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Solving Differential Equations by Using a Combination of the First Kind Chebyshev Polynomials and Adomian Decomposition Method

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Abstract. In this paper, we are going to solve a class of ordinary differential equations that its source term are rational functions. We obtain the best approximation of source term by Chebyshev polynomials of the first kind, then we solve the ordinary differential equations by using the Adomian decomposition method.

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1. Introduction

In this paper, we consider the ordinary differential equation

$$Ly + Ry + Ny = g; \tag{1}$$

where L is the highest order derivative which is assumed to be easily invertible, R is a linear differential operator of lesser order than L, Ny represents the nonlinear terms, and g is the source term.

Our main work, the solution of differential equations with the source term of the rational functions. In recent years, people like Wazwaz [12] ,Burden [3] and Atkinson [8] worked on this differential equations.

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Best polynomial approximation problem is one of the most important and applicable subjects in the approximation theory. It holds particular importance for topics such as partial differential equations, ordinary differential equations, integral equations, integro-differential equations, etc. Therefore researchers in this field obtain the characterization of the best uniform polynomial approximation for special classes of functions[1, 2, 6, 10]. Furthermore a lot of these researches were focused on classes of functions possessing a certain expansion by Chebyshev polynomials. Some of these researches are cited in Table 1. Our main goal, the determination of the best approximation of rational function g and then solve the differential equation by ADM.

This paper is organized as follows. In section 2, the basic definitions of Chebyshev polynomials of the first kind and ADM Presented. In section 3, we explain the best approximation Chebyshev polynomials in solving IVPs. In section 4, some numerical examples are solved to illustrate the importance of using combination of the first kind Chebyshev Polynomials and ADM.

Table 1. Best uniform approximation.

Class of functions	Reference
$\frac{1}{x-a};\ a>1$	[4]
$\frac{1}{1+a^2x^2}$	[6]
$\frac{1}{a^2 \pm x^2};\ a^2>1$	[3]
$\frac{1}{T_q(a) \pm T_q(x)};\ a^2>1$	[2]

2. Preliminaries

2.1 *Chebyshev polynomials*

The Chebyshev polynomial in $[-1,1]$ of degree n is denoted by T_n and is defined by [11]

$$T_n(x) = \cos(n\theta),$$

where $x = \cos\theta$.
$T_n(x)$ is a polynomial of degree n with leading coefficient 2^{n-1}. Furthermore the Chebyshev polynomials satisfy in the following recursive relation :

$$\begin{aligned} &T_0(x) = 1, \\ &T_1(x) = x, \\ &T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x). \end{aligned} \tag{2}$$

2.2 *Adomian decomposition method*

we can consider a differential equation as [12]

$$Ly + Ry + Ny = g, \tag{3}$$

where L is the highest order derivative which is assumed to be easily invertible, R is a linear differential operator of lesser order than L, Ny represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of (5), and using the given conditions we obtain

$$y = f - L^{-1}(Ry) - L^{-1}(Ny), \tag{4}$$

where the function$f$ represents the terms arising from integrating the source term g and from using the given conditions, all are assumed to be prescribed. The standard Adomian method defines the solution $y(x)$ by the series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{5}$$

where the components $y_0, y_1, y_2, ...$ are usually determined recursively by using the relation

$$y_0 = f,$$

$$y_{n+1} = -L^{-1}(Ry_n) - L^{-1}(Ny_n), \tag{6}$$

and the nonlinear term Ny can be decomposed by an infinite series of polynomials given by $Ny = \sum_{n=0}^{\infty} A_n$. The standard Adomian method defines the solution $y(x)$ by the series

$$y(x) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{n=0}^{\infty} \lambda^i y_i\right)\right]_{\lambda=0}, n = 0, 1, 2, ... \tag{7}$$

Having determined the components $y_0, y_1, y_2, ...$ the solution y in a series form defined by (7) follows immediately. Based on the ADM, we constructed the solution y as:

$$y = \lim_{m\to\infty} \varphi_m(x),$$

where

$$\varphi_m(x) = \sum_{n=0}^{m} y_n(x), \qquad m \geqslant 0. \tag{8}$$

3. The use of the best approximation in solving IVPs

Consider the ordinary differential equation

$$Ly + Ry + Ny = \frac{1}{T_q(a) \pm T_q(x)}, \tag{9}$$

where L is the highest order derivative which is assumed to be easily invertible, Ny represents the nonlinear terms; $\frac{1}{T_q(a) \pm T_q(x)}$ is the source term; $T_q(x)$ is Chebyshev polynomials of the first kind and a is a constant.

To solving the differential equation (9), first we determine the best approximation of the Chebyshev polynomials of the first kind of the rational functions $\dfrac{1}{T_q(a) \pm T_q(x)}$.
Then, by using the ADM, we find the components of the solution series.
The use of the best approximation in solving these equations, accuracy of solutions improve, it also expands convergence interval.

3.1 *Best approximation of some rational functions*

To determination the best approximation of a rational function, we present some concepts.
Definition 3.1.1.[4] For a given $f \in C[a,b]$ let Π_n be the set of all polynomials of at most n, there exists a unique polynomial $p_n^* \in \Pi_n$ such that:

$$\|f - p_n^*\|_m \leqslant \|f - p\|_m, \ \forall p \in \Pi_n.$$

we call p_n^* the best L_m polynomial approximation out of Π_n to f on $[a,b]$.

Lemma 3.1.1. For $x = \cos\theta, |t| < 1$ and p (natural number), we have:

$$(a) \ \sum_{j=0}^{\infty} t^{pj} T_{pj}(x) = \frac{1 - t^p \cos(p\theta)}{1 + t^{2p} - 2t^p \cos(p\theta)}, \tag{10}$$

$$(b) \ \sum_{n=0}^{n-1} t^{pj} T_{pj}(x) = \frac{1 - t^p \cos(p\theta) - t^{np}\cos(pn\theta) + t^{pn+p}\cos(p\theta)\cos(pn\theta) + t^{pn+p}\sin(p\theta)\sin(pn\theta)}{1 + t^{2p} - 2t^p\cos(p\theta)}. \tag{11}$$

proof. See[2].

Lemma 3.1.2. For $x \in [-1,1]$, $q > 0$ (q is integer number) and $a^2 > 1$ we have:

$$\frac{1}{T_q(a) + T_q(x)} = \frac{2t^q}{t^{2q} - 1} - \frac{4t^q}{t^{2q} - 1} \sum_{k=0}^{\infty} (-1)^k t^{qk} T_{qk}(x), \tag{12}$$

where t satisfies : $t^q = T_q(a) - \sqrt{T_q^2(a) - 1}$.
See[3].

Theorem 3.1.1. The best uniform polynomial approximation out of Π_{qn} to $f(x) = 1/(T_q(a) + T_q(x))$ on $[-1,1]$ is,

$$p_{qn}^*(x) = \frac{2t^q}{t^{2q} - 1} - \frac{4t^q}{t^{2q} - 1} \sum_{k=0}^{n-1} (-1)^k t^{qk} T_{qk}(x) + \frac{4(-1)^n t^{qn+q}}{(t^{2q} - 1)^2} T_{qn}(x), \tag{13}$$

where : $t^q = T_q(a) - \sqrt{T_q^2(a) - 1}$.
proof. See [4].

4. Examples

For comparing the best approximation method of the rational functions, in solving ordinary differential equations with previous methods in [8], we consider the following example.

Example 1.[12] Consider the first order ordinary differential equation

$$y' + y = \frac{1}{x^2+1}, y(0) = 0. \tag{14}$$

It is mentioned in [8] that the solution of the first order linear equation (14) cannot be found in a closed form.

The solution of equation (14) by ADM, is approximated by Wazwaz[8] with three components as follows:

$$\varphi_3(x) = \sum_{n=0}^{2} y_n(x) = \arctan x - x\arctan x + \frac{1}{2}\ln(1+x^2) + \frac{1}{2}(x^2-1)\arctan x + \frac{1}{2}x - x\ln(1+x^2).$$

Also the approximate solution of the Taylor Series Method is as follows [12]

$$y(x) = x - \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{23}{5!}x^5 - \frac{23}{6!}x^6 + \dots.$$

To compare the solutions of these two methods, it is better to calculate $y' + y$ term using, then we compare them with $\frac{1}{x^2+1}$. This comparison can be seen in Figure 1.

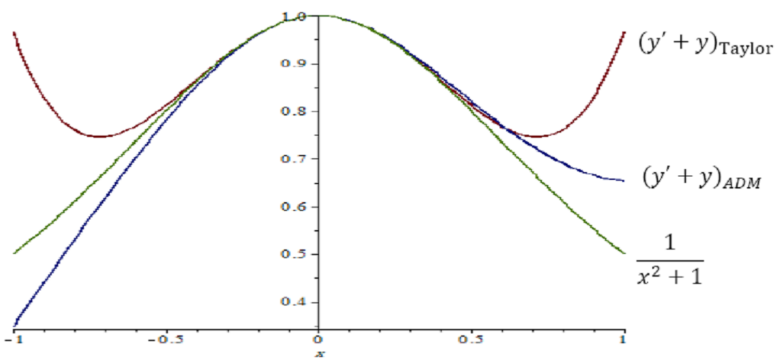

Figure 1. Comparison of two methods.

4.1 *The new method*

Eq. (14) can be re-written in an operator form as

$$Ly = \frac{1}{1+x^2} - y, \tag{15}$$

where $L = \dfrac{d}{dx}$ is the differential operator.
The inverse operator L^{-1} is an integral operator given by $L^{-1}(.) = \int_0^x(.)$.
Applying L^{-1} on Eq. (15) yields

$$L^{-1}L(y) = L^{-1}\left(\frac{1}{1+x^2} - y\right).$$

Using the initial condition we have:

$$y(x) = y(0) + L^{-1}\left(\frac{1}{1+x^2}\right) - L^{-1}(y) = L^{-1}\left(\frac{1}{1+x^2}\right) - L^{-1}(y).$$

Here, to find the best approximation of the function of $\dfrac{1}{1+x^2}$ according to (13) we have,

$$q = 2, T_2(x) = 2x^2 - 1,$$

i.e.

$$\frac{1}{1+x^2} = \frac{2}{3+(2x^2-1)},$$

in this case,

$$T_q(a) = 3, t^2 = 3 - 2\sqrt{2}.$$

Assuming $n = 2$, the best uniform polynomial approximation out of P_4 to $f(x) = \dfrac{1}{1+x^2}$ on $[-1, 1]$ is,

$$p_4^*(x) = \frac{2t^2}{t^4-1} - \frac{4t^2}{t^4-1}\sum_{k=0}^{1}(-1)^k t^{2k} T_{2k}(x) + \frac{4(-1)^2 t^6}{(t^4-1)^2}T_4(x), \tag{16}$$

This is equivalent to,

$$p_4^*(x) = \frac{4b^2}{b^4-1} - \frac{8b^2(1-b^2(2x^2-1))}{b^4-1} + \frac{8b^6(8x^4-8x^2+1)}{(b^4-1)^2},$$

where $b = 1 - \sqrt{2}$.
For detemine solution components by using the decomposition method, have

$$\sum_{i=0}^{\infty} y_i(x) = \int_0^x p_4^*(x)dx - \int_0^x \left(\sum_{i=0}^{\infty} y_i(x)\right) dx. \tag{17}$$

To find the components of the solution, we write the following recursive relation

$$y_0(x) = \int_0^x p_4^*(x)dx = 0.9926406850x - 0.2761423736x^3 + 0.06862914972x^5,$$

$$y_{j+1}(x) = -\int_0^x y_j(x)dx, \qquad j = 0, 1, 2, ...$$

Approximate solution with respect to $\varphi_3(x) = \sum_{i=0}^{3} y_i(x)$, is equal to

$$y(x) = 0.9926406850x - 0.1107022594x^3 + 0.05482203104x^5 - 0.4963203425x^2 + 0.02767556485x^4 - 0.009137005173x^6 + 0.001634027374x^7 - 0.0002042534218x^8$$

The calculations were performed with the software Maple 17.
We compare $y' + y$ with the $\dfrac{1}{x^2+1}$. This comparison can be seen in Figure 2.
According to Figure 1, Figure 2, we see accuracy of answers and the extent of

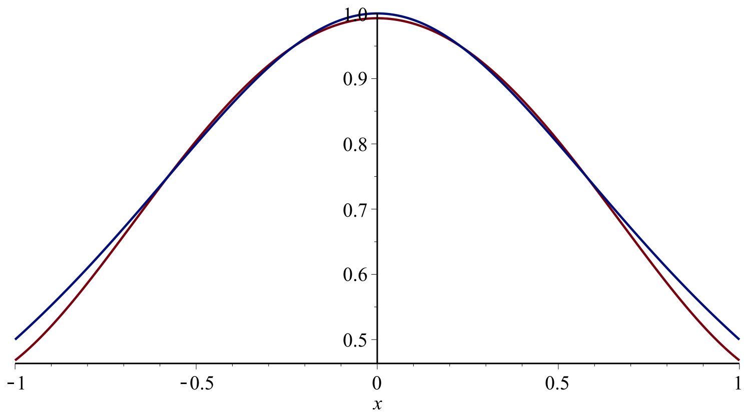

Figure 2. Comparison of $y' + y$ with $\dfrac{1}{x^2+1}$.

convergence interval.

Example 2.[8] Consider the first order ordinary differential equation

$$y' + 2y^2 = \frac{1}{x^2+1}, \qquad y(0) = 0, \tag{18}$$

where exact solution $Y(x) = \dfrac{x}{1+x^2}$.
Eq. (18) can be re-written in an operator form as

$$Ly = \frac{1}{1+x^2} - 2y^2, \tag{19}$$

where $L = \dfrac{d}{dx}$ is the differential operator.
The inverse operator L^{-1} is an integral operator given by $L^{-1}(.) = \int_0^x (.)$.
Applying L^{-1} on Eq. (19) yields

$$L^{-1}L(y) = L^{-1}\left(\frac{1}{1+x^2} - 2y^2\right).$$

Using the initial condition we have:

$$y(x) = y(0) + L^{-1}\left(\frac{1}{1+x^2}\right) - L^{-1}(2y^2) = L^{-1}\left(\frac{1}{1+x^2}\right) - 2L^{-1}(y^2).$$

According to (16), we can write the best approximation of degree four of the function $\frac{1}{x^2+1}$.

Using the decomposition method, for solution components , we have

$$\sum_{i=0}^{\infty} y_i(x) = \int_0^x p_4^*(x)dx - 2\int_0^x \left(\sum_{k=0}^{\infty} A_k\right) dx. \quad (20)$$

where A_k's are the Adomian polynomials that represent the nonlinear term y^2. Given a nonlinear term y^2, the first few polynomials are given by

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_0y_1, \\ A_2 &= 2y_0y_2 + y_1^2, \\ &\vdots \end{aligned}$$

To find the components of the solution, we write the following recursive relation

$$y_0(x) = \int_0^x p_4^* dx = 0.9926406850x - 0.2761423736x^3 + 0.06862914972x^5,$$

$$y_{j+1}(x) = -2\int_0^x A_j(x)dx, j = 0, 1, 2, ...$$

Approximate solution with respect to $\varphi_3(x) = \sum_{i=0}^{3} y_i(x)$, is equal to

$$\begin{aligned} y(x) = {} & 0.9926406850x - 0.9330327266x^3 + 0.8095621457x^5 - 0.1012445533x^{11} + \\ & 0.3108063520x^9 - 0.7079310612x^7 + 0.0006117887001x^{17} - 0.004374581536x^{15} + \\ & 0.02360108613x^{13} - 2.288190442 \times 10^{-7}x^{23} + 0.000004940643894x^{21} \\ & -0.00006657432866x^{19}. \end{aligned}$$

The calculations were performed with the software Maple 17.
For comparison between exact solution and approximate solution see Figure 3.
According to Figure 3, we see accuracy of answers and the extent of convergence interval.

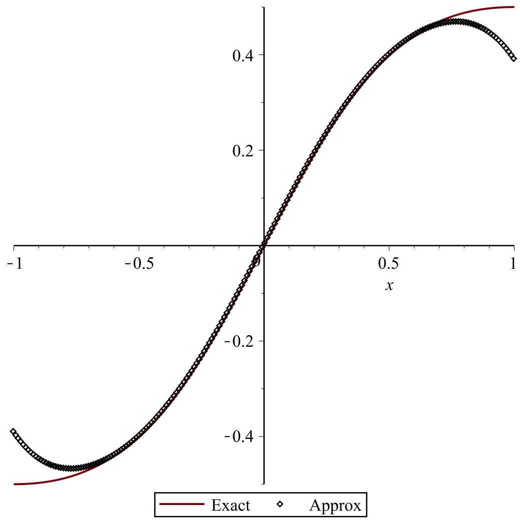

Figure 3. Comparison of $y' + y$ with $\frac{1}{x^2+1}$.

Example 3. [3] Consider the first order ordinary differential equation

$$y' = \frac{y^2}{x+2}, \qquad y(0) = \frac{-1}{\ln 2}. \tag{21}$$

Where exact solution $\frac{-1}{\ln(x+2)}$.
We first write Eq. (21) in an operator form

$$L = \frac{y^2}{x+2}, \tag{22}$$

where $L = \frac{d}{dx}$ is the differential operator.
The inverse operator L^{-1} is an integral operator given by $L^{-1}(.) = \int_0^x(.)$.
Applying L^{-1} on Eq. (22) yields

$$L^{-1}L(x) = L^{-1}\left(\frac{y^2}{x+2}\right).$$

Using the initial condition we have:

$$y(x) = y(0) + L^{-1}\left(\frac{y^2}{x+2}\right) = \frac{-1}{ln2} + L^{-1}\left(\frac{y^2}{x+2}\right).$$

Here, to find the best approximation of the function of $\frac{1}{x+2}$ according to (13) we have,

$$q = 1, T_1(x) = x, T_q(a) = 2,$$

i.e.

$$\frac{1}{T_q(x)+T_q(a)}=\frac{1}{x+2}$$

Assuming $n = 4$, the best uniform polynomial approximation out of P_4 to $f(x)=\frac{1}{x+2}$ on $[-1,1]$ is,

$$p_4^*(x)=\frac{2t}{t^2-1}-\frac{4t}{t^2-1}\sum_{k=0}^{3}(-1)^k t^k T_k(x)+\frac{4t^5}{(t^2-1)^2}T_4(x), \tag{23}$$

This is equivalent to,

$$p_4^*(x)=0.5008591278-0.2427590822x+0.1145065038x^2-0.08885599068x^3 +0.05130103237x^4.$$

Using the decomposition method, for solution components , we have

$$\sum_{i=0}^{\infty}y_i(x)=\frac{-1}{\ln 2}+\int_0^x p_4^*(x)\left(\sum_{k=0}^{\infty}A_k\right)dx. \tag{24}$$

where A_k's are the Adomian polynomials that represent the nonlinear term y^2.

To find the components of the solution, we write the following recursive relation

$$y_0(x)=\frac{-1}{\ln 2},$$

$$y_{j+1}(x)=\int_0^x p_4^*(x)A_j(x)dx, j=0,1,2,....$$

Approximate solution with respect to $\varphi_3(x)=\sum_{i=0}^{3}y_i(x)$.

The calculations were performed with the software Maple 17.
For comparison between exact solution and approximate solution see Figure 4.

According to Figure 4, we see accuracy of answers and the extent of convergence interval.

5. Conclusions

In this paper, we studied some differential equations, which includes a class of the rational functions. First, we obtained the best approximation of the first kind Chebyshev Polynomials of the rational function, then by using ADM we obtained

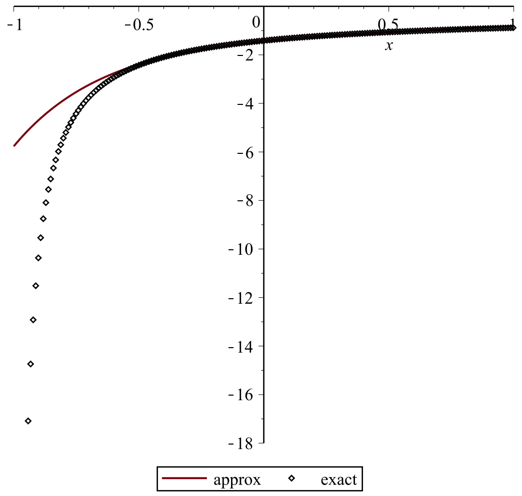

Figure 4. Comparison of our method and exact solution for Example 3.

the components of the solution. In this way, we easily obtained the components of the solution and developed the convergence region.

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