \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \tag{4}$$

$$(Re(p) \geqslant 0; Re(c) > Re(b) > 0),$$

respectively, where $(a)_n$ denotes the Pochhammer's symbol defined by [15]:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad (a)_0 = 1. \tag{5}$$

Note that

$$F_0(a,b;c;z) = {}_2F_1(a,b;c;z), \tag{6a}$$

$$\Phi_0(b;c;z) = \Phi(b;c;z) = {}_1F_1(b;c;z), \tag{6b}$$

where ${}_2F_1(a, b; c; z)$ and ${}_1F_1(b; c; z)$ (or $\Phi(b; c; z)$) are the classical Gauss hypergeometric and confluent hypergeometric functions respectively (see [15]).

In many recent works (see for example [1, 12–14]), the extended Beta function $B(x, y; p)$ and its systemic generalizations are used to introduce new extended special functions such as hypergeometric function, Appell's and Lauricella's hypergeometric functions, Mittag-Leffler function and Zeta function. For example, Özarslan and Yilmaz [12] introduced the extended Mittag-Leffler function $E_{\alpha,\beta}^{(\gamma;c)}(z; p)$ as follows:

$$E_{\alpha,\beta}^{(\gamma;c)}(z;p) = \sum_{n=0}^{\infty} \frac{B(\gamma+n, c-\gamma; p)\ (c)_n}{B(\gamma, c-\gamma)\ \Gamma(n\alpha+\beta)} \frac{z^n}{n!}. \tag{7}$$

where the classical and generalized Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are defined as (see [15]):

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \qquad E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}. \tag{8}$$

This paper is a further attempt to stress the importance of the use of extended Beta function (1) in introducing new extended special polynomials. For this aim, we recall that the 2-variable generalized Laguerre polynomials (2VGLP) ${}_GL_n^{(\alpha,\beta)}(x,y)$ are defined by Atash [3] as follows:

$${}_GL_n^{(\alpha,\beta)}(x,y) = \frac{(\alpha+1)_n(\beta+1)_n}{n!} \sum_{k=0}^{n} \frac{(-x)^k y^{n-k}}{(\alpha+\beta+1)_k\ k!(n-k)!} \tag{9}$$

and specified by the following generating function:

$$\sum_{n=0}^{\infty} {}_GL_n^{(\alpha,\beta)}(x,y)\ \frac{n!\ t^n}{(\alpha+1)_n(\beta+1)_n} = \exp(yt)\ {}_0F_1[-;\alpha+\beta+1;-xt]. \tag{10}$$

In particular, we note that

$${}_GL_n^{(0,\beta)}(x,y) = L_n^{(\beta)}(x,y), \quad {}_GL_n^{(\alpha,0)}(x,y) = L_n^{(\alpha)}(x,y), \tag{11}$$

where $L_n^{(\alpha)}(x,y)$ denotes the 2-variable associated Laguerre polynomials (2VALP) defined as [6]:

$$L_n^{(\alpha)}(x,y) = \sum_{k=0}^{n} \frac{(\alpha+1)_n\ (-x)^k y^{n-k}}{(\alpha+1)_k\ k!(n-k)!} \tag{12}$$

and specified by the following generating functions:

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x,y)\ t^n = (1-yt)^{-\alpha-1} \exp\left(\frac{-xt}{1-ty}\right), \tag{13a}$$

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x,y)\ \frac{t^n}{(\alpha+1)_n} = \Gamma(\alpha+1)\ exp(yt)\ C_\alpha(xt), \tag{13b}$$

where $C_\alpha(x)$ denotes the αth order Tricomi function defined as [15]:

$$C_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{\Gamma(\alpha+k+1)\ k!}. \tag{14}$$

Note that

$$L_n^{(0)}(x,y) = L_n(x,y), \tag{15a}$$

$$L_n^{(\alpha)}(x,1) = L_n^{(\alpha)}(x), \tag{15b}$$

$$L_n^{(0)}(x,1) = L_n(x), \tag{15c}$$

where $L_n(x,y)$ denotes the 2-variable Laguerre polynomials (2VLP) [8] and $L_n^{(\alpha)}(x)$ and $L_n(x)$ denote the classical and associated Laguerre polynomials [15], respectively.

2. An extended Laguerre polynomials of two variable

In terms of the extended Beta function $B(x,y;p)$ defined in (1), we introduce a new extension of 2-variable generalized Laguerre polynomials (E2VGLP), denoted by $L_n^{(\alpha,\beta)}(x,y;p)$, as follows:

$$L_n^{(\alpha,\beta)}(x,y;p) = \frac{(\alpha+1)_n(\beta+1)_n}{n!}\sum_{k=0}^{n}\frac{B(\alpha+k+1,\beta;p)(-x)^k y^{n-k}}{B(\alpha+1,\beta)\ (\alpha+1)_k\ k!(n-k)!}, \tag{16}$$

which for $p=0$ reduces to definition (9), i.e.,

$$L_n^{(\alpha,\beta)}(x,y;0) = {}_GL_n^{(\alpha,\beta)}(x,y). \tag{17}$$

Also, in particular, we note that

$$L_n^{(0,\beta)}(x,y;p) = L_n^{(\beta)}(x,y;p), \tag{18a}$$

$$L_n^{(\alpha,0)}(x,y;0) = L_n^{(\alpha)}(x,y), \qquad L_n^{(0,\beta)}(x,y;0) = L_n^{(\beta)}(x,y), \tag{18b}$$

$$L_n^{(0,0)}(x,y;0) = L_n(x,y). \tag{18c}$$

where $L_n^{(\beta)}(x,y;p)$ denotes the extension of 2-variable associated Laguerre polynomials (E2VALP) defined very recently by Al-Gonah [2] as:

$$L_n^{(\beta)}(x,y;p) = (\beta+1)_n\sum_{k=0}^{n}\frac{B(k+1,\beta;p)(-x)^k y^{n-k}}{B(1,\beta)\ (k!)^2(n-k)!}, \tag{19}$$

which for $p=0$ reduces to definition (12).

Now, we establish the integral representaion for the E2VGLP $L_n^{(\alpha,\beta)}(x,y;p)$ in the form of the following theorem:

Theorem 2.1. *The following integral representation for the E2VGLP $L_n^{(\alpha,\beta)}(x,y;p)$ holds ture:*

$$L_n^{(\alpha,\beta)}(x,y;p) = \frac{(\beta+1)_n}{B(\alpha+1,\beta)\ n!}\int_0^1 t^{\alpha}(1-t)^{\beta-1}\exp\left(\frac{-p}{t(1-t)}\right)L_n^{(\alpha)}(xt,y)dt. \tag{20}$$

Proof. Using definition (16) in the L.H.S. of equation (20) and then using relation (1), we get

$$L_n^{(\alpha,\beta)}(x,y;p) = \frac{(\beta+1)_n}{B(\alpha+1,\beta)\ n!}\int_0^1 t^{\alpha}(1-t)^{\beta-1}\exp\left(\frac{-p}{t(1-t)}\right)\sum_{k=0}^{n}\frac{(\alpha+1)_n(-tx)^k\ y^{n-k}}{(\alpha+1)_k\ k!(n-k)!}dt. \tag{21}$$

Using definition (12) in the R.H.S. of equation (21), we get assertion (20) of Theorem 2.1.

Remark 2.1. For $\alpha = 0$ in assertion (20) of Theorem 2.1, we get the following known result [2]:

$$L_n^{(\beta)}(x,y;p) = \frac{(\beta+1)_n}{B(1,\beta)\ n!}\int_0^1 (1-t)^{\beta-1}\exp\left(\frac{-p}{t(1-t)}\right) L_n(xt,y)dt. \tag{22}$$

Next, the integral representation (20) will be used to derive some properties for the E2VGLP $L_n^{(\alpha,\beta)}(x,y;p)$ as in the following theorem:

Theorem 2.2. *The following pure and differential recurrence relations for the E2VGLP $L_n^{(\alpha,\beta)}(x,y;p)$ hold ture:*

$$\begin{aligned} L_n^{(\alpha,\beta)}(x,y;p) &= \frac{\beta(\beta+1)}{(\alpha+\beta+1)(\beta+n+1)} L_n^{(\alpha,\beta+1)}(x,y;p) \\ &+ \frac{(\alpha+1)^2}{(\alpha+\beta+1)(\alpha+n+1)} L_n^{(\alpha+1,\beta)}(x,y;p) \\ &+ \frac{n(\alpha+1)(\alpha+2)}{(\alpha+\beta+1)(\alpha+\beta+2)(\alpha+n+1)(\alpha+n+2)} L_{n-1}^{(\alpha+2,\beta)}(x,y;p), \end{aligned} \tag{23}$$

$$\frac{\partial}{\partial x} L_n^{(\alpha,\beta)}(x,y;p) = \frac{n}{x} L_n^{(\alpha,\beta)}(x,y;p) - \frac{(\alpha+n)(\beta+n)\ y}{nx} L_{n-1}^{(\alpha,\beta)}(x,y;p), \tag{24}$$

$$\frac{\partial}{\partial y} L_n^{(\alpha,\beta)}(x,y;p) = \frac{(\alpha+n)(\beta+n)}{n} L_{n-1}^{(\alpha,\beta)}(x,y;p). \tag{25}$$

Proof. To prove (23), consider the following relation [4]:

$$B(x,y;p) = B(x,y+1;p) + B(x+1,y;p). \tag{26}$$

Applying relation (26) in the R.H.S. of definition (16), we get

$$\begin{aligned} &L_n^{(\alpha,\beta)}(x,y;p) \\ &= \frac{\Gamma(\alpha+\beta+1)(\alpha+1)_n(\beta+1)_n}{\Gamma(\beta)\ n!} \sum_{k=0}^{n} \frac{(-x)^k y^{n-k} B(\alpha+k+1,\beta+1;p)}{\Gamma(\alpha+k+1)\ k!(n-k)!} \\ &\quad + \frac{\Gamma(\alpha+\beta+1)(\alpha+1)_n(\beta+1)_n}{\Gamma(\beta)\ n!} \sum_{k=0}^{n} \frac{(\alpha+k+1)(-x)^k y^{n-k} B(\alpha+k+2,\beta;p)}{\Gamma(\alpha+k+2)\ k!(n-k)!}, \end{aligned} \tag{27}$$

which after some simplifications and in view of definition (16) yields assertion (23) of Theorem 2.2.

To prove (24) and (25)), consider the following differential recurrence relations for $L_n^{(\alpha)}(x,y)$ [10]:

$$\frac{\partial}{\partial x}L_n^{(\alpha)}(x,y)=\frac{n}{x}L_n^{(\alpha)}(x,y)-\frac{(\alpha+n)\;y}{x}L_{n-1}^{(\alpha)}(x,y), \tag{28}$$

$$\frac{\partial}{\partial y}L_n^{(\alpha)}(x,y)=(\alpha+n)L_{n-1}^{(\alpha)}(x,y). \tag{29}$$

Replacing x by xt in relation (28) and multiplying both sides by $\frac{\Gamma(\alpha+\beta+1)(\beta+1)_n}{\Gamma(\beta)\Gamma(\alpha+1)\;n!}t^{\alpha}(1-t)^{\beta-1}\exp\left(\frac{-p}{t(1-t)}\right)$ and then integrating the resultant equation with respect to t between the limits 0 to 1 and taking in account the relation $\frac{\partial}{\partial(xt)}=\frac{\partial}{t\partial x}$, we get

$$\begin{aligned}&\frac{\Gamma(\alpha+\beta+1)(\beta+1)_n}{\Gamma(\beta)\Gamma(\alpha+1)\;n!}\frac{\partial}{\partial x}\int_0^1 t^{\alpha}(1-t)^{\beta-1}\exp\left(\frac{-p}{t(1-t)}\right)L_n^{(\alpha)}(xt,y)\\&=\frac{\Gamma(\alpha+\beta+1)(\beta+1)_n\;n}{\Gamma(\beta)\Gamma(\alpha+1)\;n!\;x}\int_0^1 t^{\alpha}(1-t)^{\beta-1}\exp\left(\frac{-p}{t(1-t)}\right)L_n^{(\alpha)}(xt,y)\\&\quad-\frac{y\;(\beta+n)(\alpha+n)\;\Gamma(\alpha+\beta+1)(\beta+1)_{n-1}}{nx\;\Gamma(\beta)\Gamma(\alpha+1)\;(n-1)!}\int_0^1 t^{\alpha}(1-t)^{\beta-1}\exp\left(\frac{-p}{t(1-t)}\right)L_{n-1}^{(\alpha)}(xt,y),\end{aligned} \tag{30}$$

which on using relation (20) yields assertion (24) of Theorem 2.2.

Similarly, from relation (29) and following the same procedure leading to prove (24), we get the desired result (25) and thus the proof of Theorem 2.2 is completed.

Remark 2.2. In view of results (24) and (25), we get the following differential equations for the E2VGLP $L_n^{(\alpha,\beta)}(x,y;p)$:

$$\left(\frac{\partial}{\partial x}+\frac{y}{x}\frac{\partial}{\partial y}-\frac{n}{x}\right)L_n^{(\alpha,\beta)}(x,y;p)=0, \tag{31}$$

Remark 2.3. For $\alpha=0$ in assertion (23) of Theorem 2.2, we get the following new result for the E2VALP $L_n^{(\beta)}(x,y;p)$:

$$\begin{aligned}L_n^{(\beta)}(x,y;p)&=\frac{\beta(\beta+1)}{(\beta+1)(\beta+n+1)}L_n^{(\beta+1)}(x,y;p)+\frac{1}{(\beta+1)(n+1)}L_n^{(1,\beta)}(x,y;p)\\&\quad+\frac{2n}{(\beta+1)(\beta+2)(n+1)(n+2)}L_{n-1}^{(2,\beta)}(x,y;p).\end{aligned} \tag{32}$$

Also, For $\alpha=0$ in results (24) and (25), we get the known results given in [2].

3. Generating functions

In this section, we prove some generating functions for the E2VGLP $L_n^{(\alpha,\beta)}(x,y;p)$ in the form of the following theorems:

Theorem 3.1. *The following generating function for the E2VGLP* $L_n^{(\alpha,\beta)}(x,y;p)$

holds ture:

$$\sum_{n=0}^{\infty} L_n^{(\alpha,\beta)}(x,y;p)\frac{(\lambda)_n\ n!\ u^n}{(\alpha+1)_n(\beta+1)_n}$$
$$=\frac{(1-yu)^{-\lambda}}{B(\alpha+1,\beta)}\sum_{k=0}^{\infty}\frac{(\lambda)_k\ B(\alpha+k+1,\beta;p)}{(\alpha+1)_k\ k!}\left(\frac{-xu}{1-yu}\right)^k, \qquad (33)$$

Proof. Using definition (16) in the L.H.S. of equation (33) and then putting $n = n + k$ in the resultant equation, we get

$$\sum_{n=0}^{\infty} L_n^{(\alpha,\beta)}(x,y;p)\frac{(\lambda)_n\ n!\ u^n}{(\alpha+1)_n(\beta+1)_n}$$
$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\lambda)_k\ B(\alpha+k+1,\beta;p)}{\Gamma(\alpha+k+1)}\frac{(-xt)^k}{k!}\sum_{n=0}^{\infty}(\lambda+k)_n\frac{(yt)^n}{n!}. \qquad (34)$$

Using the following relation [15]:

$$\sum_{n=0}^{\infty}(\alpha)_n\frac{(t)^n}{n!}=(1-t)^{-\alpha} \qquad (35)$$

and relation (2) in the R.H.S. of equation (34), we get assertion (33) of Theorem 3.1.

Remark 3.1. (i) For $\lambda = \alpha + 1$ in assertion (33) of Theorem 3.1 and in view of definition (4), we get the following elegant generating function:

$$\sum_{n=0}^{\infty} L_n^{(\alpha,\beta)}(x,y;p)\frac{n!\ u^n}{(\beta+1)_n}=(1-yu)^{-\alpha-1}\ \phi_p\left(\alpha+1;\alpha+\beta+1;\frac{-xu}{1-yu}\right), \qquad (36)$$

where $\phi_p(.)$ is the extended confluent hypergeometric function defined in (4).
(ii) For $\lambda = \beta+1$ in assertion (33) of Theorem 3.1, we get the following generating function:

$$\sum_{n=0}^{\infty} L_n^{(\alpha,\beta)}(x,y;p)\frac{n!\ u^n}{(\alpha+1)_n}$$
$$=\frac{(1-yu)^{-\beta-1}}{B(\alpha+1,\beta)}\sum_{k=0}^{\infty}\frac{(\beta+1)_k\ B(\alpha+k+1,\beta;p)}{(\alpha+1)_k\ k!}\left(\frac{-xu}{1-yu}\right)^k. \qquad (37)$$

(iii) For $\lambda = \alpha + \beta + 1$ in assertion (33) of Theorem 3.1 and in view of definition (7), we get the following elegant generating function:

$$\sum_{n=0}^{\infty} L_n^{(\alpha,\beta)}(x,y;p)\frac{(\alpha+\beta+1)_n\ n!\ u^n}{(\alpha+1)_n(\beta+1)_n}$$
$$=\Gamma(\alpha+1)\ (1-yu)^{-\alpha-\beta-1}\ E_{1,\alpha+1}^{(\alpha+1;\alpha+\beta+1)}\left(\frac{-xu}{1-yu};p\right), \qquad (38)$$

where $E_{\alpha,\beta}^{(\lambda;c)}(z)$ is the extended Mittag-Leffler function defined in (7).

Remark 3.2. For $\alpha = 0$ in (33), (36) and (38), we get the known results given in [2].

Remark 3.3. (i) putting $p = 0$ in assertions (33) of Theorem 3.1 and using relation (17), we get the following known generating function for the 2VGLP ${}_GL_n^{(\alpha,\beta)}(x, y)$ [3, p.154]:

$$\sum_{n=0}^{\infty} {}_GL_n^{(\alpha,\beta)}(x,y)\frac{(\lambda)_n\ n!\ u^n}{(\alpha+1)_n(\beta+1)_n} = (1-yu)^{-\lambda}\ {}_1F_1\left(\lambda;\alpha+\beta+1;\frac{-xu}{1-yu}\right). \quad (39)$$

(ii) Putting $p = 0$ and $\alpha = 0$ in assertions (33) of Theorem 3.1 and using relation (18b), we get the following known generating function for the 2VALP $L_n^{(\beta)}(x, y)$ [9, p.878]:

$$\sum_{n=0}^{\infty} L_n^{(\beta)}(x,y)\frac{(\lambda)_n\ u^n}{(\beta+1)_n} = (1-yu)^{-\lambda}\ {}_1F_1\left(\lambda;\beta+1;\frac{-xu}{1-yu}\right), \quad (40)$$

which for $\lambda = 1$ reduces to the following generating function given in [2]:

$$\sum_{n=0}^{\infty} L_n^{(\beta)}(x,y)\frac{n!\ u^n}{(\beta+1)_n} = (1-yu)^{-1}\ E_{1,\beta+1}\left(\frac{-xu}{1-yu}\right), \quad (41)$$

where $E_{\alpha,\beta}(z)$ is the generalized mittag-Leffler function defined in (8).
(iii) For $p = 0$, $\alpha = 0$ and $\lambda = \beta + 1$, relation (33)) reduces to relation (13a).

Remark 3.4. Proceeding on the same lines of proof of Theorem 3.1, we get the following result:

Theorem 3.2. *The following generating function for the E2VGLP $L_n^{(\alpha,\beta)}(x, y; p)$ holds ture:*

$$\sum_{n=0}^{\infty} L_n^{(\alpha,\beta)}(x,y;p)\frac{n!\ u^n}{(\alpha+1)_n(\beta+1)_n} = \frac{\exp(yu)}{B(\alpha+1,\beta)}\sum_{k=0}^{\infty}\frac{B(\alpha+k+1,\beta;p)(-xu)^k}{(\alpha+1)_k\ k!}. \quad (42)$$

Remark 3.5. (i) For $\alpha = 0$ relation (42) reduces to the known result given in [2].
(ii) For $p = 0$ and $\alpha = 0$, relation (42) reduces to relation (13b).

Theorem 3.3. *The following bilinear generating function for the E2VGLP $L_n^{(\alpha,\beta)}(x, y; p)$ holds ture:*

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(n!)^3\ u^n}{\Gamma(\alpha+n+1)[(\beta+1)_n]^2}\ L_n^{(\alpha,\beta)}(x,y;p)L_n^{(\alpha,\beta)}(z,w;p)\\ &=\frac{[\Gamma(\alpha+\beta+1)]^2}{[\Gamma(\alpha+1)]^2}(1-ywu)^{-\alpha-1}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k+1)}{[\Gamma(\alpha+\beta+k+1)]^2\ k!}\left(\frac{xzt}{(1-ywt)^2}\right)^k\\ &\times\phi_p\left(\alpha+k+1;\alpha+\beta+k+1;\frac{-xwu}{1-ywt}\right)\phi_p\left(\alpha+k+1;\alpha+\beta+k+1;\frac{-yzu}{1-ywt}\right).\end{aligned} \quad (43)$$

Proof. Consider the following bilinear generating function [7]:

$$\sum_{n=0}^{\infty} \frac{n!\ u^n}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(x,y) L_n^{(\alpha)}(z,w)$$
$$= (1-ywu)^{-\alpha-1} \exp\left(-\frac{xwu+yzu}{1-ywu}\right) C_\alpha\left(-\frac{xzu}{(1-ywu)^2}\right). \tag{44}$$

Using relation (1.14) to expand the Tricmoi function in the R.H.S. of equation (44) and then replacing x and z by xt and xv respectively in the resultant equation and multiplying both sides by $t^\alpha(1-t)^{\beta-1}\exp\left(\frac{-p}{t(1-t)}\right)\ v^\alpha(1-v)^{\beta-1}\exp\left(\frac{-p}{v(1-v)}\right)$ and integrating with respect to t and v between the limits 0 to 1, we get

$$\sum_{n=0}^{\infty} \frac{n!\ u^n}{\Gamma(\alpha+n+1)} \int_0^\infty t^\alpha(1-t)^{\beta-1}\exp\left(\frac{-p}{t(1-t)}\right) L_n^{(\alpha)}(xt,y)$$
$$\times \int_0^\infty v^\alpha(1-v)^{\beta-1}\exp\left(\frac{-p}{v(1-v)}\right) L_n^{(\alpha)}(zv,w)dtdv$$
$$= (1-ywu)^{-\alpha-1} \int_0^\infty t^\alpha(1-t)^{\beta-1}\exp\left(\frac{-p}{t(1-t)}\right) \int_0^\infty v^\alpha(1-v)^{\beta-1}\exp\left(\frac{-p}{v(1-v)}\right)$$
$$\exp\left(-\frac{xwut+yzuv}{1-ywt}\right) \sum_{k=0}^{\infty} \frac{t^k\ v^k}{\Gamma(\alpha+k+1)\ k!}\left(\frac{xzt}{(1-ywt)^2}\right)^k dtdv, \tag{45}$$

which on using integral (20) in the L.H.S. and the following integral [5]:

$$\frac{1}{B(b,c-b)} \int_0^\infty t^{b-1}(1-t)^{c-b-1}\exp\left(zt-\frac{p}{t(1-t)}\right) = \phi_p(b;c;z), \tag{46}$$

in the R.H.S yields the desired result.

Remark 3.6. For $\alpha=0$ in assertion (43) of Theorem 3.3 and using relation (18a), we get the following new bilinear generating function for the E2VALP $L_n^{(\beta)}(x,y;p)$:

$$\sum_{n=0}^{\infty} \frac{(n!)^2\ u^n}{[(\beta+1)_n]^2} L_n^{(\beta)}(x,y;p) L_n^{(\beta)}(z,w;p) = [\Gamma(\beta+1)]^2(1-ywu)^{-1}\sum_{k=0}^{\infty}\left(\frac{1}{[\Gamma(\beta+k+1)]^2}\right.$$
$$\left.\times\left(\frac{xzt}{(1-ywt)^2}\right)^k \phi_p\left(k+1;\beta+k+1;\frac{-xwu}{1-ywt}\right)\phi_p\left(k+1;\beta+k+1;\frac{-yzu}{1-ywt}\right)\right). \tag{47}$$

Remark 3.7. For $\alpha=0$ in assertion (43) of Theorem 3.3 and using relation (17), we get the following new bilinear generating function for the 2VGLP ${}_GL_n^{(\alpha,\beta)}(x,y)$:

$$\sum_{n=0}^{\infty}\frac{(n!)^3\ u^n}{\Gamma(\alpha+n+1)[(\beta+1)_n]^2}\ {}_GL_n^{(\alpha,\beta)}(x,y){}_GL_n^{(\alpha,\beta)}(z,w)$$
$$=\frac{[\Gamma(\alpha+\beta+1)]^2}{[\Gamma(\alpha+1)]^2}(1-ywu)^{-\alpha-1}\sum_{k=0}^{\infty}\left(\frac{\Gamma(\alpha+k+1)}{[\Gamma(\alpha+\beta+k+1)]^2\ k!}\left(\frac{xzt}{(1-ywt)^2}\right)^k\right.$$
$$\left.\times\,{}_1F_1\left(\alpha+k+1;\alpha+\beta+k+1;\frac{-xwu}{1-ywt}\right){}_1F_1\left(\alpha+k+1;\alpha+\beta+k+1;\frac{-yzu}{1-ywt}\right)\right), \tag{48}$$

which for $\beta=0$ reduces to relation (44).

4. Integral formulae

We prove the following results:

Theorem 4.1. *The following integral involving the E2VGLP $L_n^{(\alpha,\beta)}(x,y;p)$ holds ture:*

$$\frac{n!}{(\alpha+1)_n}\int_0^{\infty}L_n^{(\alpha,\beta)}(-x,y;p)\ \exp(-sx)\ x^{\alpha}dx$$
$$=\frac{\Gamma(\alpha+1)(\beta+1)_n\ y^n}{s^{\alpha+1}}F_p\left(\beta+1+n,\alpha+1;\alpha+\beta+1;\frac{1}{s}\right), \tag{49}$$

where $F_p(.)$ is the extended Hypergeometric function defined in (3).

Proof. Using definition (16) in the L.H.S. of (49) and interchanging the order of integration and summation, we get

$$\frac{n!}{(\alpha+1)_n}\int_0^{\infty}L_n^{(\alpha,\beta)}(-x,y;p)\ \exp(-sx)\ x^{\alpha}dx=\frac{\Gamma(\alpha+\beta+1)(\beta+1)_n}{\Gamma(\beta)}$$
$$\times\sum_{k=0}^{n}\frac{y^{n-k}B(\alpha+k+1,\beta;p)}{\Gamma(\alpha+k+1)\ k!(n-k)!}\int_0^{\infty}\exp(-sx)x^{\alpha+k}dx. \tag{50}$$

Using the following relation [15]:

$$\int_0^{\infty}\exp(-st)\ t^{\alpha-1}dt=\frac{\Gamma(\alpha)}{s^{\alpha}}, \tag{51}$$

in the R.H.S. of equation (50) and then putting $n=n+k$ and using relation (2), we get

$$\frac{n!}{(\alpha+1)_n}\int_0^{\infty}L_n^{(\alpha,\beta)}(-x,y;p)\ \exp(-sx)\ x^{\alpha}dx=\frac{\Gamma(\alpha+1)(\beta+1)_n\ y^n}{s^{\alpha+1}}$$
$$\times\sum_{k=0}^{n}\frac{(\beta+1+n)_kB(\alpha+1+k,\beta;p)}{B(\alpha+1,\beta)\ k!}\left(\frac{1}{s}\right)^k, \tag{52}$$

which in view of definition (3), we get assertion (49) of Theorem 4.1.

Remark 4.1. For $s = 1$ in assertions (49) of Theorem 4.1, we get the following result:

$$\frac{n!}{(\alpha+1)_n}\int_0^\infty L_n^{(\alpha,\beta)}(-x,y;p)\ \exp(-x)\ x^\alpha dx$$
$$= \Gamma(\alpha+1)(\beta+1)_n\ y^n\ F_p(\beta+1+n,\alpha+1;\alpha+\beta+1;1), \tag{53}$$

which on using the following relation [16, p.485]:

$$F_p(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\ {}_2\Psi_1\left[\begin{matrix}(b,-1),(c-b-a,-1); \\ (c-a,-2);\end{matrix}\ -p\right], \tag{54}$$

in the R.H.S. gives

$$\frac{n!}{(\alpha+1)_n}\int_0^\infty L_n^{(\alpha,\beta)}(-x,y;p)\ \exp(-x)\ x^\alpha dx$$
$$= \frac{\Gamma(\alpha+\beta+1)(\beta+1)_n\ y^n}{\Gamma(\beta)}\ {}_2\Psi_1\left[\begin{matrix}(\alpha+1,-1),(-n-1,-1); \\ (\alpha-n,-2);\end{matrix}\ -p\right], \tag{55}$$

where ${}_2\Psi_1(.)$ is the Wright function given in [15].

Remark 4.2. For $p = 0$ in assertion (49) of Theorem 4.1 and using (18a) and (6a), we get the following new result for the 2VGLP ${}_GL_n^{(\alpha,\beta)}(x,y)$:

$$\frac{n!}{(\alpha+1)_n}\int_0^\infty {}_GL_n^{(\alpha,\beta)}(-x,y)\ \exp(-sx)\ x^\alpha dx$$
$$= \frac{\Gamma(\alpha+1)(\beta+1)_n\ y^n}{s^{\alpha+1}}{}_2F_1\left(\beta+1+n,\alpha+1;\alpha+\beta+1;\frac{1}{s}\right), \tag{56}$$

which on putting $s = 1$ in the R.H.S. and using the following relation [14]:

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}, \tag{57}$$

yields the following result for the 2VGLP ${}_GL_n^{(\alpha,\beta)}(x,y)$:

$$\frac{n!}{(\alpha+1)_n}\int_0^\infty {}_GL_n^{(\alpha,\beta)}(-x,y)\ \exp(-x)\ x^\alpha dx$$
$$= \frac{y^n\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(-n-1)(\beta+1)_n}{\Gamma(\alpha-n)\Gamma(\beta)}. \tag{58}$$

Theorem 4.2. *The following integral involving the E2VGLP $L_n^{(\alpha,\beta)}(x,y;p)$ holds ture:*

$$\frac{n!}{(\alpha+1)_n(\beta+1)_n}\int_0^\infty\int_0^\infty L_n^{(\alpha,\beta)}(x,y;p)\ \exp(-rx-sy)\ x^\alpha dxdy$$
$$= \frac{\Gamma(\alpha+1)}{r^{\alpha+1}s^{n+1}}\Phi_p\left(\alpha+1;\alpha+\beta+1;-\frac{s}{r}\right). \tag{59}$$

Proof. Using definition (16) in the L.H.S. of (59) and interchanging the order of integration and summation, we get

$$\frac{n!}{(\alpha+1)_n(\beta+1)_n}\int_0^\infty\int_0^\infty L_n^{(\alpha,\beta)}(x,y;p)\ \exp(-rx-sy)\ x^\alpha dxdy=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta)}$$
$$\times\sum_{k=0}^n\frac{(-1)^kB(\alpha+k+1,\beta;p)}{\Gamma(\alpha+k+1)\ k!(n-k)!}\int_0^\infty\exp(-rx)x^{\alpha+k}dx\int_0^\infty\exp(-sy)y^{n-k}dy,\quad(60)$$

which on using relation (51) in the R.H.S. and then in view of definition (4) yields assertion (59) of Theorem 4.2.

Remark 4.3. For $p=0$ in assertion (59) of Theorem 4.2 and using relations (17) and (6b), we get the following new result for the 2VGLP ${}_GL_n^{(\alpha,\beta)}(x,y)$:

$$\frac{n!}{(\alpha+1)_n(\beta+1)_n}\int_0^\infty\int_0^\infty {}_GL_n^{(\alpha,\beta)}(x,y)\ \exp(-rx-sy)\ x^\alpha dxdy$$
$$=\frac{\Gamma(\alpha+1)}{r^{\alpha+1}s^{n+1}}\ {}_1F_1\left(\alpha+1;\alpha+\beta+1;-\frac{s}{r}\right).\quad(61)$$

Theorem 4.3. *The following integral involving the E2VGLP $L_n^{(\alpha,\beta)}(x,y;p)$ holds ture:*

$$\int_0^\infty L_n^{(\alpha,\beta)}(x,y;p)\ dp=\frac{\beta(\beta+1)(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta+2)(\alpha+n+1)}$$
$$\times\left(\frac{(\alpha+1)\ {}_GL_n^{(\alpha+1,\beta+1)}(x,y)}{(\beta+n+1)}-\frac{x\ (\alpha+2)\ {}_GL_{n-1}^{(\alpha+2,\beta+1)}(x,y)}{n\ (\alpha+\beta+3)}\right).\quad(62)$$

Proof. Using definition (20) in the L.H.S. of equation (62), we get

$$\int_0^\infty L_n^{(\alpha,\beta)}(x,y;p)\ dp=\frac{\Gamma(\alpha+\beta+1)(\alpha+1)_n(\beta+1)_n}{\Gamma(\beta)\ n!}\sum_{k=0}^n\frac{(-x)^ky^{n-k}}{\Gamma(\alpha+k+1)\ k!(n-k)!}$$
$$\times\int_0^\infty B(\alpha+k+1,\beta;p)\ dp.\quad(63)$$

Using the following relations [4]:

$$\int_0^\infty B(\alpha,\beta;p)\ dp=B(\alpha+1,\beta+1),\quad(64)$$

in the R.H.S. of equation (63) and then using relation (2) in the resultant equation, we get

$$\int_0^\infty L_n^{(\alpha,\beta)}(x,y;p)\ dp$$
$$=\frac{\Gamma(\alpha+\beta+1)(\alpha+1)_n(\beta+1)_n}{\Gamma(\beta)\ n!}\sum_{k=0}^n\frac{(\alpha+k+1)\Gamma(\beta+1)(-x)^ky^{n-k}}{\Gamma(\alpha+\beta+k+3)\ k!(n-k)!},\quad(65)$$

which after some simplification by using Pochhammers properties and in view of definition (9) yields assertion (62) of Theorem 4.3.

Remark 4.4. For $\alpha = 0$ in assertion (62) of Theorem 4.3, we get the following new result for the E2VALP $L_n^{(\beta)}(x, y; p)$:

$$\int_0^\infty L_n^{(\beta)}(x, y; p)\ dp = \frac{\beta}{(\beta+2)(n+1)} \left(\frac{{}_G L_n^{(1,\beta+1)}(x,y)}{(\beta+n+1)} - \frac{2x\ {}_G L_{n-1}^{(2,\beta+1)}(x,y)}{n\ (\beta+3)} \right). \tag{66}$$

Note that, for $\alpha = 0$ in the above main results in this section, we get the known results given in [2].

5. Summation formulae

First, we prove the following summation formulae by using integral representation (20):

Theorem 5.1. *The following summation formulae for the E2VGLP $L_n^{(\alpha,\beta)}(x, y; p)$ hold ture:*

$$L_n^{(\alpha,\beta)}\left(\frac{x}{1-uy^{-1}}, y; p\right) = (1-uy^{-1})^{-n} \sum_{k=0}^{n} L_{n-k}^{(\alpha,\beta)}(x, y; p) \frac{(-\alpha-n)_k(-\beta-n)_k\ u^k}{(-n)_k\ k!}, \tag{67}$$

$$L_n^{(\alpha,\beta)}(xy, z; p) = \sum_{k=0}^{n} \frac{(\alpha+1)n(\beta+1)_n[z(1-y)]^{n-k} y^k r!}{(\alpha+1)_k(\beta+1)_k(n-k)!n!} L_k^{(\alpha,\beta)}(x, z; p). \tag{68}$$

Proof. To prove (67), consider the following relation [11]:

$$(1-uy^{-1})^n\ L_n^{(\alpha)}\left(\frac{x}{1-uy^{-1}}, y\right) = \sum_{k=0}^{\infty} L_{n-k}^{(\alpha)}(x, y) \frac{(-\alpha-n)_k\ u^k}{k!}. \tag{69}$$

Replacing x by xt in relation (69) and multiplying both sides by $t^\alpha(1-t)^{\beta-1}\exp\left(\frac{-p}{t(1-t)}\right)$ and integrating the resultant equation with respect to t between the limits 0 to 1, we get

$$\begin{aligned}(1-uy^{-1})^n &\int_0^1 t^\alpha(1-t)^{\beta-1}\exp\left(\frac{-p}{t(1-t)}\right) L_n^{(\alpha)}\left(\frac{xt}{1-uy^{-1}}, y\right) dt \\ &= \sum_{k=0}^{\infty} \frac{(-\alpha-n)_k\ u^k}{k!} \int_0^1 t^\alpha(1-t)^{\beta-1}\exp\left(\frac{-p}{t(1-t)}\right) L_{n-k}^{(\alpha)}(xt, y)dt,\end{aligned} \tag{70}$$

which in view of relation (20) and after some simplifications yields assertion (67) of Theorem 5.1.

Similarly, proceeding on the same lines of proof of result (67) and using the following relation [9, p.879]:

$$L_n^{(\alpha)}(xy,z)=\sum_{k=0}^{n}\frac{(\alpha+1)_n[z(1-y)]^{n-k}y^k}{(\alpha+1)_k(n-k)!}L_k^{(\alpha)}(x,z), \tag{71}$$

we get assertion (68) of Theorem 5.1, thus the proof of Theorem 5.1 is completed.

Remark 5.1. For $p=0$ in assertions (67) and (68) of Theorem 5.1 and using relation (17), we get the following new summation formulae for the 2VGLP ${}_GL_n^{(\alpha,\beta)}(x,y)$:

$${}_GL_n^{(\alpha,\beta)}\left(\frac{x}{1-uy^{-1}},y\right)=(1-uy^{-1})^{-n}\sum_{k=0}^{n}{}_GL_{n-k}^{(\alpha,\beta)}(x,y)\frac{(-\alpha-n)_k(-\beta-n)_k\ u^k}{(-n)_k\ k!}, \tag{72}$$

$${}_GL_n^{(\alpha,\beta)}(xz,y)=\sum_{k=0}^{n}\frac{(\alpha+1)n(\beta+1)_n[z(1-y)]^{n-k}y^kr!}{(\alpha+1)_k(\beta+1)_k(n-k)!n!}{}_GL_k^{(\alpha,\beta)}(x,z). \tag{73}$$

Next, we prove the following summation formulae by using generating function (36):

Theorem 5.2. *The following summation formula for the E2VGLP* $L_n^{(\alpha,\beta)}(x,y;p)$ *hold ture:*

$$L_n^{(\alpha,\beta)}(x,y;p)=\sum_{k=0}^{n}\frac{(\beta+1)_n\ k!}{(\alpha+2)_k\ n!}L_{n-k}^{(\alpha-\beta)}(x,y)L_k^{(\beta-1,\alpha+1)}(-x,y;p). \tag{74}$$

Proof. Using the following relation [5]:

$$\Phi_p(b;c;z)=\exp(z)\ \Phi_p(c-b;c;-z), \tag{75}$$

in the R.H.S. of relation (36), we obtain

$$\sum_{n=0}^{\infty}L_n^{(\alpha,\beta)}(x,y;p)\frac{n!\ u^n}{(\beta+1)_n}$$
$$=(1-yu)^{-(\alpha-\beta)-1}\ \exp\left(\frac{-xu}{1-yu}\right)(1-yu)^{-\beta}\phi_p\left(\beta;\alpha+\beta+1;\frac{xu}{1-yu}\right). \tag{76}$$

Using relations (13a) and (36) in the R.H.S. of the above equation and putting $n=n-k$ and then equating the coefficients of u^n, we get the desired result.

Remark 5.2. For $p=0$ in assertion (74) of Theorem 5.2 and using relation (17), we get the following new summation formula for the 2VGLP ${}_GL_n^{(\alpha,\beta)}(x,y)$:

$${}_GL_n^{(\alpha,\beta)}(x,y)=\sum_{k=0}^{n}\frac{(\beta+1)_n\ k!}{(\alpha+2)_k\ n!}L_{n-k}^{(\alpha-\beta)}(x,y)\ {}_GL_k^{(\beta-1,\alpha+1)}(-x,y). \tag{77}$$

Note that, for $\alpha=0$ in the above main results in this section, we get the known results given in [2].

6. Conclusion

By using the extended Beta function, we have extended the two variable generalized Laguerre polynomials. We have investigated some properties of these extended polynomials, most of which are analogous with the original polynomials. The approach presented in this paper is general and can be extended to introduce other new families of special polynomials. Therefore, the corresponding extensions of several other familiar special polynomials are expected to be investigated.

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