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A Simple and Efficient Method for Solving Multi-Objective Programming Problems and Multi-Objective Optimal Controls

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Abstract. In this paper, a new approach based on weighted sum algorithm is applied to solve multi-objective optimal programming problems (MOOPP) and multi-objective optimal control problems (MOOCP). In this approach, first, we change the problem into a new one whose optimal solution is obtained by solving some single-objective problems simply. Then, we prove that the optimal solutions of the two problems are equal. Numerical examples are presented to show the efficiency of the given approach.

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1. Introduction

In many practical or real life problems, we normally need to optimize several objectives simultaneously which can even be in conflict to each other. Minimizing the cost while maximizing comfort when buying a car, and maximizing performance whilst minimizing fuel consumption and emission of pollutants of a vehicle are examples of multi-objective optimization problems involving two and three objectives, respectively. In these problems, it is difficult to find an optimal solution to achieve the extreme value of every objective function so that the decision maker is searching for the compromise solution. Based on this idea, the concepts of Pareto

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optimal solution and weakly Pareto optimal solution are introduced into multi-objective programming problem [6]. Therefore, different researchers have defined the term "solving a multi-objective optimization problem" in various ways [11].

In the area of control engineering, multi-objective optimization has been discussed used by the control engineers (See e.g. [4]). These objectives often involve conflict situations such as control energy, tracking performance, robustness, and etc. A suitable introduction on the concepts of MOOCP may be found in [5]. Also, one may find an overview on multi-objective optimization applications in control engineering in [8]. Over the years, some indirect and direct approaches have been presented to extract analytical and approximate Pareto solutions of MOOCP 's [2] and [3]. But, these approaches are facing some difficulties. For instance, convexity of the objectives is a basic requirement which limits the scope of applications of such methods [10].

In this paper, we express a simple and efficient method for solving multi-objective programming problems and multi-objective optimal controls. In this method, we express a problem equivalent to the multi-objective problem. The solution of this new problem is obtained by solving some single-objective problems. Finally, we prove that the solutions obtained this method are optimal.

The paper is organized as follows. In Section 2, the multi-objective problem and the equivalent problem are formulated and we prove that the optimal solution of these two problems is equal. In Section 3, some famous test functions are expressed and we solve these problems to demonstrate the efficiency of the method in the final section.

2. Problem formulation

In this section, we introduce a multi-objective optimal problem and its equivalent problem. Then, the necessary theorem is proved.
Multi-objective programming problem is described as follow:

$$\begin{aligned} &\min Z(X) = (Z_1(X), Z_2(X), ..., Z_k(X)) \\ &\text{s.t. } g_j(X) \leqslant (\geqslant or =) b_j, \quad j = 1, 2, ..., m, \\ &\quad x_r \geqslant 0, \quad r = 1, 2, ..., n, \\ &\quad X = (x_1, x_2, ..., x_n). \end{aligned} \tag{1}$$

First, we consider the required concepts.

Definition 2.1 (Dominates). The feasible vector X^* is said to be a Pareto minimum or nondominated solution, if there does not exist feasible vector X such that

$$\begin{aligned} &Z_i(X) \leqslant Z_i(X^*) \quad for\ all \quad i = 1, 2, ..., k \\ &Z_j(X) < Z_j(X^*) \quad for\ one \quad j \in \{1, 2, ..., k\} \end{aligned}$$

and denote it as $X^* \preceq X$. On the other hand, X^* is said to be a weak Pareto minimum, if there does not exist X such that

$$Z_i(X) < Z_i(X^*) \quad for\ all \quad i = 1, 2, ..., k$$

Theorem 2.2 *Consider multi-objective programming problem (1), a vector $Z' = (z'_1, z'_2, ..., z'_k)$ is an optimal solution for problem (1) corresponding to X', if and* *only if $Z^{'}$ is the optimal solution of the following problem:*

$$\begin{aligned} &\min Z(X) = M_1(Z_1(X) - z_1^*)^2 + M_2(Z_2(X) - z_2^*)^2 + ... + M_k(Z_k(X) - z_k^*)^2 \\ &\text{ s.t. } g_j(X) \leqslant (\geqslant \, or =) b_j, \quad j = 1, 2, ..., m, \\ &\quad x_r \geqslant 0, \quad r = 1, 2, ..., n, \\ &\quad X = (x_1, x_2, ..., x_n), \end{aligned} \tag{2}$$

when $z_1^*, ..., z_k^*$ is the optimal solution of the following single problems:

$$\begin{aligned} &\min Z(X) = Z_i(X) \\ &\text{ s.t. } g_j(X) \leqslant (\geqslant \, or =) b_j, \quad j = 1, 2, ..., m, \\ &\quad x_r \geqslant 0, \quad r = 1, 2, ..., n, \\ &\quad X = (x_1, x_2, ..., x_n), \end{aligned} \tag{3}$$

and $M_1, ..., M_k$ are positive and very large.

Proof Suppose we solve k single-objective problems (3) and get the solutions $z_1^*, ..., z_k^*$.
If $Z^{'} = (z_1^{'}, z_2^{'}, ..., z_k^{'})$ is a nondominated solution for problem (1) corresponding to $X^{'}$, then,

$$\begin{pmatrix} z_1^{'} \\ z_2^{'} \\ \vdots \\ z_k^{'} \end{pmatrix} \preceq \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{pmatrix} \qquad x_r \in \mathbb{R}^n, \quad \forall z_j \in \mathbb{R}$$

Since $X^{'}$ is a feasible solution for problem (1), it is a feasible solution for problem (2) and also

$$\begin{pmatrix} z_1^{'} - z_1^* \\ z_2^{'} - z_2^* \\ \vdots \\ z_k^{'} - z_k^* \end{pmatrix} \preceq \begin{pmatrix} z_1 - z_1^* \\ z_2 - z_2^* \\ \vdots \\ z_k - z_k^* \end{pmatrix}$$

Since z_i^* is the optimal solution for single-objective problems (3) and they are the best possible solutions for these problems, then:

$$z_i^* \leqslant z_i^{'} \Rightarrow -z_i^* \geqslant -z_i^{'}$$

and

$$z_i^{'} - z_i^* \geqslant z_i^{'} - z_i^{'} = 0$$

therefore:

$$\begin{aligned}
&M_1(z_1^{'} - z_1^*)^2 \leqslant M_1(z_1 - z_1^*)^2\\
&M_2(z_2^{'} - z_2^*)^2 \leqslant M_2(z_2 - z_2^*)^2\\
&\vdots\\
&M_k(z_k^{'} - z_k^*)^2 \leqslant M_k(z_k - z_k^*)^2\\
&\Rightarrow M_1(z_1^{'} - z_1^*)^2 + ... + M_k(z_k^{'} - z_k^*)^2 \leqslant M_1(z_1 - z_1^*)^2 + ... + M_k(z_k - z_k^*)^2\\
&\Rightarrow Z^{'} \preceq Z \qquad \forall Z \in \mathbb{R}^k
\end{aligned}$$

that means the nondominated solution of problem (1) is the same nondominated solution of problem (2).
Contrariwise, suppose that $Z^{'} = (z_1^{'}, z_2^{'}, ..., z_k^{'})$ is a nondominated solution for problem (2) corresponding to $X^{'}$. As in the previous part, $X^{'}$ is the feasible solution of problem (1). We need to prove that $X^{'}$ and the objective vector $Z^{'} = (z_1^{'}, z_2^{'}, ..., z_k^{'})$ are the optimal solution to problem (1).
Ad absurdum: Suppose that the optimal solution of problem (2) is not the optimal solution of problem (1), in this case, there exists a vector $\hat{X}$ and corresponding objective vector $\hat{Z} = (\hat{z}_1, \hat{z}_2, ..., \hat{z}_k)$ so that:

$$\begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \vdots \\ \hat{z}_k \end{pmatrix} \preceq \begin{pmatrix} z_1^{'} \\ z_2^{'} \\ \vdots \\ z_k^{'} \end{pmatrix}$$

also

$$\begin{pmatrix} \hat{z}_1 - z_1^* \\ \hat{z}_2 - z_2^* \\ \vdots \\ \hat{z}_k - z_k^* \end{pmatrix} \preceq \begin{pmatrix} z_1^{'} - z_1^* \\ z_2^{'} - z_2^* \\ \vdots \\ z_k^{'} - z_k^* \end{pmatrix}$$

In this case, as in the previous part

$$z_i^* \leqslant \hat{z}_i \Rightarrow -z_i^* \geqslant -\hat{z}_i$$

and

$$\hat{z}_i - z_i^* \geqslant \hat{z}_i - \hat{z}_i = 0$$

So

$$\begin{aligned}
&M_1(\hat{z}_1 - z_1^*)^2 \leqslant M_1(z_1^{'} - z_1^*)^2\\
&M_2(\hat{z}_2 - z_2^*)^2 \leqslant M_2(z_2^{'} - z_2^*)^2\\
&\vdots\\
&M_k(\hat{z}_k - z_k^*)^2 \leqslant M_k(z_k^{'} - z_k^*)^2\\
&\Rightarrow M_1(\hat{z}_1 - z_1^*)^2 + ... + M_k(\hat{z}_k - z_k^*)^2 \leqslant M_1(z_1^{'} - z_1^*)^2 + ... + M_k(z_k^{'} - z_k^*)^2\\
&\Rightarrow \hat{Z} \preceq Z^{'} \qquad \hat{Z} \in \mathbb{R}^k, \quad Z^{'} \in \mathbb{R}^k
\end{aligned}$$

This is a contradiction, so $Z^{'} = (z_1^{'}, z_2^{'}, ..., z_k^{'})$ and $X^{'}$ are the optimal solution of problem (1). ■

Now, consider the following multi-objective optimal control problem:

$$\begin{aligned} \min \ & \{\varphi_1(X(.),U(.),T), ..., \varphi_j(X(.),U(.),T), ..., \varphi_k(X(.),U(.),T)\} \\ \text{s.t.} \ & F(t,X(t),\dot{X}(t),U(t)) = 0, \\ & h(t,X(t),U(t)) \geqslant 0, \\ & r(X(0),X(T)) = 0. \end{aligned} \tag{4}$$

Here, function F represents the model equation with X as model states, U as control inputs, and T as the final time.
Also, φ_j denotes the j-th individual objective function as

$$\varphi_j(X(.),U(.),T) = \int_0^T f_j(t,X(t),U(t))dt,$$

with continuous function f_j, while h and r are the path constraints and boundary constraints of the optimal control problem.
We can rewrite the problem using the above-mentioned method in the form of (2) and solve $k+1$ single-objective problems using Pontryagin's minimum principle (PMP) method or other methods.
With respect to multi-objective optimal control problems, this issue assumes more importance because solving single-objective optimal control problems is much simpler and there are famous and efficient methods in this regard.

Theorem 2.3 *Consider multi-objective programming problem (4), a vector* $\Phi^{'} = (\varphi_1^{'}, \varphi_2^{'}, ..., \varphi_k^{'})$ *is an optimal solution for problem (4) corresponding to* $\left(X^{'}(t), U^{'}(t)\right)$*, if and only if* $\Phi^{'}$ *is the optimal solution of the following problem:*

$$\begin{aligned} \min \ & \Phi\left(X,U,t\right) = M_1(\varphi_1\left(X(.),U(.),t\right) - \varphi_1^*)^2 + \cdots + M_k(\varphi_k\left(X(.),U(.),t\right) - \varphi_k^*)^2 \\ \text{s.t.} \ & F(t,X(t),\dot{X}(t),U(t)) = 0, \\ & h(t,X(t),U(t)) \geqslant 0, \\ & r(X(0),X(T)) = 0, \end{aligned} \tag{5}$$

when $\varphi_1^*, \varphi_2^*, ..., \varphi_k^*$ *is the optimal solution of the following single problems:*

$$\begin{aligned} \min \ & \Phi(X) = \varphi_i(X) \\ \text{s.t.} \ & F(t,X(t),\dot{X}(t),U(t)) = 0, \\ & h(t,X(t),U(t)) \geqslant 0, \\ & r(X(0),X(T)) = 0. \end{aligned} \tag{6}$$

and $M_1, ..., M_k$ *are positive and very large.*

Proof Suppose we solve k single-objective problems (6) and get the solutions $\varphi_1^*, \varphi_2^*, ..., \varphi_k^*$.
If $\Phi^{'} = (\varphi_1^{'}, \varphi_2^{'}, ..., \varphi_k^{'})$ is a nondominated solution for problem (4) corresponding

to $\left(X^{'}(t), U^{'}(t)\right)$, then,

$$\begin{pmatrix} \varphi_1^{'} \\ \varphi_2^{'} \\ \vdots \\ \varphi_k^{'} \end{pmatrix} \preceq \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_k \end{pmatrix} \qquad t \in [0, T], \qquad \forall \varphi_j \in \mathbb{R}.$$

Since $\left(X^{'}(t), U^{'}(t)\right)$ is a feasible solution for problem (4), it is a feasible solution for problem (5) and also

$$\begin{pmatrix} \varphi_1^{'} - \varphi_1^* \\ \varphi_2^{'} - \varphi_2^* \\ \vdots \\ \varphi_k^{'} - \varphi_k^* \end{pmatrix} \preceq \begin{pmatrix} \varphi_1 - \varphi_1^* \\ \varphi_2 - \varphi_2^* \\ \vdots \\ \varphi_k - \varphi_k^* \end{pmatrix}$$

Since φ_i^* is the optimal solution for single-objective problems (6) and they are the best possible solutions for these problems, then:

$$\varphi_i^* \leqslant \varphi_i^{'} \Rightarrow -\varphi_i^* \geqslant -\varphi_i^{'}$$

and

$$\varphi_i^{'} - \varphi_i^* \geqslant \varphi_i^{'} - \varphi_i^{'} = 0$$

therefore:

$$\begin{aligned}
&M_1(\varphi_1^{'} - \varphi_1^*)^2 \leqslant M_1(\varphi_1 - \varphi_1^*)^2 \\
&M_2(\varphi_2^{'} - \varphi_2^*)^2 \leqslant M_2(\varphi_2 - \varphi_2^*)^2 \\
&\vdots \\
&M_k(\varphi_k^{'} - \varphi_k^*)^2 \leqslant M_k(\varphi_k - \varphi_k^*)^2 \\
&\Rightarrow M_1(\varphi_1^{'} - \varphi_1^*)^2 + ... + M_k(\varphi_k^{'} - \varphi_k^*)^2 \leqslant M_1(\varphi_1 - \varphi_1^*)^2 + ... + M_k(\varphi_k - \varphi_k^*)^2 \\
&\Rightarrow \Phi^{'} \preceq \Phi \qquad \forall \Phi \in \mathbb{R}^k
\end{aligned}$$

that means the nondominated solution of problem (4) is the same nondominated solution of problem (5).

Contrariwise, suppose that $\Phi^{'} = (\varphi_1^{'}, \varphi_2^{'}, ..., \varphi_k^{'})$ is a nondominated solution for problem (5) corresponding to $\left(X^{'}(t), U^{'}(t)\right)$. As in the previous part, $\left(X^{'}(t), U^{'}(t)\right)$ is the feasible solution of problem (4). We need to prove that $\left(X^{'}(t), U^{'}(t)\right)$ and the objective vector $\Phi^{'} = (\varphi_1^{'}, \varphi_2^{'}, ..., \varphi_k^{'})$ are the optimal solution to problem (4). Ad absurdum: Suppose that the optimal solution of problem (5) is not the optimal solution of problem (4), in this case, there exists a vector $\left(\hat{X}, \hat{U}\right)$ and corresponding objective vector $\hat{\Phi} = (\hat{\varphi}_1, \hat{\varphi}_2, ..., \hat{\varphi}_k)$ so that:

$$\begin{pmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \\ \vdots \\ \hat{\varphi}_k \end{pmatrix} \preceq \begin{pmatrix} \varphi_1^{'} \\ \varphi_2^{'} \\ \vdots \\ \varphi_k^{'} \end{pmatrix}$$

also

$$\begin{pmatrix} \hat{\varphi}_1 - \varphi_1^* \\ \hat{\varphi}_2 - \varphi_2^* \\ \vdots \\ \hat{\varphi}_k - \varphi_k^* \end{pmatrix} \preceq \begin{pmatrix} \varphi_1^{'} - \varphi_1^* \\ \varphi_2^{'} - \varphi_2^* \\ \vdots \\ \varphi_k^{'} - \varphi_k^* \end{pmatrix}$$

In this case, as in the previous part

$$\varphi_i^* \leqslant \hat{\varphi}_i \Rightarrow -\varphi_i^* \geqslant -\hat{\varphi}_i$$

and

$$\hat{\varphi}_i - \varphi_i^* \geqslant \hat{\varphi}_i - \hat{\varphi}_i = 0$$

So

$$\begin{aligned}
&M_1(\hat{\varphi}_1 - \varphi_1^*)^2 \leqslant M_1(\varphi_1^{'} - \varphi_1^*)^2 \\
&M_2(\hat{\varphi}_2 - \varphi_2^*)^2 \leqslant M_2(\varphi_2^{'} - \varphi_2^*)^2 \\
&\vdots \\
&M_k(\hat{\varphi}_k - \varphi_k^*)^2 \leqslant M_k(\varphi_k^{'} - \varphi_k^*)^2 \\
&\Rightarrow M_1(\hat{\varphi}_1 - \varphi_1^*)^2 + ... + M_k(\hat{\varphi}_k - \varphi_k^*)^2 \leqslant M_1(\varphi_1^{'} - \varphi_1^*)^2 + ... + M_k(\varphi_k^{'} - \varphi_k^*)^2 \\
&\Rightarrow \hat{\Phi} \preceq \Phi^{'} \qquad \hat{\Phi} \in \mathbb{R}^k, \quad \Phi^{'} \in \mathbb{R}^k
\end{aligned}$$

This is a contradiction, so $\Phi^{'} = (\varphi_1^{'}, \varphi_2^{'}, ..., \varphi_k^{'})$ and $\big(X^{'}(t), U^{'}(t)\big)$ are the optimal solution of problem (4). ■

Note: In single objective optimization problems, if we want to compute a proven global optimal solution to an optimization problem with nonlinear conditions, we have basically two families of methods. Stochastic (also metaheuristic, genetic, ...) methods are easier to apply, quite performant and do not require cost and constraint functions to have some properties like being twice continuously differentiable or something like that. The drawback is that there is no guarantee that we converge to the global solution in finite time. Deterministic methods converge to the solution in finite time. But they heavily depend on the type of nonlinearity: in case we have continuous variables only (that is, no integer conditions) and convex constraints, then any local optimum is global already. In case we have integer variables and/or non-convex constraints, we have to use branch-and-bound methods. Here the original problem is split into a sequence of subproblems, each being continuous and convex (maybe even linear).

3. Numerical results

Now, to show the efficiency of our method and to explain how it works, we solve numerical examples with two evolutionary algorithms. It is worth mentioning that these examples are famous test functions in order for the readers to be able to compare and contrast the two methods.

Example 3.1 Consider a bi-objective programming problem (Binh and Korn function):

$$\min \begin{cases} f_1(x,y) = 4x^2 + 4y^2 \\ f_2(x,y) = (x-5)^2 + (y-5)^2 \end{cases}$$
$$\text{s.t.} \begin{cases} g_1(x,y) = (x-5)^2 + y^2 \leqslant 25 \\ g_2(x,y) = (x-8)^2 + (y+3)^2 \geqslant 7.7 \end{cases}$$
$$0 \leqslant x \leqslant 5,\ 0 \leqslant y \leqslant 3.$$

As mentioned in the previous section, we first solve two single-objective problems using *fmincon* Matlab function:

$$\begin{aligned} &\min f_1(x,y) = 4x^2 + 4y^2 \\ &\text{s.t.} \begin{cases} g_1(x,y) = (x-5)^2 + y^2 \leqslant 25 \\ g_2(x,y) = (x-8)^2 + (y+3)^2 \geqslant 7.7 \end{cases} \\ &\qquad 0 \leqslant x \leqslant 5,\ 0 \leqslant y \leqslant 3. \end{aligned} \tag{7}$$

Optimal solution of problem (7) is $(x^*, y^*) = (0, 0)$ and $f_1^* = 0$.

$$\begin{aligned} &\min f_2(x,y) = (x-5)^2 + (y-5)^2 \\ &\text{s.t.} \begin{cases} g_1(x,y) = (x-5)^2 + y^2 \leqslant 25 \\ g_2(x,y) = (x-8)^2 + (y+3)^2 \geqslant 7.7 \end{cases} \\ &\qquad 0 \leqslant x \leqslant 5,\ 0 \leqslant y \leqslant 3. \end{aligned} \tag{8}$$

Optimal solution of problem (8) is $(x^*, y^*) = (5, 3)$ and $f_2^* = 4$.
Now, we minimize the following objective function with the constraints of the main problem:

$$\min \quad f = 10^6((4x^2 + 4y^2) - 0)^2 + 10^6(((x-5)^2 + (y-5)^2) - 4)^2$$

Optimal solution of this problem is $(x^*, y^*) = (1.3655, 1.3655)$ and $f_1^* = 14.9167, f_2^* = 26.4192$. According to the Pareto solution of this problem shown in Figure 1, obtain using Genetic algorithm [1], this solution is Pareto and optimal.

Example 3.2 Consider the following problem (Chakong and Haimes function):

$$\min \begin{cases} f_1(x,y) = 2 + (x-2)^2 + (y-1)^2 \\ f_2(x,y) = 9x - (y-1)^2 \end{cases}$$
$$\text{s.t.} \begin{cases} g_1(x,y) = x^2 + y^2 \leqslant 225 \\ g_2(x,y) = x - 3y + 10 \leqslant 10 \end{cases}$$
$$-20 \leqslant x, y \leqslant 20$$

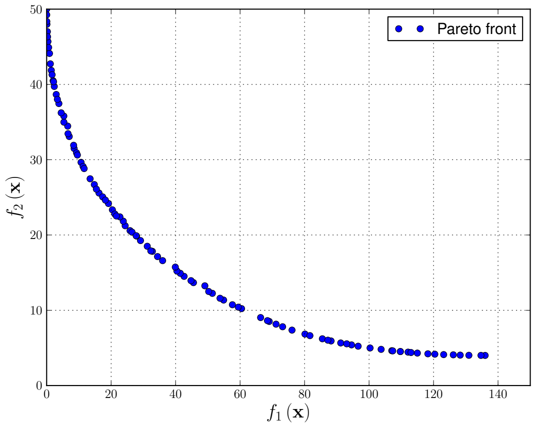

Figure 1. The Pareto front of Example 1.

We first solve two single-objective problems using *fmincon* Matlab function:

$$\begin{aligned} &\min\ f_1(x,y) = 2 + (x-2)^2 + (y-1)^2 \\ &\text{s.t.} \begin{cases} g_1(x,y) = x^2 + y^2 \leqslant 225 \\ g_2(x,y) = x - 3y + 10 \leqslant 10 \end{cases} \\ &\qquad -20 \leqslant x, y \leqslant 20 \end{aligned} \tag{9}$$

Optimal solution of problem (9) is $(x^*, y^*) = (2, 1)$ and $f_1^* = 2$.

$$\begin{aligned} &\min\ f_2(x,y) = 9x - (y-1)^2 \\ &\text{s.t.} \begin{cases} g_1(x,y) = x^2 + y^2 \leqslant 225 \\ g_2(x,y) = x - 3y + 10 \leqslant 10 \end{cases} \\ &\qquad -20 \leqslant x, y \leqslant 20 \end{aligned} \tag{10}$$

Optimal solution of problem (10) is $(x^*, y^*) = (-14.2302, -4.7434)$ and $f_2^* = -161.0591$.
Now, we minimize the following objective function with the constraints of the main problem:

$$\min \quad f = 10^6((2 + (x-2)^2 + (y-1)^2) - 2)^2 + 10^6((9x - (y-1)^2) + 161.0591)^2$$

Optimal solution of this problem is $(x^*, y^*) = (-5.5790, -1.8597)$ and $f_1^* = 67.6191, f_2^* = -58.3889$. According to the Pareto solution of this problem shown in Figure 2, obtain using genetic algorithm [1], this solution is Pareto and optimal.

Example 3.3 Consider tumour anti-angiogenesis as follow [9]:
State and control variables:
p : primary tumour volume (mm3)
q : carrying capacity, or endothelial support (mm3)
u: anti-angiogenic agent

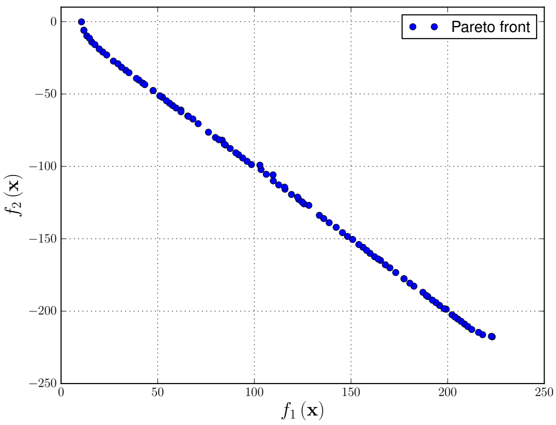

Figure 2. The Pareto front of Example 2.

$$\begin{cases} \min \quad (p(t_f), p(t_f) + 140y(t_f)) \\ \text{s.t. } \dot{p} = -0.084\, p \ln\left(\frac{p}{q}\right), \quad p(0) = 8000, \\ \dot{q} = 5.85q^{2/3} - 0.00873q^{4/3} - 0.02q - 0.15qu, \quad q(0) = 10000, \\ \dot{y} = u, \qquad y(0) = 0, \\ y(t_f) \leqslant 45, \quad 0 \leqslant u(t) \leqslant 15. \end{cases}$$

Control u appears linearly: bang-bang and singular arcs.
We first solve two single-objective problems:

$$\begin{cases} \min \quad p(t_f) \\ \text{s.t. } \dot{p} = -0.084\, p \ln\left(\frac{p}{q}\right), \quad p(0) = 8000, \\ \dot{q} = 5.85q^{2/3} - 0.00873q^{4/3} - 0.02q - 0.15qu, \quad q(0) = 10000, \\ \dot{y} = u, \qquad y(0) = 0, \\ y(t_f) \leqslant 45, \quad 0 \leqslant u(t) \leqslant 15. \end{cases} \tag{11}$$

Optimal solution of problem (11) is $(t_f^*, p(t_f)^*) = (8.101, 2856)$.

$$\begin{cases} \min \quad p(t_f) + 140y(t_f) \\ \text{s.t. } \dot{p} = -0.084\, p \ln\left(\frac{p}{q}\right), \quad p(0) = 8000, \\ \dot{q} = 5.85q^{2/3} - 0.00873q^{4/3} - 0.02q - 0.15qu, \quad q(0) = 10000, \\ \dot{y} = u, \qquad y(0) = 0, \\ y(t_f) \leqslant 45, \quad 0 \leqslant u(t) \leqslant 15. \end{cases} \tag{12}$$

Optimal solution of problem (12) is $(t_f^*, Z_2^*) = (8.705, 8000)$.
Now, we minimize the following objective function with the constraints of the main problem:

$$\min \quad f = 10^6(p(t_f) - 2856)^2 + 10^6((p(t_f) + 140y(t_f) - 8000)^2$$

Optimal solution of this problem is $(t_f^*, z^*) = (9.378, 1502336)$ and $f_1^* = 4000, f_2^* = 7560$. According to the Pareto solution of this problem shown in Figure 3, this solution is Pareto and optimal.

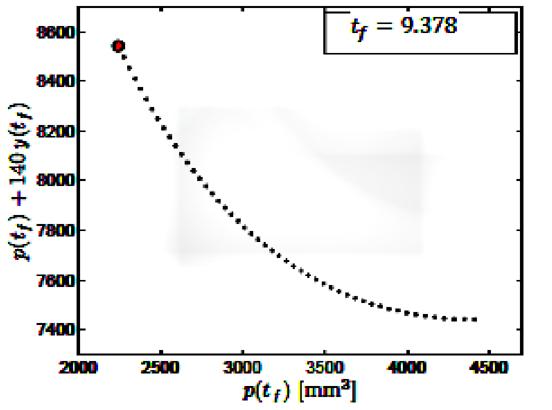

Figure 3. The Pareto front of Example 3.

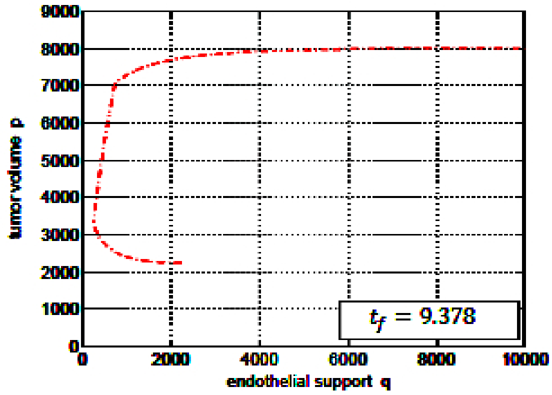

Figure 4. The optimal tumor volume p in Example 3.

4. Conclusion

This paper proposed a practical approach based on weighted sum algorithm for obtaining the solution to general multi-objective optimal programming problems and multi-objective optimal control problems. Compared with other methods, this approach is more practical since the results are obtained by solving single-objective problems. Furthermore, the new problem can be solved easily with the help of efficients algorithms. It is also especially practical and accurate enough for systems with nonlinear terms.

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