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Using Reproducing Kernel for Solving the Lane-Emden Equations, Arising in Astrophysics

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Abstract. In this paper, we utilize an algorithm for solving nonlinear ordinary differential equations of Lane-Emden type on a semi-infinite interval. The algorithm is based on an iterative technique and the reproducing kernel Hibert method. We give the convergence analysis for the proposed method. The validity and applicability of the proposed method are demonstrated by some numerical examples. The obtained results and comparison with other methods provide confirmation for the validity of our numerical method.

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1. Introduction

1.1 *Lane-Emden equation and engineering applications*

Many problems in the literature of mathematical physics which occur on semi-infinite interval are related to the diffusion of heat perpendicular to the parallel planes. This problem can be distinctively are modeled by the heat equation

$$x^{-2}\frac{d(x^2\frac{dy}{dx})}{dx}+\kappa f(x)g(y)=V(x), x\geqslant 0, \tag{1}$$

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where $f(x)$ and $g(y)$ are some given functions of x and y, respectively.
For the case of steady- state, and for $V(x) = 0$, Eq. (1) is the generalized Emden-Fowler equation [8] given by

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \kappa f(x)g(y) = 0, x \geqslant 0, \tag{2}$$

subject to the conditions:

$$y(0) = A, y^{'}(0) = B. \tag{3}$$

When $f(x) = 1$ the Lane-Emden type equations becomes

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \kappa g(y) = 0, y(0) = A, y^{'}(0) = B, x \geqslant 0, \tag{4}$$

with specializing $g(y)$ several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere, and theory of thermionic currents. One of the important fields of application of this equation is the analysis of the diffusive transport and chemical reaction of species inside a porous catalyst particle. These equations are also one of the basic equations in the theory of stellar structure and have been the focus of many studies [3, 24].
Choosing $f(x) = 1$, $g(y) = y^M$, $A = 1$ and $B = 0$, we get the standard Lane-Emden equation of index n

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \kappa y^M(x) = 0, x \geqslant 0, \tag{5}$$

where $M > 0$ is constant.
The Lane-Emden equation can be analytically solved only for a few special, integer values of the index n. The Lane-Emden equation has analytical solutions for $M = 0, 1$, and 5. In other cases, there is not any analytical solution for the standard Lane-Emden equation.
For a polytropic system, the relation of pressure P and density ϱ is given by

$$P = K\varrho^{\gamma} \equiv K\varrho^{1+\frac{1}{M}}, \tag{6}$$

where K is the polytropic constant, γ is the adiabatic index (a parameter characterizing the behavior of the specific heat of a gas), and M is called the polytropic index.
We begin with the equation of mass continuity and the equation of hydrostatic equilibrium

$$\frac{1}{\varrho}\frac{dQ}{dr} = 4\pi r^2 \varrho, \tag{7}$$

$$\frac{1}{\varrho}\frac{dP}{dr} = -\frac{GQ(r)}{r^2} \tag{8}$$

where G is the constant of gravitation, $P(r)$ denote the hydrostatic pressure at a distance r from the center of a spherical cloud of gas and $Q(R)$ is the mass of the sphere at a certain radius r and ϱ is the density, at a distance r from the center of the sphere.

Eliminating $Q(r)$ between the continuity equation and the condition for hydrostatic equilibrium, we get

$$\frac{1}{r^2}\frac{d}{dr}(\frac{r^2}{\varrho}\frac{dP}{dr}) = -4\pi G\varrho \tag{9}$$

Replacing P by $K\varrho^{\gamma}$, we obtain that

$$\frac{1}{r^2}\frac{d}{dr}(\frac{r^2 K}{\varrho}\gamma\varrho^{\gamma-1}\frac{d\varrho}{dr}) = -4\pi G\varrho \tag{10}$$

Now, we define the following quantities

$$\varrho \equiv \lambda y^M, \gamma \equiv \frac{M+1}{M}. \tag{11}$$

Using the quantities (11), the Eq. (10) can be written as:

$$[\frac{(M+1)}{4\pi G}K\lambda^{\frac{1}{M}-1}]\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dy}{dr}) = -y^M. \tag{12}$$

Eq. 12 can be partially alleviated by the introduction of a radial variable x, is given by

$$x \equiv \frac{r}{\alpha}, \alpha \equiv \sqrt{[\frac{(M+1)}{4\pi G}K\lambda^{\frac{1}{M}-1}]}. \tag{13}$$

Inserting these relations into our previous equations, we derive the standard Lane-Emden equation with $g(y) = y^M(x)$,

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dy} = -y^M(x), \tag{14}$$

where x and y denote the independent and dependent variables, respectively. It must be solved with the original central conditions:

$$y(0) = 1, \quad \left.\frac{dy}{dx}\right|_{x=0} = 0, \tag{15}$$

which will ensure the regularity of the solution at the center.

1.2 *Methods have been proposed to solve Lane-Emden type equations*

The solution of the Lane-Emden type equations as well as a variety of nonlinear problems in astrophysics and quantum mechanics such as the scattering length calculations in the variable phase approach are numerically challenging because of singularity behavior at the origin.

The Lane-Emden type equations have been investigated due to their mathematical importance and the potential for applications in diversified applied sciences. For instance, we refer here to a few. The approximate analytical solutions to the Lane-Emden equations were presented by Mandelzweig and Tabakin [18], the authors have compared the results obtained by the quasilinearization method with the exact solutions. In 1993, Shawagfeh derived a nonperturbative approximate analytical solution for Lane-Emden equation, using the Adomian Decomposition Method, the solution obtained by Shawagfeh [25] is in the form of a power series with easily computable coefficients. In 2009, Chowdhury et al. presented a reliable algorithm based on the homotopy perturbation method to solve singular IVPs of time-independent Emden-Fowler type equations [4]. In 2008, Van Gorder et al. [26] handled the Homotopy Analysis Method (HAM) to calculate numerical solutions for Lane-Emden type equations, the author's illustrated that the series solutions obtained by the HAM converge in a larger interval than in the case of the corresponding traditional series solutions. Iqbal and Javed [17] applied the optimal homotopy asymptotic method for the analytic solution of singular Lane-Emden type equation. Authors in [9] applied the variational iteration method to approximate solution of a differential equation arising in astrophysics. In [29], Yildirim used the variational iteration method for solving singular IVPs of Lane-Emden type. Parand et al. [22], proposed the sinc-collocation method for solving astrophysics equations, the authors demonstrated that Sinc procedure converges with the solution at an exponential rate. Authors of [21] proposed a collocation method for solving some well-known classes of LaneEmden type equations. Bhrawy and Alofi [2] introduced a shifted JacobiGauss collocation spectral method for solving nonlinear Lane-Emden type equations, The spatial approximation is based on shifted Jacobi polynomials $P_{T,n}^{(\alpha,\beta)}(x)$ with $\alpha, \beta \in (-1, \infty)$, $T > 0$ and n is the polynomial degree. In 2012, Gokdogan et al. solved Lane-Emden equations arising in astrophysics using truncated shifted Chebyshev series together with the operational matrix [20]. Adibi and Rismani [1] used efficient and accurate a numerical scheme based on the Legendre-spectral method for solving singular IVPs of Lane-Emden type. Also, Parand et al. [23] obtained another approximate solution for nonlinear differential equations of Lane-Emden type based on rational Legendre pseudospectral approach.

1.3 *Motivation of paper*

Reproducing kernel theory has been recently emerged as a powerful framework in numerical analysis, computational mathematics, image processing, machine learning, finance, and probability and statistics. In recent years, the reproducing kernel method has increased its popularity and has been applied for a large spectrum of distinct problems, there are broader interests in using reproducing kernels for the solutions to several linear and nonlinear problems such as singular nonlinear second-order periodic boundary value problems [14], nonlinear system of second-order boundary value problems [13], one-dimensional variable-coefficient Burgers equation [12], the coefficient inverse problem of differential [7], nonlinear age-structured population equation [5], singular second order three-point boundary value problems [11], one-dimensional variable-coefficient Burgers equation [6], the generalized regularized long-wave equation [19], nonlinear delay differential equations of fractional order [15], variational problems depending on indefinite integrals [10]. In this paper, we will give the approximate of (2)-(3) in the reproducing kernel space. The advantages of this method are as follows:

- The conditions (3) can be imposed on the reproducing kernel space and therefore the reproducing kernel satisfying the conditions for determining the approximate solution can be calculated.
- We can prove that the approximate solution obtained by the presented method and its derivative are both uniformly convergent.
- The method can be easily implemented and its algorithm is simple and efficient to approximate the solution.

1.4 *Structure of paper*

The structure of this paper is organized as follows. In section 2, we present some standard definitions and results used throughout this paper. In section 3, we present our main results concerning to our method. In Section 4, we report our numerical finding to demonstrate the accuracy and applicability of the proposed method by considering five examples. Finally, we end the paper with few concluding remarks in Section 5.

2. Reproducing kernel spaces

In this section, we refer the recent work of [10, 15] and present some useful materials.

Definition 2.1 For a nonempty set $\mathcal{X}$, let $(\mathcal{H}, \langle ., . \rangle_{\mathcal{H}})$ be a Hilbert space of real-valued functions on some set $\mathcal{X}$. A function $K : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ is said to be the *reproducing kernel* of $\mathcal{H}$ if and only if

(1) $K(x, .) \in \mathcal{H}$, $\forall x \in \mathcal{X}$,
(2) $\langle \xi, K(x, .)\rangle_{\mathcal{H}} = \xi(x)$, $\forall \xi \in \mathcal{H}$, $\forall x \in \mathcal{X}$, (reproducing property).

Also, a Hilbert space of functions $(\mathcal{H}, \langle ., . \rangle_{\mathcal{H}})$ that possesses a reproducing kernel K is a reproducing kernel Hilbert space and we denote it by $(\mathcal{H}, \langle ., . \rangle_{\mathcal{H}}, K)$. In the following we often denote by K_x the function $K(x, .) : t \longmapsto K(x, t)$.

Definition 2.2 $W_2^3[0, R] = \{\xi \mid \xi''$ is an absolute continuous real-valued function on the interval $[0, R]$, $\xi''' \in L^2[0, R]$, $\xi(0) = \xi^{'}(0) = 0\}$. The inner product and the norm in the function space $W_2^3[0, R]$ are defined as follows:

$$\langle \xi, \zeta \rangle_{W_2^3} = \xi''(0)\zeta''(0) + \int_0^R \xi'''(x)\zeta'''(x)dx, \ ||\xi||_{W_2^3} = \sqrt{\langle \xi, \xi \rangle_{W_2^3}}.$$

Suppose that function $K_x \in W_2^3[0, R]$ satisfies the following generalized differential equations

$$\begin{cases} (-1)^3 \frac{\partial^6 K_x(t)}{\partial t^6} = \delta(t - x), \\ \frac{\partial^4 K_x(0)}{\partial t^4} = 0, \ \frac{\partial^2 K_x(0)}{\partial t^2} - \frac{\partial^3 K_x(0)}{\partial t^3} = 0, \ \frac{\partial^4 K_x(R)}{\partial t^4} = 0, \ \frac{\partial^3 K_x(R)}{\partial t^3} = 0. \end{cases} \tag{16}$$

where δ is Dirac delta function, therefore the following theorem holds.

Theorem 2.1 Under the assumptions of Eq. (16), Hilbert space $W_2^3[0, R]$ is a RKHS with the reproducing kernel function K_x, namely for any $\xi \in W_2^3[0, R]$ and each fixed $x \in [0, R]$,

$$\langle \xi, K_x \rangle_{W_2^3} = \xi(x).$$

Proof since $K_x \in W_2^3[0,R]$, applying integration by parts three times, we have

$$\begin{aligned}\langle \xi, K_x\rangle_{W_2^3} &= \xi''(0)\frac{\partial^2 K_x(0)}{\partial t^2} + \int_0^R \xi'''(t)\frac{\partial^3 K_x(t)}{\partial t^3}dt \\ &= \xi'(0)\frac{\partial^4 K_x(0)}{\partial t^4} + \xi''(0)[\frac{\partial^2 K_x(0)}{\partial t^2} - \frac{\partial^3 K_x(0)}{\partial t^3}] \\ &+ \xi''(R)\frac{\partial^3 K_x(R)}{\partial t^3} - \xi'(R)\frac{\partial^4 K_x(R)}{\partial t^4} - \int_0^R \xi(t)\frac{\partial^6 K_x(t)}{\partial t^6}\,dt. \end{aligned} \tag{17}$$

Therefore, Eq. (16) implies that

$$\langle \xi, K_x\rangle_{W_2^3} = \int_0^R \xi(t)\delta(t-x)\,dt = \xi(x).$$

■

While $x \neq t$, the function $K_x(t)$ is the solution of the following constant linear homogeneous differential equation with 6 order,

$$\frac{\partial^6 K_x(t)}{\partial t^6} = 0, \tag{18}$$

with the boundary condition:

$$\begin{cases} \frac{\partial^4 K_x(0)}{\partial t^4} = 0, \frac{\partial^3 K_x(R)}{\partial t^3} = 0, \\ \frac{\partial^2 K_x(0)}{\partial t^2} - \frac{\partial^3 K_x(0)}{\partial t^3} = 0, \frac{\partial^4 K_x(R)}{\partial t^4} = 0. \end{cases} \tag{19}$$

We know that Eq. (18) has characteristic equation $\lambda^6 = 0$, and the eigenvalue $\lambda = 0$ is a root whose multiplicity is 6. Hence, the general solution of Eq. (16) is

$$K_x(t) = \begin{cases} \sum_{i=1}^6 c_i(x)t^{i-1}, \; t \leqslant x, \\ \sum_{i=1}^6 d_i(x)t^{i-1}, \; t > x. \end{cases} \tag{20}$$

Now, we are ready to calculate the coefficients $c_i(x)$ and $d_i(x),\ i = 1, \ldots, 6$. Since

$$\frac{\partial^6 K_x(t)}{\partial t^6} = -\delta(t-x),$$

we have

$$\begin{cases} \frac{\partial^k K_x(x^+)}{\partial t^k} = \frac{\partial^k K_x(x^-)}{\partial t^k}, k = 0, \ldots, 4, \\ \frac{\partial^5 K_x(x^+)}{\partial t^5} - \frac{\partial^5 K_x(x^-)}{\partial t^5} = -1. \end{cases} \tag{21}$$

Then, using Eqs. (19) and (21), the unknown coefficients of Eq. (20) are uniquely obtained. Therefore,

$$K(x,t) = \begin{cases} k(x,t), t \leqslant x, \\ k(t,x), t > x, \end{cases}$$

where

$$k(x,t) = \frac{1}{4}x^2t^2 + \frac{1}{12}x^2t^3 - \frac{1}{24}xt^4 + \frac{1}{120}t^5.$$

3. The new implementation of the method

In this section, we shall give the exact or approximate solution of Eq. (2) in the reproducing kernel space $W_2^3[0,R]$. We introduce the following transformation

$$\xi(x) = y(x) - A - Bx. \tag{22}$$

Using the transformations (22), the equivalent problem of (4) can be written as:

$$\xi^{''}(x) + \frac{2}{x}\xi^{'}(x) + \kappa g(\xi(x)) = 0,\ \xi(0) = 0, \xi^{'}(0) = 0. \tag{23}$$

After multiplying Eq. (23) by x, we find that

$$x\xi^{''}(x) + 2\xi^{'}(x) + \kappa x g(\xi(x)) = 0. \tag{24}$$

We consider Eq. (24) as

$$\text{Ł}\xi(x) = F(x,\xi(x)), \quad 0 \leqslant x \leqslant R, \tag{25}$$

where $\text{Ł}\xi(x) = x\frac{d^2\xi}{dx^2} + 2\frac{d\xi}{dx}$ and $F(x,\xi) = -x\kappa f(x)g(\xi)$. We suppose that Eq. (25) has a unique solution. In order to represent the analytical solution of Eq. (2), it is easy to show that $\text{Ł} : W_2^3[0,R] \to W_2^1[0,R]$ is a bounded linear operator. Choosing a countable dense subset $\{x_i\}_{i=1}^{\infty}$ in the domain $[0,R]$, setting $\rho_i(x) = \text{Ł}^*K_{x_i}(x),\ i = 1,2,\cdots$.

Lemma 3.1 For (25), if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,R]$, then $\{\rho_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,R]$ and $\rho_i(x) = \text{Ł}_tR(x,x_i)$, where the subscript t of the operator Ł indicates that the operator Ł applies to the function of t.

The orthonormal system $\{\overline{\rho}_i(x)\}_{i=1}^{\infty}$ can be derived from the Gram-Schmidt orthogonalization process of $\{\rho_i(x)\}_{i=1}^{\infty}$

$$\overline{\rho}_i(x) = \sum_{k=1}^{i}\beta_{ik}\rho_k(x), \quad (\beta_{ii} > 0,\ i = 1,2,...). \tag{26}$$

So $\{\overline{\rho}_i(x)\}_{i=1}^{\infty}$ is the orthonormalized sequence and β_{ik} are orthogonal coefficients. Let $S = \overline{\{\overline{\rho}_i(x)\}_{i=1}^{\infty}}$ and $S^{\perp}$ be the orthogonal complement of S in $W_2^3[0,R]$, thus $W_2^3[0,R] = S \bigoplus S^{\perp}$.

Theorem 3.1 Suppose that the following conditions are satisfied

- $\{x_i\}_{i=1}^{\infty}$ be a countable dense subset in the domain $[0,R]$,
- The solution of Eq. (25) be unique.

Then the exact solution of Eq. (25) in $W_2^3[0,R]$ is given by

$$\xi(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}[F(x_k, \xi(x_k)]\overline{\rho}_i(x), \tag{27}$$

Representation of approximate solution:
For numerical computation, we give initial function $\xi_0 \in W_2^3[0,R]$ and by using (25), an iterative sequence is constructed as

$$\begin{cases} \text{Ł}[z_n(t)] = F(x, \xi_{n-1}(x)), \\ \xi_n(x) = P_n z_n(x), \end{cases} \tag{28}$$

where $z_n \in W_2^3[0,R]$ is the solution of (25) and $P_n : W_2^3[0,R] \to span\{\overline{\rho}_1(x), \overline{\rho}_2(x), ..., \overline{\rho}_n(x)\}$ is an orthogonal projection operator.

Theorem 3.2 Suppose that the following conditions are satisfied

- $\{x_i\}_{i=1}^{\infty}$ be a countable dense subset in the domain $[0,R]$,
- The solution of Eq. (2) be unique.

Then the solution of Eq. (25) is given by

$$z_n(x) = \sum_{i=1}^{\infty} \mathbf{H}_i \overline{\rho}_i(x), \quad n = 1, 2, ..., \tag{29}$$

where $\mathbf{H}_i = \sum_{k=1}^{i} \beta_{ik}[F(x, \xi_{n-1}(x)]$.

Proof The proof is similar to Theorem 3.1. ■

Therefore considering the numerical computation, we define the n-term approximation $\xi_n(x)$ to $\xi(x)$ by

$$\xi_n(x) = P_n z_n(x) = \sum_{i=1}^{n} \mathbf{H}_i \overline{\rho}_i(x), \quad n = 1, 2, ..., \tag{30}$$

where

$$\begin{cases} \mathbf{H}_1 = \beta_{11} F(x_1, \xi_0(x_1)), \\ \mathbf{H}_2 = \sum_{k=1}^{2} \beta_{2k} F(x_k, \xi_{k-1}(x_k)), \\ \mathbf{H}_3 = \sum_{k=1}^{3} \beta_{3k} F(x_k, \xi_{k-1}(x_k)), \\ \vdots \end{cases} \tag{31}$$

3.1 *The existence of solution*

Now, we will prove that the solution of (25) exists, and $\{\xi_n\}_{n=1}^{\infty}$ in the iterative formula (30) is convergent to the exact solution $\xi(x)$.

Lemma 3.2 (see [10]) For any $\xi(x) \in W_2^3[0,R]$, we have the following statement

$$\|\xi\|_{\infty} \leqslant \alpha \, \|\xi\|_{W_2^3[0,R]}, \tag{32}$$

where α, is a real constant.

Lemma 3.3 Suppose that, $\|\xi\|_{W_2^3[0,R]}$, is bounded, then there exists constant γ, such that

$$\|\xi\|_\infty \leqslant \gamma. \tag{33}$$

Proof Since $\|\xi\|_{W_2^3[0,R]}$ is bounded, by Lemma 3.2, $\|\xi\|_\infty$ is also bounded. ∎

Lemma 3.4 $B = \{\xi_n(x)| \ \|\xi_n\|_{W_2^3[0,R]} \leqslant \gamma\} \subset C[0,R]$, is a bounded set, where γ, is a real constant.

Proof From Lemma 3.3, there exists a positive constant $\gamma < \infty$, such that $\|\xi_n\|_\infty \leqslant \gamma$, for each $x \in [0,R]$ and each $\xi_n(x) \in B$. ∎

Lemma 3.5 $B = \{\xi_n(x)| \ \|\xi_n\|_{W_2^3[0,R]} \leqslant \gamma\} \subset C[0,R]$, is equicontinuous set, where γ is a real constant.

Proof Based on Lemma 3.4, for an arbitrary $\xi_n \in B$, we deduce

$$\begin{aligned} |\xi_n(x^{'}) - \xi_n(x)| &= |\langle \xi_n(t), R_{x'}(t) - R_{x''}(t)\rangle_{W_2^3[0,R]}| \\ &\leqslant \|\xi_n\|_{W_2^3[0,R]} \ \|R_{x'} - R_{x''}\|_{W_2^3[0,R]} \\ &\leqslant \|\xi_n\|_{W_2^3[0,R]} \ \|\frac{d}{dx}R_t|_{x\in[x',x'']}\|_{W_2^3[0,R]} \ |x^{'} - x^{''}| \\ &\leqslant \omega|x^{'} - x^{''}|, \end{aligned} \tag{34}$$

where ω is a positive constant.
Choosing

$$\delta = \frac{\epsilon}{\omega},$$

gives that for all $x^{'}, x^{''} \in [0,R]$ with $|x^{'} - x^{''}| < \delta$, we have

$$|\xi_n(x^{'}) - \xi_n(x^{''})| < \epsilon, \tag{35}$$

hence B is equicontinuous set. ∎

Theorem 3.3 Suppose that the following conditions hold

- $\{x_i\}_{i=1}^\infty$ be a countable dense subset in the domain $[0,R]$,
- $B = \{\xi_n(x)| \ \|\xi_n\|_{W_2^3[0,R]} \leqslant \gamma\} \subset C[0,R]$,
- Ł is a invertible operator of $\xi(x)$,
- $f(x,\xi)$ is continuous as $x \in [0,R]$ and $\xi = \xi(x) \in \mathbf{R}$.

Then there exists subsequence $\{\xi_{n_\kappa}\}_{\kappa=1}^\infty \subseteq B$, which $\lim_{\kappa\to\infty} \|\xi_{n_\kappa} - \xi\|_\infty = 0$, where

$$\xi(x) = Ł^{-1}F(x, \xi(x) \in \ W_2^3[0,R].$$

Proof Using (28), we have

$$Ł[\xi_n(x_k)] = F(x_k, \xi_{n-1}(x_k), n = 1, 2, \cdots \tag{36}$$

It follows by Lemmas 3.4 and 3.5 that B, is a pre-compact set. Then, any sequence in B, has a uniformly convergent subsequence whose limit belongs to B. Applying this principle we find that there exists sequence $\{n_\kappa\}_{\kappa=1}^{\infty}$, such that subsequence $\{y_{n_\kappa}\}_{\kappa=1}^{\infty}$, is uniformly convergent and $\lim_{\kappa\to\infty}\|\xi_{n_\kappa}-\xi\|_\infty=0$. Using (36), we have

$$Ł[\xi_{n_\kappa}(x)]|_{x=x_k}=F(x_k,\xi_{n_\kappa-1}(x_k)),\kappa=1,2,\dots. \tag{37}$$

Since Ł and f are continuous and $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,R]$, after taking limits for both sides of (37), we have

$$Ł[\xi(x)]=F(x,\xi(x)).,as\kappa\to\infty.$$

It follows that

$$\xi(x)=Ł^{-1}F(x,\xi(x)),$$

from the existence of $Ł^{-1}$.
Now, we prove that $\xi(x)\in\ W_2^3[0,R]$.
For arbitrary $\epsilon>0$, choose $\delta=\frac{\epsilon}{M}$. Let $\{(a_k,b_k)\}_{k=1}^{n}$ be a set of mutually disjoint open intervals $(a_k,b_k)\subset[0,R]$, satisfying $\sum_{k=1}^{n}(b_k-a_k)<\delta$. For $\xi(x)$, we have

$$\begin{aligned}|\xi(b_k)-\xi(a_k)|&=\lim_{\kappa\to\infty}|\xi_{n_\kappa}(b_k)-\xi_{n_\kappa}(a_k)|\\&=\lim_{\kappa\to\infty}|\langle\xi_{n_\kappa}(t),K_x(t)|_{x=b_k}-K_x(t)|_{x=a_k}\rangle|\\&\leqslant\lim_{\kappa\to\infty}\|\xi_{n_\kappa}\|_{W_2^3[0,R]}\|\partial_xK_x|_{x_k\in[a_k,b_k]}\|_{W_2^3[0,R]}|b_k-a_k|.\end{aligned}$$

Note that $\|\partial_x^jK_x(t)\|_{W_2^3[0,R]}\leqslant c$, $\lim_{\kappa\to\infty}\|\xi_{n_\kappa}\|_{W_2^3[0,R]}<\infty$ and therefore

$$|\xi(b_k)-\xi(a_k)|\leqslant M|b_k-a_k|. \tag{38}$$

Then, we have

$$\sum_{k=1}^{n}|\xi(b_k)-\xi(a_k)|\leqslant\sum_{k=1}^{n}M|b_k-a_k|<\epsilon. \tag{39}$$

So, ξ is absolutely continuous function.
Therefore $f(x,\xi(x))$ is absolutely continuous function.
Furthermore $\partial_xF(x,\xi(x))\in L^2[0,R]$. Then $F(x,\xi(x))\in W_2^1[0,R]$. Consequently $Ł^{-1}F(x,\xi(x))\in\ W_2^3[0,R]$, and we must have

$$\xi(x)\in W_2^3[0,R]\ .$$

This completes the proof. ■

Theorem 3.4 Suppose that the conditions of Theorem 3.3 hold, and the solution of (25) is existence and uniqueness, then

$$\|\xi_n-\xi\|_\infty\to0,\ as\ n\to\infty. \tag{40}$$

Proof Suppose $\{\xi_n\}_{n\geqslant1}\subset B$ doesn't converge to ξ. Then there exists a positive number ϵ_0, and a subsequence $\{\xi_{n_\kappa}(x)\}_{\kappa\geqslant1}\subset B$, such that

$$\|\xi_{n_\kappa}-\xi\|_\infty\geqslant\epsilon_0,\quad\kappa=1,2,... \tag{41}$$

Since $\{\xi_n(x)\}_{L\geqslant 1} \subset B$ is a pre-compact set, there exists subsequence of $\{\xi_{n_\kappa}(x)\}_{\kappa\geqslant 1}$ in which converges uniformly to $\widehat{y}$. Without of generality, we may assume that $\{\xi_{n_\kappa}(x)\}_{\kappa\geqslant 1}$ converges uniformly to $\widehat{\xi}$:

$$\|\xi_{n_\kappa} - \widehat{\xi}\|_\infty \to 0, \ as\ \kappa \to \infty, \tag{42}$$

Since the solution of Eq. (25) is unique, we have $\xi = \widehat{\xi}$, and so (42) contradicts (41). So the proof of Theorem 3.4 is completed. ∎

Theorem 3.5 Suppose that the conditions of Theorem 3.4 hold, and the solution of (25) is existence and uniqueness, then

$$\|\xi_n^{'} - \xi^{'}\|_\infty \to 0, \ \|\xi_n^{''} - \xi^{''}\|_\infty \to 0, \ as\ n \to \infty. \tag{43}$$

Proof The proof of this theorem follows from the proof of Theorem 3.4. ∎

4. Numerical examples and discussion

The method presented in this paper is applied on four examples to illustrate the efficiency and the applicability of the proposed method. The LaneEmden problem domain is $[0,\infty)$, we truncated the problem domain to $[0,R]$. The set of points in the domain $[0,R]$, are defined by

$$x_i = \frac{R}{n} i, \ i = 0,1,...,n, \tag{44}$$

where n is the number of interior points in the domain $[0,R]$.
The exact solution are available for Examples 4.1, 4.2 and 4.3. Therefore, we report the absolute error values which are defined as:

$$Er_n^{(i)}(x) = |y_n^{(i)}(x) - y^{(i)}(x)| = |\xi_n^{(i)}(x) - \xi^{(i)}(x)|, \ i = 0,1,2,$$

The exact solution is not available for Examples 4.4 and 4.5. Therefore, we report the averaged residual error of the nth-order approximation which is defined by:

$$\widetilde{\Xi}_n = \sqrt{\frac{1}{n+1}\sum_{i=1}^{n}[\Delta_n(x_i]^2}, \quad x_i = \frac{R}{n} i, \tag{45}$$

All the results are calculated by using the symbolic calculus software Maple. Results obtained by the method are compared with the exact solution, and with classical Tau method [20], the Hermite functions collocation method [21], the Sinc-Collocation method [22] and the Adomian's decomposition method [27].

Example 4.1 (see [21]) For $\kappa = 1$, $f(x) = 1$ and $g(y(x)) = -6y(x) - 4y(x)ln(y(x))$ Eq. (2) will be one of the Lane-Emden type equations that is

$$y^{''}(x) + \frac{2}{x} y^{'}(x) - 6y(x) - 4y(x)ln(y(x)) = 0, \tag{46}$$

subject to the boundary conditions

$$y(0) = 1, y^{'}(0) = 0. \tag{47}$$

The exact solution of (46) is given by $y(x) = e^{x^2}$.
We introduce the following transformation

$$\xi(x) = y(x) - 1. \tag{48}$$

Using the transformations (48), the equivalent problem of (46)-(47) can be written as:

$$\xi^{''}(x) + \frac{2}{x}\xi^{'}(x) - 6(\xi(x)+1) - 4(\xi(x)+1)ln(\xi(x)+1) = 0, \ \xi(0) = 0, \xi^{'}(0) = 0. \tag{49}$$

After multiplying Eq. (49) by x, we obtain

$$x\xi^{''}(x) + 2\xi^{'}(x) - 6x(\xi(x)+1) - 4x(\xi(x)+1)ln(\xi(x)+1) = 0. \tag{50}$$

Using the proposed method, we choose 150 points and 170 points on $[0, 7]$. For a numerical computation, we define initial function $\xi_0(x_0) = 0$. We calculate the error values $Er_n^{(i)}(x),\ i = 0, 1, 2$ in W_2^3, the computational error values are plotted in Figure 1. In Table 1 we compare the error values of the method with the method [21]. The error values $Er_n^{(i)}(x),\ i = 1, 2$ for $n = 150, 170$ obtained by proposed method are given in Table 2. From the numerical results, it is clear that the approximate solution is in good agreement with the exact solution.

Table 1. The error values of Example 1 for the x values and CPU time of the method for $n = 150, 170$.

x	$Er_{150}(x)$	$Er_{170}(x)$	Ref. [22]	
0.00	0.00000E+0	0.00000E+0	0.000E+0	
0.01	0.76552E-7	0.54395E-7	0.224E-7	
0.02	0.89710E-7	0.61427E-7	0.158E-7	
0.05	0.96673E-7	0.66480E-7	0.212E-7	
0.10	0.10409E-6	0.75756E-7	0.179E-7	
0.20	0.15333E-6	0.12705E-6	0.215E-7	
0.50	0.39256E-6	0.37210E-6	0.305E-7	
0.70	0.39256E-5	0.13451E-5	0.423E-7	
0.80	0.57360E-5	0.44859E-5	0.514E-7	
0.90	0.13812E-4	0.10870E-4	0.929E-7	
1.00	0.28907E-4	0.22765E-4	0.881E-7	
CPU-time (s)	22.901(s)	30.857(s)	-	-

Example 4.2 (see [21]) Consider the following homogeneous nonlinear Lane-Emden equation

$$\xi^{''}(x) + \frac{2}{x}\xi^{'}(x) - 4(2e^{\xi(x)} + e^{\frac{\xi(x)}{2}}) = 0, \tag{51}$$

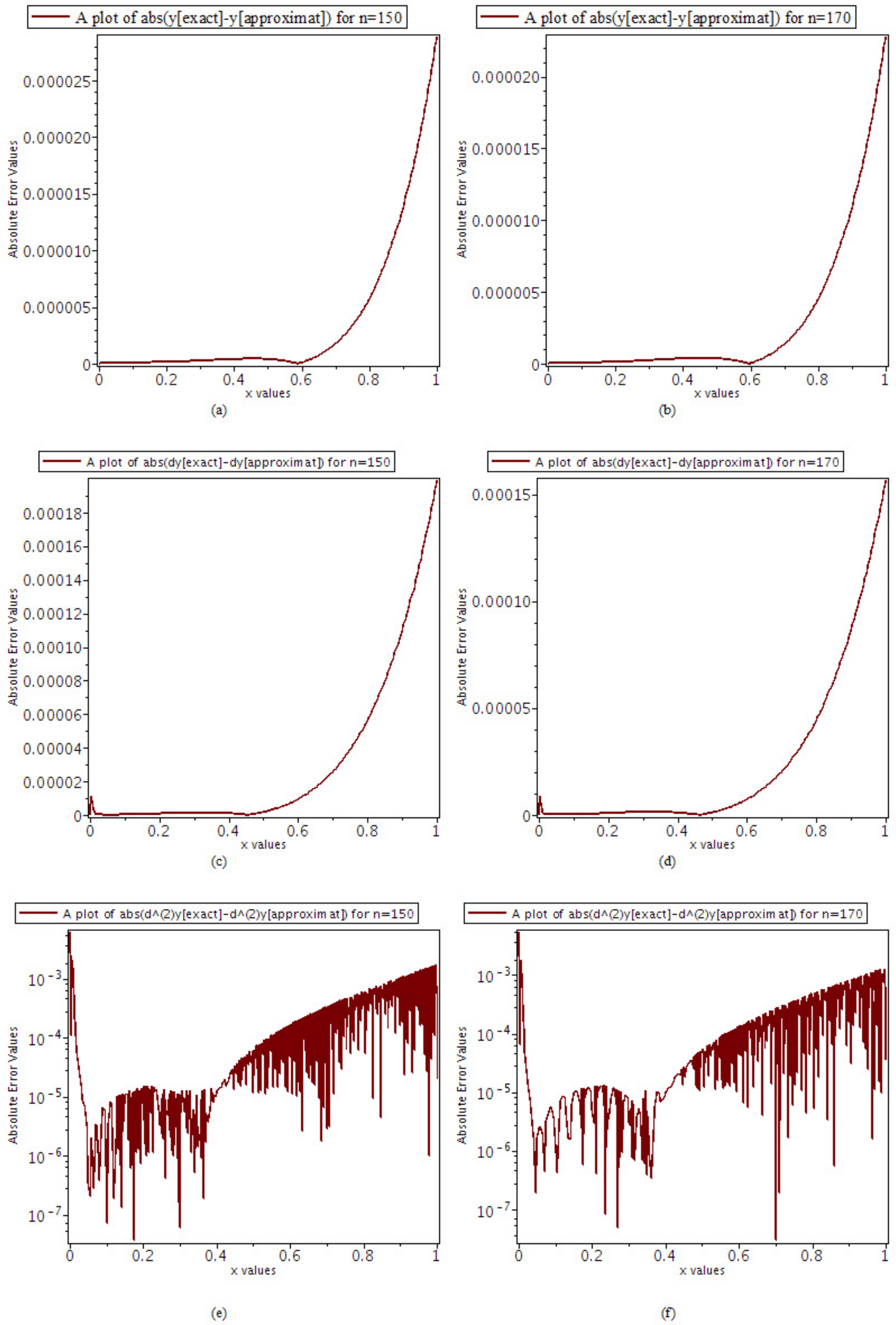

Figure 1. (a) The error value $Er_{150}(x)$; (b) The error value $Er_{170}(x)$; (c) The error value $Er^{'}_{150}(x)$; (d) The error value $Er^{'}_{170}(x)$; (e) The error value $Er^{''}_{150}(x)$, (f) The error value $Er^{''}_{170}(x)$.

Table 2. The error values $Er_n^{(i)}(x),\ i=1,2$ for $n=150,170$ obtained by using the proposed method.

x	$Er'_{150}(x)$	$Er'_{170}(x)$	$Er''_{150}(x)$	$Er''_{170}(x)$
0.00	0.00000E+0	0.00000e+0	0.62087E-2	0.54611E-2
0.01	0.332231E-5	0.16539E-5	0.98416E-3	0.55577E-3
0.02	0.56013E-6	0.33127E-6	0.55371E-4	0.31941E-4
0.05	0.12117E-6	0.12837E-6	0.27804E-6	0.16291E-5
0.10	0.22201E-6	0.26305E-6	0.29680E-5	0.18707E-5
0.20	0.800827E-6	0.80695E-6	0.81457E-5	0.57705E-5
0.50	0.17076E-5	0.12613E-5	0.11034E-4	0.10443E-4
0.70	0.25778E-4	0.20467E-4	0.59189E-5	0.26857E-5
0.80	0.56384E-4	0.44672E-4	0.20104E-4	0.13151E-4
0.90	0.10981E-3	0.86707E-4	0.27919E-4	0.11772E-4
1.00	0.19953E-3	0.15707E-3	0.19891E-5	0.32898E-4

subject to the boundary conditions

$$\xi(0)=0, \xi'(0)=0. \tag{52}$$

The exact solution of (51) is given by $\xi(x)=-2ln(1+x^2)$.
After multiplying Eq. (51) by x, we obtain

$$x\xi''(x)+2\xi'(x)-4x(2e^{\xi(x)}+e^{\frac{\xi(x)}{2}})=0,\ \xi(0)=0, \xi'(0)=0. \tag{53}$$

Using proposed method, we choose 190 points and 210 points on $[0,10]$, and define initial function $\xi_0(x_0)=0$. We calculate the error values $Er_n^{(i)}(x),\ i=0,1,2$ for $n=190,210$, the computational error values are plotted in Figure 2. In Table 3, we compare our results with those reported in [21]. The error values $Er_n^{(i)}(x),\ i=1,2$ for $n=190,210$ obtained by proposed method are given in Table 4. The numerical results show that the proposed method give us an approximate solution with a high degree of accuracy for $n=190,210$.

Example 4.3 (see [20]) For $\kappa=1$, $f(x)=1$ and $g(y(x))=y^5(x)$ Eq. (2) yields the standard Lane-Emden equation that was originally used to model several phenomena in astrophysics and mathematical physics.

$$y''(x)+\frac{2}{x}y'(x)+y^5(x)=0,$$

subject to the boundary conditions

$$y(0)=1, y'(0)=0. \tag{54}$$

The exact solution of (54) is given by $y(x)=(1+\frac{x^2}{3})^{-\frac{1}{2}}$. We introduce the following transformation

$$\xi(x)=y(x)-1. \tag{55}$$

Using the transformations (55), the equivalent problem of (54)-(54) can be written

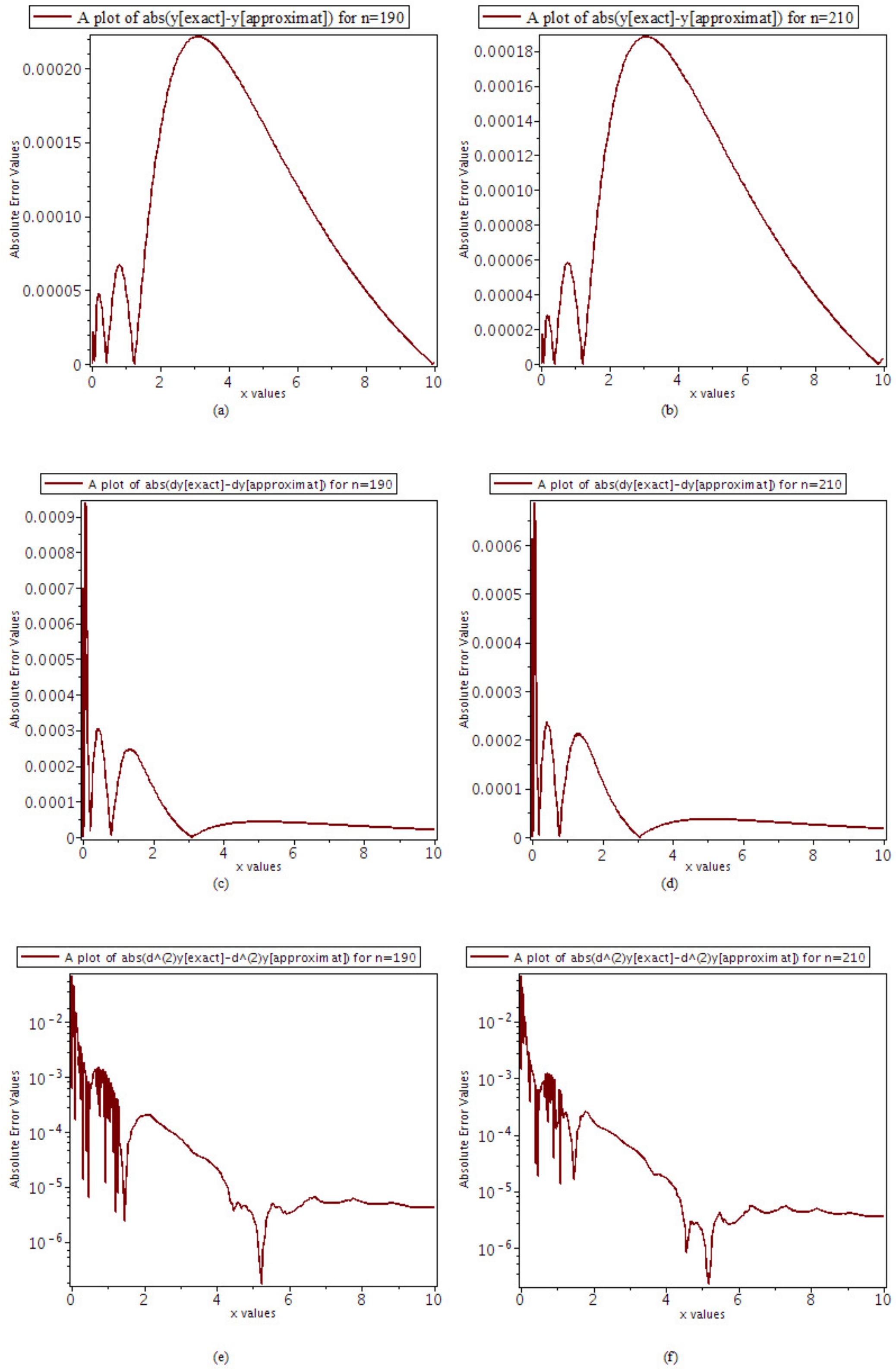

Figure 2. (a) The error value $Er_{190}(x)$; (b) The error value $Er_{210}(x)$; (c) The error value $Er^{'}_{190}(x)$; (d) The error value $Er^{'}_{210}(x)$; (e) The error value $Er^{''}_{190}(x)$, (f) The error value $Er^{''}_{210}(x)$.

Table 3. The error values of Example 4.2 for the x values and CPU time of the method for $n = 190, 210$.

x	$Er_{190}(x)$	$Er_{210}(x)$	Ref. [22]
0.00	0.00000E+0	0.00000E+0	0.000E+0
0.01	0.28645E-5	0.26382E-5	0.293E-5
0.10	0.14263E-4	0.10298E-4	0.394E-5
0.50	0.18204E-4	0.53098E-4	0.302E-5
1.00	0.49996E-4	0.40102E-4	0.931E-6
2.00	0.15816E-3	0.13788E-3	0.500E-6
3.00	0.22097E-3	0.18808E-3	0.810E-6
4.00	0.20263E-3	0.17083E-3	0.769E-6
5.00	0.16249E-3	0.13601E-3	0.664E-6
6.00	0.12036E-3	0.99975E-4	0.548E-6
7.00	0.82185E-4	0.67532E-4	0.170E-6
8.00	0.49364E-4	0.39757E-4	0.109E-5
9.00	0.21802E-4	0.16492E-4	0.121E-4
10.00	0.10762E-5	0.27728E-5	0.383E-4
CPU-time (s)	36.520(s)	45.833(s)	- -

Table 4. The error values $Er_n^{(i)}(x),\ i = 1, 2$ for $n = 190, 210$ obtained by using the proposed method.

x	$Er'_{190}(x)$	$Er'_{210}(x)$	$Er''_{190}(x)$	$Er''_{210}(x)$
0.00	0.00000E+0	0.00000e+0	0.69854E-1	0.65291E-1
0.01	0.51277E-3	0.46795E-3	0.33774E-1	0.29477E-1
0.10	0.85706E-4	0.52299E-3	0.11844E-1	0.12107E-1
0.50	0.28899E-4	0.21687E-3	0.26552E-3	0.27421E-3
1.00	0.15856E-3	0.14826E-3	0.24594E-3	0.22923E-3
2.00	0.14004E-3	0.11538E-3	0.25668E-3	0.21717E-3
3.00	0.69695E-5	0.35569E-5	0.67307E-4	0.55759E-3
4.00	0.34579E-4	0.30456E-4	0.12515E-4	0.98748E-5
5.00	0.42783E-4	0.36773E-4	0.24680E-5	0.25377E-5
6.00	0.40612E-4	0.34585E-4	0.62396E-5	0.54695E-5
7.00	0.35555E-4	0.30120E-4	0.65974E-5	0.56917E-5
8.00	0.30113E-4	0.25432E-4	0.60072E-5	0.51116E-5
9.00	0.25107E-4	0.21147E-4	0.04560E-5	0.42697E-5
10.00	0.20747E-4	0.17422E-4	0.42394E-5	0.35803E-5

as:

$$\xi''(x) + \frac{2}{x}\xi'(x) + (\xi(x)+1)^5 = 0,\ \ \xi(0) = 0, \xi'(0) = 0. \tag{56}$$

After multiplying Eq. (56) by x, we obtain

$$x\xi''(x) + 2\xi'(x) + x(\xi(x)+1)^5 = 0,\ \ \xi(0) = 0, \xi'(0) = 0. \tag{57}$$

Using proposed method, we choose 80 points and 100 points on $[0, 7]$, and define initial function $\xi_0(x_0) = 0$. We calculate the error values $Er_n^{(i)}(x),\ i = 0, 1, 2$ for $n = 80, 100$, the computational error values are plotted in Figure 3. In Table 5, we compare our results with those reported in [20]. The error values $Er_n^{(i)}(x),\ i = 1, 2$ for $n = 80, 100$ obtained by proposed method are given in Table 6. The numerical results show that the proposed method give us an approximate solution with a high degree of accuracy for $n = 80, 100$.

Table 5. The error values of Example 4.3 for the x values and CPU time of the method for $n = 80, 100$.

x	$Er_{80}(x)$	$Er_{100}(x)$	Ref. [20]($N_e = 6$)	Ref. [20]($N_e = 7$)
0.00	0.00000E+0	0.00000E+0	0.550000E-8	0.50000E-11
0.20	0.48061E-5	0.84502E-6	0.123534E-5	0.103920E-7
0.40	0.96997E-5	0.95173E-6	0.170817E-5	0.641001E-7
0.60	0.14544E-5	0.41714E-5	0.779963E-6	0.106438E-6
0.80	0.98436E-5	0.10207E-4	0.013216E-6	0.779280E-7
1.00	0.18497E-4	0.14113E-4	0.204581E-7	0.45062E-10
1.50	0.16480E-4	0.90643E-5	-	-
2.00	0.63368E-5	0.75983E-5	-	-
2.50	0.29412E-4	0.23052E-4	-	-
3.00	0.45314E-4	0.33206E-4	-	-
3.50	0.54214E-4	0.38585E-4	-	-
4.00	0.58139E-4	0.40683E-4	-	-
4.50	0.58927E-4	0.40764E-4	-	-
5.00	0.57869E-4	0.39690E-4	-	-
5.50	0.55794E-4	0.38001E-4	-	-
6.00	0.53209E-4	0.36025E-4	-	-
6.50	0.50423E-4	0.33958E-4	-	-
7.00	0.47622E-4	0.31913E-4	-	-
CPU-time (s)	10.233(s)	33.275(s)	-	-

Table 6. The error values $Er_n^{(i)}(x),\ i = 1, 2$ for $n = 80, 100$ obtained by using the proposed method.

x	$Er'_{80}(x)$	$Er'_{100}(x)$	$Er''_{80}(x)$	$Er''_{100}(x)$
0.00	0.00000E+0	0.00000E+0	0.10071E-1	0.85317E-3
0.50	0.44444E-4	0.25647E-4	0.21752E-3	0.32418E-4
1.00	0.30865E-4	0.11477E-4	0.18191E-3	0.10597E-3
1.50	0.32437E-4	0.27060E-4	0.52937E-3	0.80761E-4
2.00	0.50964E-4	0.34893E-4	0.17798E-4	0.82889E-5
2.50	0.39685E-4	0.25909E-4	0.22927E-4	0.19846E-4
3.00	0.24188E-4	0.15067E-4	0.28192E-4	0.19531E-4
3.50	0.12130E-4	0.69876E-5	0.17537E-4	0.11589E-4
4.00	0.41691E-5	0.18375E-5	0.12996E-4	0.74320E-5
4.50	0.60101E-6	0.12110E-5	0.84191E-4	0.48755E-5
5.00	0.33293E-5	0.29007E-5	0.34413E-5	0.29804E-5
5.50	0.47994E-5	0.37522E-5	0.16027E-5	0.15329E-5
6.00	0.54482E-5	0.40979E-5	0.12325E-5	0.85376E-6
6.50	0.56247E-5	0.41453E-5	0.18108E-6	0.21510E-6
7.00	0.55345E-4	0.40200E-5	0.79007E-6	0.62098E-6

Example 4.4 (see [22]) For $\kappa = 1,\ f(x) = 1$ and $g(y(x)) = y^3(x)$ Eq. (2) will be

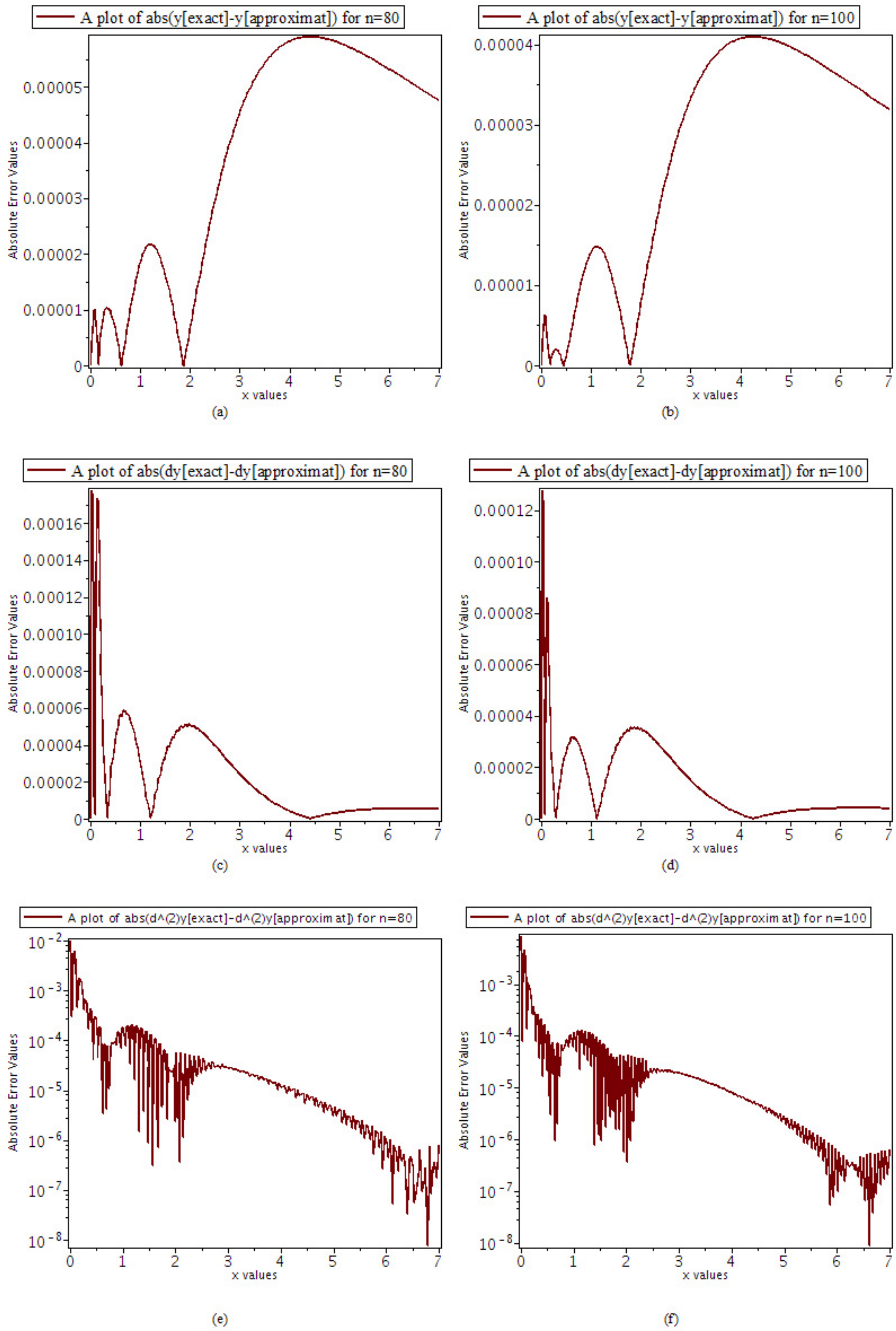

Figure 3. (a) The error value $Er_{80}(x)$; (b) The error value $Er_{100}(x)$; (c) The error value $Er'_{80}(x)$; (d) The error value $Er'_{100}(x)$; (e) The error value $Er''_{80}(x)$, (f) The error value $Er''_{100}(x)$.

one of the standard Lane-Emden type equations that is

$$y^{''}(x) + \frac{2}{x}y^{'}(x) + y^3(x) = 0, \tag{58}$$

subject to the boundary conditions

$$y(0) = 1, y^{'}(0) = 0. \tag{59}$$

We introduce the following transformation

$$\xi(x) = y(x) - 1. \tag{60}$$

Using the transformations (60), the equivalent problem of (58)-(59) can be written as:

$$\xi^{''}(x) + \frac{2}{x}\xi^{'}(x) + \xi^3(x) = 0,\ \xi(0) = 0, \xi^{'}(0) = 0. \tag{61}$$

After multiplying Eq. (61) by x, we obtain

$$x\xi^{''}(x) + 2\xi^{'}(x) + x\xi^3(x) = 0. \tag{62}$$

Using proposed method, we choose 130 points and 150 points on $[0, 6]$, and define initial function $\xi_0(x_0) = 0$. Table 7 shows the values of $y(x)$ obtained by the proposed method for standard Lane-Emden equation, and those obtained by Parand [22] and Horedt [16]. The resulting graph of LaneEmden equation for $n = 130, 150$ are shown in Figure 4.

Table 7. Approximation of $y(x)$ for the present method and solutions obtained by Parand [22] and Horedt [16] and CPU time of the method for $n = 130, 150$.

x	$y_{130}(x)$	$y_{150}(x)$	Ref. [22]	Ref. [16]
0.000	1.000000	1.000000	1.000000	1.000000
0.010	0.998338	0.998337	0.998354	0.998336
0.500	0.959841	0.959841	0.959951	0.959839
1.000	0.855063	0.855062	0.855165	0.855058
1.500	0.719508	0.719506	-	-
2.000	0.582852	0.582851	-	-
2.500	0.461120	0.461121	-	-
3.000	0.359214	0.359216	-	-
3.500	0.276245	0.276249	-	-
4.000	0.209261	0.209266	-	-
4.500	0.155047	0.155052	-	-
5.000	0.110797	0.110802	0.110618	0.110820
5.500	0.074263	0.074268	-	-
6.000	0.043715	0.043720	0.043688	0.043738
CPU-time (s)	15.553(s)	20.576(s)	-	-

Remark 4.1 In Example 4.4, when $n = 130$, the averaged residual error is $0.44662E - 8$. When $n = 150$, the averaged residual error is $0.93561E - 9$.

Example 4.5 (see [27]) For $\kappa = 1$, $f(x) = 1$ and $g(y(x)) = \sinh(y(x))$ Eq. (2) will be one of the Lane-Emden type equations that we solve

$$y^{''}(x) + \frac{2}{x}y^{'}(x) + \sinh(y(x)) = 0, \tag{63}$$

subject to the boundary conditions

$$y(0) = 1, y^{'}(0) = 0. \tag{64}$$

We introduce the following transformation

$$\xi(x) = y(x) - 1. \tag{65}$$

Using the transformations (65), the equivalent problem of (63)-(64) can be written as:

$$\xi^{''}(x) + \frac{2}{x}\xi^{'}(x) + \sinh(\xi(x)) = 0,\ \xi(0) = 0, \xi^{'}(0) = 0. \tag{66}$$

After multiplying Eq. (66) by x, we obtain

$$x\xi^{''}(x) + 2\xi^{'}(x) + x\sinh(\xi(x)) = 0. \tag{67}$$

Using proposed method, we choose 90 points and 110 points on $[0, 3]$, and define initial function $\xi_0(x_0) = 0$. Table 8 shows the values of $y(x)$ obtained by the proposed method for standard Lane-Emden equation, and those obtained by Wazwaz [27]. The resulting graph of LaneEmden equation for $n = 90, 110$ are shown in Figure 5.

Table 8. Approximation of $y(x)$ for the present method and solutions obtained by Wazwaz [27] CPU time of the method for $n = 90, 110$.

x	$y_{90}(x)$	$y_{110}(x)$	Ref. [27]
0.000	1.000000000	1.000000000	1.000000000
0.400	0.969045293	0.969044650	0 .969043758
0.450	0.960949308	0.960948655	0 .960947741
0.500	0.951962708	0.951962041	0 .951961101
0.600	0.931398824	0.931398119	0 .931397142
0.700	0.907532465	0.907531707	0 .907530823
0.800	0.880566466	0.880565644	0 .880565336
0.900	0.850724118	0.850723222	0 .850724891
0.920	0.844431359	0.844430447	0.844432883
1.000	0.818245502	0.818244525	0.818251666
1.300	0.707569410	0.707568187	0.707679500
1.500	0.625442463	0.625441109	0.625891607
1.510	0.621209567	0.621208208	0.621688632
2.000	0.406626820	0.406625380	0.413669103
2.500	0.193297045	0.193295903	-
3.000	0.011274538	0.011274023	-
CPU-time (s)	7.176(s)	11.076(s)	-

Remark 4.2 In Example 4.5, when $n = 90$, the averaged residual error is $0.10304E-6$. When $n = 110$, the averaged residual error is $0.89256E-7$.

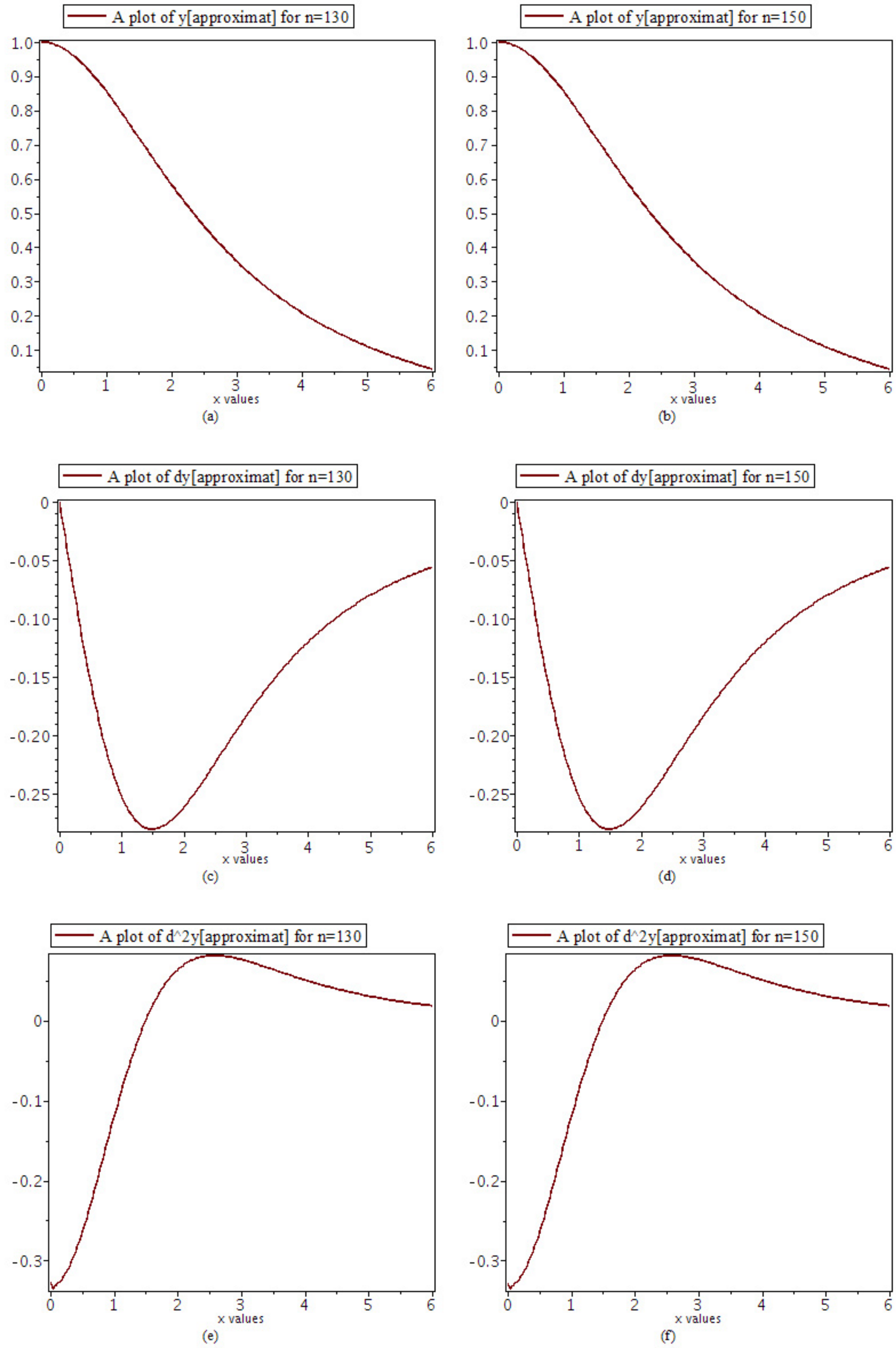

Figure 4. Approximation of $y(x)$ for the present method and its derivatives for $n = 130, 150$.

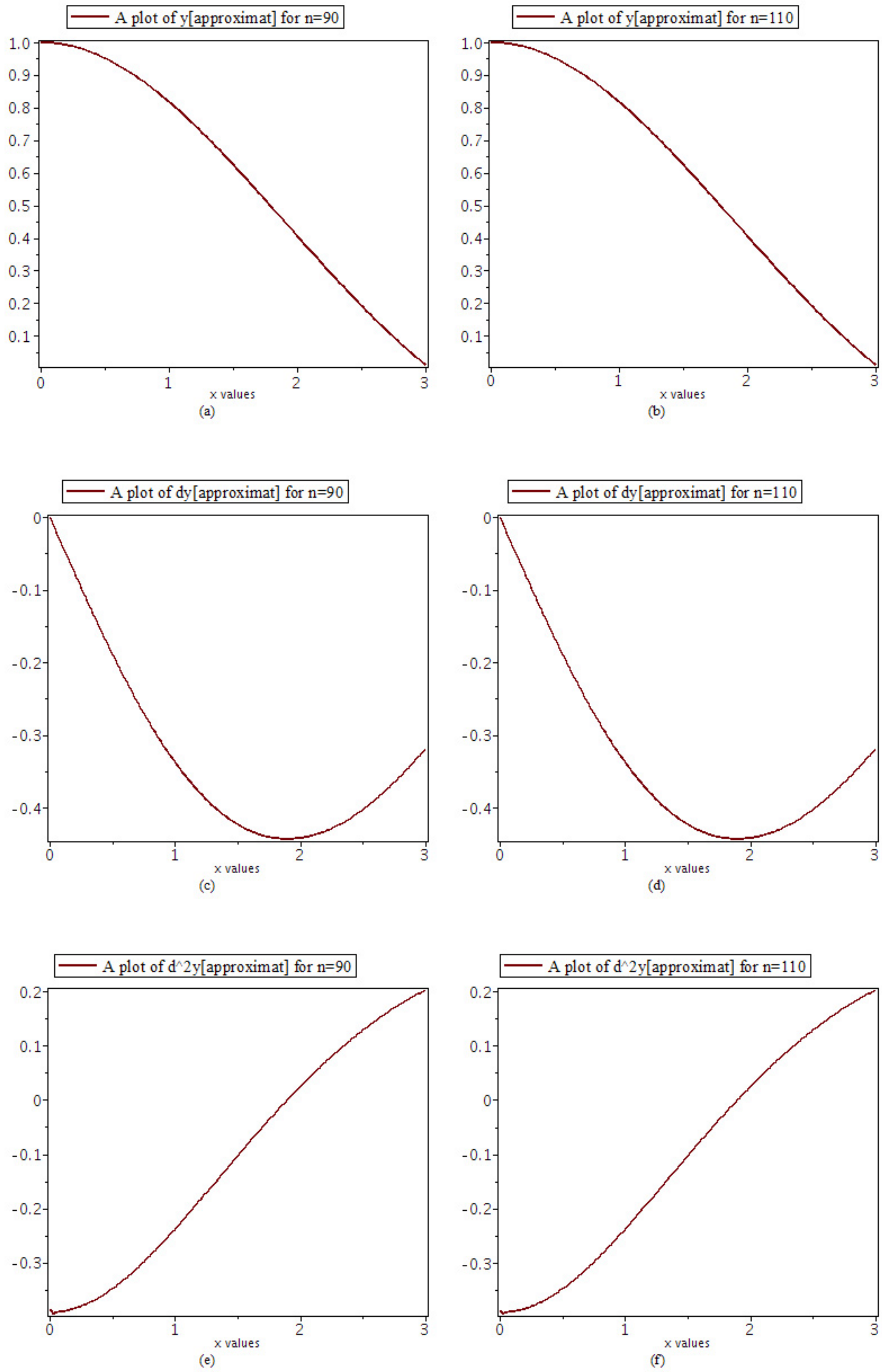

Figure 5. Approximation of $y(x)$ for the present method and its derivatives for $n = 90, 110$.

5. Conclusions

In this manuscript an efficient method in reproducing kernel space is developed to solve Lane-Emden equations. The numerical examples are presented to show the accuracy of the proposed method. Based on the obtained results of the proposed method for illustrative examples, we have the following conclusions:

- By using this method, we introduced an iterative sequence which converges uniformly to exact solution. The results of the proposed method for Lane-Emden equations clearly indicate that method is accurate even when the singularity occurs at the boundary.
- The results obtained by using the proposed method compared with the existing method show that the present method is valid.
- The numerical results demonstrate the relatively rapid convergence of the proposed method.
- By increasing the value of n we get better results.
- We should also point out that the examples studied in the previous section show that method is very effective and convenient for solving astrophysics equations.

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